STUDIA MATHEMATICA, T. XC. (1988)

Inductive limit of operators and its applications

by

J. JANAS (Kraków)

Abstract. The paper deals mainly with spectral properties of inductive limits of operators in Hilbert spaces. Applications to inductive limits of Riesz and hyponormal as well as to other operators are also given. In particular, extensions of Marchenko theorems to normal operators are shown.

Notation. All Hilbert spaces considered in the following are complex. For a complex Hilbert space H with a scalar product $(\cdot, \cdot)$, $L(H)$ denotes the space of all linear and bounded operators on H. If T is a closed densely defined operator in H, then $\sigma(T)$ (respectively $\sigma_\pi(T)$) stands for the spectrum of T (respectively the approximate point spectrum of T) and $D(T)$ for the domain of T.

I. Let us recall the notion of inductive limit of Hilbert spaces. Suppose we are given a sequence of Hilbert spaces H_k, $k = 1, 2, \ldots$ We say that a Hilbert space H is an *inductive limit* of the H_k if there are isometries $\gamma_k^l\colon H_k \to H_l$ $(k \leqslant l)$ and $\gamma_k\colon H_k \to H$ such that the following conditions are satisfied:

(a) γ_k^k is the identity on H_k.
(b) $\gamma_k^m = \gamma_l^m \circ \gamma_k^l$ if $k \leqslant l \leqslant m$.
(c) $\gamma_k = \gamma_l \circ \gamma_k^l$ if $k \leqslant l$.
(d) $H = \bigvee_{k \geqslant 1} \gamma_k H_k$.

By the above definition there is no loss of generality in denoting by $\|\cdot\|$ the norm in every H_k and also in H. We write $H = \underrightarrow{\lim}_\gamma H_k$.

The starting point of our work concerns a generalization of a result of Marchenko [3]. Namely, for a given sequence of selfadjoint operators A_n in H_n he has found a sufficient condition for the essential selfadjointness of the operator

$$A_\infty \varphi = \lim_{m\to\infty} \gamma_m A_m \gamma_n^m \varphi_n, \qquad \varphi = \gamma_n \varphi_n,$$

on $D_\infty = \bigcup_n \gamma_n D_n$, where $D_n =$ the domain of A_n. It turns out that a similar condition imposed on general closed densely defined operators L_k in H_k also guarantees the existence of

$$(1)\qquad L_\infty\varphi = \lim_{m\to\infty}\gamma_m L_n \gamma_n^m \varphi_n, \qquad \varphi = \gamma_n\varphi_n,$$

on $D_\infty = \bigcup_n \gamma_n D_n$, where D_n = the domain of L_n. To be more precise, suppose that the above sequence L_n satisfies the following condition:

$$\text{(i)}\qquad \gamma_n^{n+1} D_n \subset D_{n+1}, \qquad \gamma_n^{n+1} D_n^* \subset D_{n+1}^*,$$

where D_n^* = the domain of L_n^*.

THEOREM 1.1. *Let L_n be a sequence of densely defined closable operators in H_N with domains D_N satisfying the above condition* (i). *Assume that for any $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for every $m > n \geqslant n_0(\varepsilon)$ and any $\varphi \in D_n$, $\psi \in D_n^*$*

$$(*)\qquad \|(L_m\gamma_n^m - \gamma_n^m L_n)\varphi\| \leqslant \varepsilon(\|\varphi\| + \|L_m\gamma_n^m\varphi\| + \|L_n\varphi\|),$$

$$(**)\qquad \|(L_m^*\gamma_n^m - \gamma_n^m L_n^*)\psi\| \leqslant \varepsilon(\|\psi\| + \|L_m^*\gamma_n^m\psi\| + \|L_n^*\psi\|).$$

Then equality (1) *defines a closable densely defined operator L_∞ on D_∞. Moreover, for any $\psi \in D_\infty^* = \bigcup_n \gamma_n D_n^*$ the limit*

$$A_\infty\psi \overset{\text{df}}{=} \lim_{m\to\infty}\gamma_m L_m^*\gamma_n^m\psi_n, \qquad \psi = \gamma_n\psi_n,$$

exists and $A_\infty \subset L_\infty^*$.

Proof. Let $\varphi = \gamma_n\varphi_n$, $\psi = \gamma_n\psi_n$. The existence of $\lim_{m\to\infty}\gamma_m L_m\gamma_n^m\varphi_n = L_\infty\varphi$ and $\lim_{m\to\infty}\gamma_m L_m^*\gamma_n^m\psi_n = A_\infty\psi$ can be proved exactly in the same way as for selfadjoint operators [3].

Now we will show that L_∞ is closable on D_∞. First note that $\bar{D}_\infty = H$ and so L_∞^* exists. Moreover, we have $D_\infty^* \subset D(L_\infty^*)$.

Indeed, for $\varphi = \gamma_n\varphi_n \in D_\infty$ and $\psi = \gamma_s\psi_s \in D_\infty^*$ we write (with $m > s$ and $m > n$)

$$(\gamma_n\varphi_n, \gamma_m L_m^*\gamma_s^m\psi_s) = (\gamma_m\gamma_n^m\varphi_n, \gamma_m L_m^*\gamma_s^m\psi_s) = (\gamma_n^m\varphi_n, L_m^*\gamma_s^m\psi_s)$$
$$= (L_m\gamma_n^m\varphi_n, \gamma_s^m\psi_s) = (\gamma_m L_m\gamma_n^m\varphi_n, \gamma_m\gamma_s^m\psi_s) = (\gamma_m L_m\gamma_n^m\varphi_n, \gamma_s\psi_s).$$

Hence

$$(\varphi, A_\infty\psi) = \lim_{m\to\infty}(\gamma_n\varphi_n, \gamma_m L_m^*\gamma_s^m\psi_s) = \lim_{m\to\infty}(\gamma_m L_m\gamma_n^m\varphi_n, \gamma_s\psi_s) = (L_\infty\varphi, \psi)$$

and so $\psi \in D(L_\infty^*)$.

Now applying a result of von Neumann [4] we know that L_∞ is closable (and $L_\infty = L_\infty^{**}$). The inclusion $A_\infty \subset L_\infty^*$ is also evident by the above equalities. ∎

Remark 1.2. If $L_n \in L(H_n)$ and if the norms $\|L_n\|$ are uniformly bounded and satisfy condition $(*)$ then L_∞ can be extended to a bounded operator L on H.

The operator $\bar{L}_\infty \overset{\text{df}}{=} L$ shares some properties of the approximating sequence L_n. For example: if the L_n are symmetric (hyponormal) then L is also symmetric (hyponormal). Here T is called *hyponormal* if T is densely defined, closed, $D(T) \subset D(T^*)$ and $\|T^*x\| \leqslant \|Tx\|$, $x \in D(T)$. (We do not assume that $D(T) = D(T^*)$, the condition required in [4]).

The proofs are straightforward and left to the reader.

Remark 1.3. Theorem 1.1 has an obvious extension to linear mappings $L_n\colon H_n \to K_n$ (closed, densely defined) between Hilbert spaces H_n, K_n. Conditions $(*)$ and $(**)$ also have natural interpretation in this context. In particular, $(*)$ is trivially satisfied if $L_{n+1}\gamma_n^{n+1} = \tau_n^{n+1}L_n$, where $\tau_n^{n+1}\colon K_n \to K_{n+1}$ denotes an isometric embedding.

Now we will give a few applications (examples) of Theorem 1.1.

EXAMPLE 1.4. Let A_i be a sequence of closed densely defined operators in H_i, a Hilbert space with norm $\|\cdot\|_i$. Suppose we are given a sequence $e_i \in D(A_i) \cap D(A_i^*)$, $\|e_i\|_i = 1$, such that

$$(\alpha)\qquad \sum_i \|A_i e_i\|_i < +\infty, \qquad \sum_i \|A_i^* e_i\|_i < +\infty.$$

Then the expression

$$\sum_i I_1 \otimes \ldots \otimes I_{i-1} \otimes A_i \otimes I_{i+1} \otimes \ldots,$$

where I_i denotes the identity on H_i, has a meaning and defines a closable densely defined operator in the infinite tensor product $H = \bigotimes_{1(e_i)}^\infty H_i$ with the stabilizing sequence (e_i).

Here to apply Theorem 1.1 we put

$$L_n = \sum_1^n I_1 \otimes \ldots \otimes I_{i-1} \otimes A_i \otimes I_{i+1} \otimes \ldots \otimes I_n,$$

$$H_n = \bigotimes_1^n H_i, \qquad \gamma_n^{n+1} f = f \otimes e_{n+1}.$$

Conditions $(*)$ and $(**)$ of Theorem 1.1 can be verified, using (α), in the same way as in [3] for selfadjoint operators.

EXAMPLE 1.5 (a problem). Let $R_{jn}\colon L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$, $1 \leqslant j \leqslant n$, be the Riesz transforms. Set $d\mu_n = \alpha_n|x|^{-n}dx$, $\alpha_n = \Gamma(n/2)/\pi^{n/2}$. A straightforward reasoning shows that $L_n = R_{1n}$ is a densely defined closable operator in

$L^2(\mu_n)$. By a result of [12] there exists a unique (up to a constant) measure μ on the Borel subsets of $\boldsymbol{R}^\infty = \boldsymbol{R} \times \boldsymbol{R} \times \ldots$ such that:

(a) $\mu(\lambda E) = \mu(E)$, $\lambda > 0$.

(b) $\mu(UE) = \mu(E)$ for every unitary U: $l^2 \to l^2$ which maps $\boldsymbol{R}_0^\infty \to \boldsymbol{R}_0^\infty$, where $\boldsymbol{R}_0^\infty$ denotes the real sequences with all but a finite number of coordinates different from zero.

(c) If p_n: $\boldsymbol{R}^\infty \to \boldsymbol{R}^n$ is the canonical projection, then for any $E = p_n^{-1}(E_n)$, $\mu(E) = \mu_n(E_n)$.

We claim that the L_n satisfy condition (*) of Theorem 1.1, where

$$\gamma_n^{n+1}: f \to f, \quad \gamma_n: f \to f \in L^2(\mu).$$

It is easy to check that both above mappings are isometric. In fact, for $f \in L^2(\mu_n)$ we have

$$\|\gamma_n^{n+1} f\|^2 = \int |f|^2 \, d\mu_{n+1} = \alpha_{n+1} \int_{\boldsymbol{R}^n} |f|^2 \int_{-\infty}^{\infty} \frac{dx_{n+1}\, dx}{(|x|^2 + x_{n+1}^2)^{(n+1)/2}}$$

$$= \int_{\boldsymbol{R}^n} |f|^2 \frac{\sqrt{\pi}}{|x|^n} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} dx = \int |f|^2 \, d\mu_n = \|f\|^2.$$

It is clear that $L^2(\mu) = \varinjlim_\gamma L^2(\mu_n)$. A careful reasoning shows that

$$L_{n+1} \gamma_n^{n+1} f = \gamma_n^{n+1} L_n f, \quad f \in L^1(\mu_n) \cap L^2(\mu_n).$$

It follows that

$$L_m \gamma_n^m f = \gamma_n^m L_n f$$

for any $m > n$, and so condition (*) of Theorem 1.1 is trivially satisfied. In order to apply Th. 1.1 we should also verify condition (**) for R_{1n}^*. Unfortunately, we were not able to prove or disprove this condition.

Example 1.6. Suppose we are given a sequence T_i of bounded operators in H_i. Theorem 1.1 can also be applied to obtain the well-known sufficient condition for the existence of the countable tensor product of the T_i. Assume that there exists a sequence $e^{(i)}$ of unit vectors, $\|e^{(i)}\| = 1$, such that

$$\sum_i \|e^{(i)} - T_i e^{(i)}\| < +\infty.$$

Let $K = \bigotimes_{1(e^{(i)})}^\infty H_i$ be the countable tensor product of the H_i with the stabilizing sequence $(e^{(i)})$. Let $K_n = \bigotimes_1^n H_i$, $L_n = \bigotimes_1^n T_i$ and define γ_n^{n+1}: $K_n \to K_{n+1}$, γ_n: $K_n \to K$ by

$$\gamma_n^{n+1} f = f \otimes e^{(n+1)}, \quad \gamma_n f = f \otimes e^{(n+1)} \otimes e^{(n+2)} \otimes \ldots$$

By a straightforward computation one can check that

$$L_m \gamma_n^m - \gamma_n^m L_n \text{ satisfies } (*).$$

If there exists a constant C such that

$$\prod_1^n \|T_i\| < C, \quad n = 1, 2, \ldots,$$

then $\|L_n\| \leqslant C$ and applying Theorem 1.1 we have

$$\bigotimes_1^\infty T_i f = \lim_{m \to \infty} \gamma_m L_m \gamma_n^m f_n, \quad f = \gamma_n f_n.$$

Now we shall formulate a few simple facts concerning mostly unbounded hyponormal (cohyponormal) operators. We shall use some of them in the next section. Recall that a densely defined closed operator T is called *cohyponormal* if T^* is hyponormal.

Proposition 1.7. *For a hyponormal (cohyponormal) operator T (W) the following properties hold:*

(a) $T^*|_{D(T)} = KT$, $\|K\| \leqslant 1$.

(b) *If T^{-1} exists, then T^{-1} is also hyponormal.*

(c) $\sigma(W) = \sigma_\pi(W)$.

(d) *For any isometry V the operator VTV^* is hyponormal.*

(e) *If A and B are bounded hyponormal operators, then $A \otimes B$ is also hyponormal.*

Proof. (a) The proof is similar to the one given for bounded hyponormal operators in [1].

(b) follows easily from (a).

(c) can be checked directly by the definitions.

(d) is a straightforward calculation by using the inclusion

$$D(VT^*V^*) \supset D(VTV^*).$$

(e) We have $A \otimes B = (A \otimes I)(I \otimes B) = A_1 B_1$. Therefore it is enough to show that A_1, B_1 are hyponormal (note that A_1 and B_1 doubly commute). Let us check the hyponormality of A_1. If $f = \sum_k a_k \otimes b_k$ and $b_k = \sum_s \beta_{ks} e_s$, where (e_s) is an orthonormal basis, then we compute:

$$\|A_1^* f\|^2 = \|A_1^*(\sum_k a_k \otimes b_k)\|^2 = \|\sum_{k,s} \beta_{ks} A^* a_k \otimes e_s\|^2 = \sum_s \|\sum_k \beta_{ks} A^* a_k\|^2$$

$$= \sum_s \|A^* \sum_k \beta_{ks} a_k\|^2 \leqslant \sum_s \|A \sum_k \beta_{ks} a_k\|^2$$

$$= \|\sum_{k,s} A a_k \otimes \beta_{ks} e_s\|^2 = \|\sum_k A a_k \otimes b_k\|^2 = \|A_1 f\|^2.$$

COROLLARY 1.8. *Let A_i be a bounded hyponormal (cohyponormal) operator in H_i. Assume that there exist unit vectors $e_i \in H_i$ with $\sum_i \|e_i - A_i e_i\| < +\infty$. Then*

$$T = \bigotimes_1^{\infty} A_i \colon \bigotimes_{1(e_i)}^{\infty} H_i \to \bigotimes_{1(e_i)}^{\infty} H_i$$

is hyponormal (cohyponormal).

Proof. In fact, if we define $T_n = \bigotimes_{i=1}^n A_i \otimes I \otimes \ldots$, then by Proposition 1.7(e) we know that T_n is hyponormal (the presence of infinitely many I causes no problem). Since $T_n - T_m$ is also hyponormal and $T_n \to T$ strongly, T_n^* must tend strongly to T^*. Thus T is hyponormal.

We end this section by giving a few examples of unbounded hyponormal operators.

EXAMPLE 1.9. Let (a_i) be a sequence of complex numbers such that $|a_i|$ is increasing. Define T= weighted shift with weights a_i. Then T is hyponormal. Note that in general $D(T) \neq D(T^*)$. By a direct computation one can check that $D(T) = D(T^*)$ if and only if there exists $C > 0$ such that

$$|a_k| \geqslant C|a_{k+1}|, \quad k = 1, 2, \ldots$$

EXAMPLE 1.10. Let $(Cf)(x) = 2^{-1/2}(xf(x) - df/dx)$ be the creation operator in $L^2(\boldsymbol{R})$. By direct use of the Bargmann model for C (or applying the results of [7, p. 252]) we know that $C^* f(x) = 2^{-1/2}(xf(x) + df/dx)$, $D(C) = D(C^*)$ and $\|C^* f\| \leqslant \|Cf\|$.

EXAMPLE 1.11. A straightforward computation shows that CC^*C is also hyponormal (as a weighted shift with increasing weights).

II. In case the L_n of Theorem 1.1 are selfadjoint Marchenko has found (under condition $(*)$) the following nice formula:

$$\sigma(L) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(L_m)} \quad \text{(the closure)}. \tag{2}$$

We shall prove an analogous formula for $\sigma_\pi(L)$ in terms of $\sigma_\pi(L_n)$ under certain assumptions on L_n.

THEOREM 2.1. *Let $H = \varinjlim_\gamma H_k$. Suppose we are given a sequence L_n of closed densely defined operators satisfying conditions $(*)$ and $(**)$ of Theorem 1.1. Assume that $\sigma(L_m) = \sigma_\pi(L_m)$ and*

$$\|(\lambda I - L_m)^{-1}\| \leqslant F_m(\operatorname{dist}(\lambda, \sigma(L_m))), \quad m = 1, 2, \ldots, \tag{3}$$

where the $F_m(t)$ are uniformly bounded for $t \geqslant \delta > 0$ (for any $\delta > 0$). Then

$$\sigma(L) = \sigma_\pi(L) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma_\pi(L_m)}.$$

Proof. Write $\sigma_\pi(L_m) = \sigma_m$, $\sigma_\pi(L) = \sigma$, $\bigcap_{n=1}^{\infty} \overline{\bigcup_{n=m}^{\infty} \sigma_m} = \sigma_\infty$. We have to show that $\sigma = \sigma_\infty$. We shall follow the ideas of [3].

a) $\sigma \supset \sigma_\infty$. Since $\sigma(L_m) = \sigma_\pi(L_m)$, we can repeat step by step the reasoning given in [3] for L_n selfadjoint and obtain the desired inclusion.

b) $\sigma \subset \sigma_\infty$. We shall prove that $\boldsymbol{C} \backslash \sigma_\infty \subset \boldsymbol{C} \backslash \sigma$. If $\lambda \notin \sigma_\infty$, then there exists n such that $\lambda \notin \overline{\bigcup_{m \geqslant n} \sigma_m}$, i.e.

$$\operatorname{dist}(\lambda, \overline{\bigcup_{m \geqslant n} \sigma_m}) \geqslant \delta \quad \text{for some } \delta > 0,$$

and so $\operatorname{dist}(\lambda, \sigma_m) \geqslant \delta$, $m \geqslant n$. Hence by (3)

$$\|(\lambda I - L_m)^{-1}\| \leqslant F_m(\operatorname{dist}(\lambda, \sigma_m)) \leqslant C, \quad m \geqslant n. \tag{4}$$

We want to check that $\lambda \notin \sigma_\infty$. It is enough to check that:

(α) $\gamma_m (\lambda I - L_m)^{-1} \gamma_n^m \varphi_n$ is converging in H to $S_\lambda \varphi$ for any $\varphi = \gamma_n \varphi_n$, where S_λ is a bounded operator in H.

(β) $S_\lambda(\lambda I - L_\infty)\varphi = \varphi$, $\varphi \in D_\infty$, $(\lambda I - L_\infty) S_\lambda \psi = \psi$, $\psi \in H$.

Ad (α). Write $S_m = (\lambda I - L_m)^{-1}$. We claim that $\gamma_m S_m \gamma_n^m \varphi_n$ satisfies the Cauchy condition. Indeed, for $l > m \geqslant \max(n_0(\varepsilon), n)$ we have

$$\begin{aligned} \|(\gamma_l S_l \gamma_n^l - \gamma_m S_m \gamma_n^m)\varphi_n\| &= \|(S_l \gamma_m^l - \gamma_m^l S_m)\gamma_n^m \varphi_n\| = \|S_l(\gamma_m^l L_m - L_l \gamma_m^l) S_m \gamma_n^m \varphi_n\| \\ &\leqslant F_l(\operatorname{dist}(\lambda, \sigma_l))\varepsilon(\|S_m \gamma_n^m \varphi_n\| + \|L_m S_m \gamma_n^m \varphi_n\| \\ &\quad + \|L_l \gamma_m^l S_m \gamma_n^m \varphi_n\|), \quad \text{by } (*) \text{ and } (4). \end{aligned}$$

The last expression is majorized by $\varepsilon C(\lambda, \varphi_n)$, as one checks directly. Hence for every $\varphi = \gamma_n \varphi_n$ the limit

$$\lim_{m \to \infty} \gamma_m S_m \gamma_n^m \varphi_n \stackrel{\text{df}}{=} S_\lambda \varphi$$

exists. Since $\|\gamma_m S_m \gamma_n^m \varphi_n\| \leqslant C\|\varphi\|$, $m \geqslant n$ (by (4)), we have $S_\lambda \in L(H)$.

Ad (β). Let $\tilde{\varphi} = \gamma_n \varphi \in D_\infty$. Consider the sequence

$$\varphi_{l,m} = \gamma_l S_l \gamma_m^l (\lambda - L_m) \gamma_n^m \varphi.$$

Note that $\varphi_{l,m} \to \tilde{\varphi}$ as $m \to \infty$ and $l > m$. In fact, for $l > m > \max(n_0(\varepsilon), n)$ we have

$$\begin{aligned}\|\varphi_{l,m} - \tilde{\varphi}\| &\leqslant F_l(\operatorname{dist}(\lambda, \sigma_l)) \|(\gamma_m^l(\lambda - L_m) - (\lambda - L_l)\gamma_m^l)\gamma_n^m \varphi\| \\ &\leqslant C\|(L_l\gamma_m^l - \gamma_m^l L_m)\gamma_n^m \varphi\| \\ &\leqslant C\varepsilon(\|\varphi\| + \|L_l\gamma_m^l \varphi\| + \|L_m\gamma_n^m \varphi\|) \leqslant C(\varphi)\varepsilon.\end{aligned}$$

Thus the claim is true and letting first $l \to \infty$ and next $m \to \infty$ we obtain

$$\begin{aligned}\varphi_{\infty,m} &= \lim_{l\to\infty} \gamma_l S_l \gamma_m^l (\lambda - L_m)\gamma_n^m \varphi \\ &= S_\lambda \gamma_m (\lambda - L_m)\gamma_n^m \varphi \to S_\lambda(\lambda - L_\infty)\tilde{\varphi} \quad \text{as } m \to \infty.\end{aligned}$$

Hence

$$\tilde{\varphi} = \lim_{m\to\infty} \varphi_{\infty,m} = S_\lambda(\lambda - L_\infty)\tilde{\varphi}.$$

A similar reasoning proves that

$$\psi_{l,m} = \gamma_l(\lambda - L_l)\gamma_m^l S_m \gamma_n^m \psi \to \tilde{\psi} \quad \text{as } m \to \infty,$$

where $\tilde{\psi} = \gamma_n \psi$. Thus

$$\tilde{\psi} = \lim_{m\to\infty} \psi_{\infty,m} = \lim_{m\to\infty} (\lambda - L_\infty)\gamma_m S_m \gamma_n^m \psi = (\lambda I - L_\infty) S_\lambda \tilde{\psi}. \ \blacksquare$$

COROLLARY 2.2. *Let $H = \lim_{\to\gamma} H_k$. Suppose we are given a sequence T_n (W_n) of hyponormal (cohyponormal) operators in H_n satisfying conditions (*) and (**) of Theorem 1.1. Then*

$$\sigma(T) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(T_m)} \quad \Big(\sigma(W) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(W_m)}\Big). \tag{5}$$

Proof. By Proposition 1.7(b), (c) we know that $(\lambda I - W_m)^{-1}$ is cohyponormal and $\sigma(W_m) = \sigma_\pi(W_m)$. On the other hand,

$$\|(\lambda I - W_m)^{-1}\| = 1/\operatorname{dist}(\lambda, \sigma(W_m)),$$

and so $F_m(r) = 1/r$ in Theorem 2.1. Since W is also cohyponormal the second equality in (5) holds by Theorem 2.1. The first equality of (5) is also obvious because T_m^*, T^* are cohyponormal and $\sigma(T^*) = \overline{\sigma(T)}$. $\blacksquare$

Later we shall give specific examples of applications of the above corollary. The next corollary concerns inductive limits of certain finite-dimensional operators.

Let $H = \lim_{\to\gamma} H_k$ be the inductive limit of an increasing sequence of its finite-dimensional subspaces H_k, i.e. γ_n^{n+1} and γ_n are canonical embeddings.

Suppose we are given operators A_k on H_k. We shall identify A_k with $A_k = A_k \oplus 0$, according to the decomposition $H = H_k \oplus H_k^\perp$. Let P_n: $H \to H_n$ be the orthogonal projection. Fix a uniform cross norm $\|\cdot\|_0$ on the set $\mathscr{F}(H)$ of finite-dimensional operators in H. Assume that $\|\cdot\|_0$ is not equivalent on $\mathscr{F}(H)$ to the operator norm $\|\cdot\|$. Write $r_n = \operatorname{dist}(\lambda, \sigma(A_n))$. The following estimate has been proved in [6]:

$$\|(\lambda - A_n)^{-1}\| \leqslant 3r_n^{-1} \exp[39\|A_n\|_0 r_n^{-1} \tau(r_n/(6\|A_n\|_0))], \tag{6}$$

where τ is a decreasing function in $(0, \infty)$ depending on $\|P_n\|_0$. Applying Theorem 2.1 we have

COROLLARY 2.3. *Let $H, H_n, A_n, \|\cdot\|_0$ be as above. Suppose that the sequence A_n satisfies (*) of Theorem 1.1 and there exists $C > 0$ such that $\|A_n\|_0 \leqslant C$, $n = 1, 2, \ldots$ Then*

$$\sigma(A) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(A_m)},$$

where $A = \lim_{\to\gamma} A_n$.

Proof. Since $\|A_n\|_0 \leqslant C$, $n = 1, 2, \ldots$, by (6) we can put in Theorem 2.1

$$F_n(r) = 3r^{-1} \exp[39\|A_n\|_0 r^{-1} \tau(r/(6\|A_n\|_0))]. \ \blacksquare$$

In the next section we shall give an example of application of Corollary 2.3 to trace class operators.

III. This section contains a few applications of the results of the previous one.

α) Let L be a trace class operator in a separable Hilbert space H. Suppose that there exists an orthonormal basis (e_k) in which L can be written as a tridiagonal matrix $a_{ij} = (Le_j, e_i)$. Let $H_n = \operatorname{span}(e_1, \ldots, e_n)$ and denote by P_n the orthogonal projection on H_n. Define L_n to be the compression of L to H_n, i.e. $L_n = P_n L P_n$. Let γ_n^{n+1}: $H_n \to H_{n+1}$, γ_n: $H_n \to H$ be the inclusion embeddings. If $\varphi = \sum_1^n \varphi_i e_i$, then by a direct computation we find

$$\|(L_m\gamma_n^m - \gamma_n^m L_n)\varphi\| = |a_{n+1,n}\varphi_n|, \quad m > n. \tag{7}$$

Since L is of trace class, $a_{n+1,n} \to 0$ as $n \to \infty$.

Now choose the cross norm $\|T\|_0 = \operatorname{tr}(TT^*)^{1/2}$. Since $L_m \to L$, there exists $M > 0$ such that $\|L_m\|_0 \leqslant M$, $m = 1, 2, \ldots$ On the other hand, (7) and the convergence to zero of $a_{n+1,n}$ imply condition (*) of Theorem 1.1. Applying Corollary 2.3 we obtain the (surely known)

PROPOSITION 3.1. *Let L be a tridiagonal trace class operator and let L_m be as above. Then*

$$\sigma(L) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(L_m)}.$$

Remark 3.2. Note that for a tridiagonal operator $L=(a_{ij})$ the convergence of

$$\sum_k |a_{kk}|, \quad \sum_k |a_{k+1,k}|, \quad \sum_k |a_{k,k+1}|$$

guarantees that L is of trace class.

β) The next application concerns cohyponormal operators. Suppose we are given a sequence of cohyponormal operators T_i in H_i. Assume that there exist $e_i \in H_i$, $\|e_i\| = 1$, with

$$(8) \qquad \sum_i \|T_i^* e_i - e_i\| < +\infty.$$

Then as we know $\otimes_1^\infty T_i$: $\otimes_{1(e_i)}^\infty H_i \to \otimes_{1(e_i)}^\infty H_i$ is also cohyponormal (by Corollary 1.8 adapted for arbitrary cohyponormal operators). In order to find $\sigma(\otimes_i T_i)$ we apply Corollary 2.2. Namely, we define

$$K_n = \bigotimes_1^n H_i, \quad L_n = \bigotimes_1^n T_i, \quad \gamma_n^{n+1}\colon K_n \ni f \to f \otimes e_{n+1} \in K_{n+1},$$

and we obtain

PROPOSITION 3.3. *Let T_i be a sequence of cohyponormal operators in H_i. If there exists a sequence $e_i \in D(T_i^*)$ of unit vectors such that (8) is satisfied, then*

$$\sigma(\bigotimes_1^\infty T_i) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(L_m)},$$

where $L_m = \otimes_1^m T_i$.

Proof. By (8) and direct computation one can check that for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for $m > n \geq n_0$ and $\varphi_n \in D(L_n)$

$$\|(L_n \gamma_n^m - \gamma_n^m L_n)\varphi_n\| \leq \varepsilon \|L_n \varphi_n\|.$$

Thus condition (*) of Theorem 1.1 holds. The same computation gives an analogous inequality for L_n^* and so condition (**) also holds. Applying Corollary 2.2 gives the desired result. ∎

Remark 3.4. In case the T_i are bounded and $\prod_1^\infty \|T_i\| < +\infty$ it is enough to assume that $\sum_i \|e_i - T_i e_i\| < +\infty$ instead of (8).

Here are two immediate applications of Proposition 3.3.

a) Suppose we are given cohyponormal operators $S_k \in L(H_k)$ such that $\|S_k\| \leq C$, $k = 1, 2, \ldots$ Let $\operatorname{Exp} S_k$ be the exponent of S_k in $\operatorname{Exp} H_k$ (see [2] for the definitions). Since $\operatorname{Exp}(\oplus_1^n S_k) = \otimes_1^n \operatorname{Exp} S_k$ we have

$$(9) \qquad \prod_1^n \|\operatorname{Exp} S_k\| = \|\bigotimes_1^n \operatorname{Exp} S_k\| = \|\operatorname{Exp}(\bigoplus_1^n S_k)\| \leq \exp\|\bigoplus_1^n S_k\| \leq e^C.$$

Let $\mathbf{1} = 1 \oplus 0 \oplus \ldots \in \operatorname{Exp} H_k$. Then $\operatorname{Exp} S_k \mathbf{1} = \mathbf{1}$. Hence by (9) one can define the (bounded) operator

$$S = \bigotimes_1^\infty \operatorname{Exp} S_k\colon \bigotimes_{1(1)}^\infty \operatorname{Exp} H_k \to \bigotimes_{1(1)}^\infty \operatorname{Exp} H_k.$$

Note that $S = \operatorname{Exp} \oplus_1^\infty S_k$.

Now we put $T_k = \operatorname{Exp} S_k$ and applying Proposition 3.3 we have

$$\sigma(S) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(S_1)\ldots\sigma(S_m)},$$

where $\sigma(S_1)\ldots\sigma(S_m) = \{\lambda\colon \lambda = \lambda_1 \ldots \lambda_m,\ \lambda_k \in \sigma(S_k)\}$.

b) Let $d\mu = (2\pi)^{-1/2}\exp(-x^2/2)\,dx$ be the Gaussian measure on $\mathbf{R}$ and let $0 < |\alpha| \leq 1$, $\alpha \in \mathbf{R}$. Following Mlak we define the operator C_α in $L^2(\mu)$ by $(C_\alpha f)(x) = f(\alpha x)$. It turns out that C_α is bounded and $\sigma(C_\alpha) = \{z\colon |z| \leq |\alpha|^{-1/2}\}$ (see [5]). Moreover, for any sequence (α_i), $0 < |\alpha_i| \leq 1$, such that $c = \prod_1^\infty |\alpha_i|^{-1/2} < +\infty$ Mlak defined the operator

$$\bigotimes_1^\infty C_{\alpha_i}\colon \bigotimes_{1(1)}^\infty L^2(\mu) \to \bigotimes_{1(1)}^\infty L^2(\mu).$$

It can be identified with the operator

$$C_{(\alpha_i)}\colon L^2(\tilde{\mu}) \to L^2(\tilde{\mu})$$

given by $(C_{(\alpha_i)} f)(x) = f(\alpha_1 x_1, \alpha_2 x_2, \ldots)$, where $\tilde{\mu} = \otimes_1^\infty d\mu$ is the Gaussian measure on the Borel subsets of $\mathbf{R}^\infty$. By a direct computation we check that the C_{α_i} are cohyponormal. Hence applying Proposition 3.3 (see Remark 3.4) we have

$$(10) \qquad \sigma(C_{(\alpha_i)}) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(L_m)},$$

where $L_n = \otimes_1^n C_{\alpha_i}$. Since $\sigma(L_n) = \{z\colon z = z_1 \ldots z_n,\ |z_k| \leq |\alpha_k|^{-1/2}\}$, by (10) we have

$$\sigma(C_{(\alpha_i)}) = \{z\colon |z| \leq c\}.$$

IV. Concluding comments and results. First note the following remarks.

Remark 4.1. Theorem 2.1 remains true for inductive limits of linear mappings L_n: $E_n \to F_n$ between Banach spaces under the same assumptions on L_n as before.

Remark 4.2. One can also consider inductive limits of C^*-algebras A_k with $*$-monomorphisms γ_k^{k+1}: $A_k \to A_{k+1}$ and continuous operators L_k: $A_k \to A_k$ satisfying conditions analogous to the previous ones. In particular, by applying this procedure to Calkin algebras $A_n = A(H_n)$ (over Hilbert spaces H_n) and $L_n a = b_n a$ one obtains a result corresponding to Theorem 2.1 or its corollaries. The details are left to the interested reader.

Now we shall extend Theorem 2.1 to commuting tuples of normal operators. In order to do this we first modify slightly the form of conditions $(*)$ and $(**)$ using the notion of bounded vector. Recall that for a densely defined operator A a vector f is said to be a *bounded vector* for A if there are $c_f > 0$ and $M_f > 0$ such that

$$\text{(B)} \qquad \|A^k f\| \leqslant M_f c_f^k, \qquad k = 1, 2, \ldots$$

Denote by $B(A)$ the set of all bounded vectors for A and by $B_c(A)$ the set of all those $f \in B(A)$ which satisfy (B) with $c_f = c$. It is clear that

$$B(A) = \bigcup_{k=1}^{\infty} B_k(A).$$

Now let $H = \lim\limits_{\to}{}_{\gamma} H_k$ and let L_k be a sequence of closed densely defined operators in H_k. The sequence L_k is said to satisfy $(*)_1$ or $(**)_1$, respectively, if $(*)$ or $(**)$ hold for L_k and vectors in $B(L_k)$ or $B(L_k^*)$, respectively. It turns out that for normal operators, under certain natural assumptions on γ_n^m, condition $(*)_1$ implies $(**)_1$. Namely, we have

Lemma 4.3. *Let N_k be a sequence of normal operators in H_k. Suppose that $\forall_{c>0}\ \exists_{c_1 \geqslant c}$ such that*

$$\gamma_n^m B_c(N_n) \subset B_{c_1}(N_m), \qquad \forall_{m \geqslant n}.$$

If the sequence N_k satisfies $(*)_1$, then it satisfies $(**)_1$.

Proof. We shall suitably modify the method used by Rosenblum in his proof of the Putnam–Fuglede theorem [8]. In fact, we shall use (and prove) a slightly modified form of $(*)_1$ (and $(**)_1$). Namely, suppose that

$$\forall_{\varepsilon>0}\ \exists_{n_0}\ \forall_{m>n\geqslant n_0}\ \forall_{f\in B_c(N_n)} \qquad \|(N_m \gamma_n^m - \gamma_n^m N_n) f\| \leqslant \varepsilon(1+c)\|f\|.$$

By the above assumptions and induction we have

$$\|(N_m^k \gamma_n^m - \gamma_n^m N_n^k) f\| \leqslant k\varepsilon c_1^k (1+c)\|f\|, \qquad k = 1, 2, \ldots, f \in B_c(N_n),\ m > n \geqslant n_0.$$

Hence by a direct computation we have for $m > n \geqslant n_0$

$$\|(e^{i\bar{\lambda}N_m}\gamma_n^m - \gamma_n^m e^{i\lambda N_n}) f\| \leqslant \varepsilon|\lambda| c_1 (1+c) e^{|\lambda|c_1}\|f\|, \qquad \lambda \in \boldsymbol{C},\ f \in B_c(N_n).$$

Similarly one can check that

$$\text{(11)} \qquad \|e^{i\lambda N_m^*}(e^{i\lambda N_m}\gamma_n^m - \gamma_n^m e^{i\bar{\lambda}N_n}) e^{-i\bar{\lambda}N_n} e^{-i\lambda N_n^*} f\|$$
$$\leqslant \varepsilon e^{|\lambda|c_1}(1+c)|\lambda| c_1 e^{|\lambda|(c_1+2c)}\|f\|, \qquad m > n \geqslant n_0,\ f \in B_c(N_n).$$

Since $e^{aN_k + bN_k^*} = e^{aN_k} e^{bN_k^*}$, $a, b \in \boldsymbol{C}$, $e^{i(\bar{\lambda}N_m + \lambda N_m^*)}$ is unitary and by (11) we have

$$\text{(12)} \qquad \|e^{i\lambda N_m^*}\gamma_n^m e^{-i\lambda N_n^*} f\| \leqslant \|f\| + \varepsilon c_2(|\lambda|)\|f\|,$$

where $c_2(|\lambda|) = e^{|\lambda|c_1}(1+c)|\lambda| c_1 e^{|\lambda|(c_1+2c)}$.

Let $F_f^{mn}(\lambda) = e^{i\lambda N_m^*}\gamma_n^m e^{-i\lambda N_n^*} f$ be an entire vector-valued function, where $f \in B_c(N_n)$. Write $C = \{z \in \boldsymbol{C}: |z| = r\}$. Then by the Cauchy formula (we omit the indices m, n)

$$F_f'(0) = \frac{1}{2\pi i}\int_C \frac{F(u)}{u^2}\,du.$$

Hence

$$\|F_f'(0)\| \leqslant \frac{1}{r}\max_{|z|=r}|F_f(z)|,$$

and so by (12) we have

$$\|F_f'(0)\| \leqslant \frac{1}{r}(1+\varepsilon c_2(r))\|f\|.$$

But $F_f'(0) = i(N_m^*\gamma_n^m - \gamma_n^m N_n^*) f$, so

$$\text{(13)} \qquad \|(N_m^*\gamma_n^m - \gamma_n^m N_n^*) f\| \leqslant (1/r + \varepsilon c_2(r)/r)\|f\|.$$

Now for any $\eta > 0$ put $r = 2/\eta$ and take $\varepsilon = \varepsilon(\eta)$ so small that $\varepsilon c_2(r)/r \leqslant \eta/2$. It follows (by (13)) that

$$\|(N_m^*\gamma_n^m - \gamma_n^m N_n^*) f\| \leqslant \eta\|f\|$$

for $m > n \geqslant n_0(\varepsilon)$ and $f \in B_c(N_n)$. Thus $(**)_1$ holds. ■

Corollary 4.4. *Let $H = \lim\limits_{\to}{}_{\gamma} H_k$ and let N_k be a sequence of normal operators satisfying $(*)_1$. If $\gamma_n^m B_c(N_n) \subset B_{c_1}(N_m)$, $\forall_{m>n}$, then the operator $N = \bar{N}_\infty$ is normal in H (N_∞ is the same as in Theorem 1.1).*

Proof. By Lemma 4.3, $(**)_1$ holds for N_k so it is obvious that $\gamma_m N_m^* \gamma_n^m \varphi_n$ tends to $N^*\varphi$ as $m \to \infty$, where $\varphi = \gamma_n \varphi_n$, $\varphi_n \in B(N_n)$. Let $\bigcup_k \gamma_k B(N_k) = X$. As we know $D(N_\infty) \subset D(N_\infty^*)$ (see the proof of Th. 1.1). It is also clear that $\|N_\infty f\| = \|N_\infty^* f\|$ for $f \in D(N_\infty)$. Moreover, X is dense in $D(N_\infty)$, $N_\infty X \subset X$, $N_\infty^* X \subset X$, so $\bar{N}_\infty = N$ must be normal (see [11]).

Now we shall extend Theorem 2.1 to commuting normal operators. Suppose we are given a sequence of normal commuting operators $N_{1k}, \ldots, N_{sk}$ in H_k, i.e. $N_{pk} = \int \lambda\, dE_{pk}$ and the spectral measures E_{pk} pairwise commute. By the joint spectrum $\sigma(N_{1k}, \ldots, N_{sk})$ we mean the joint approximate point spectrum of $N_{1k}, \ldots, N_{sk}$ (see [10] for the definition). Let $H = \varinjlim_\gamma H_k$. Write $B_c(N_k) = B_c(N_{1k}) \cap \ldots \cap B_c(N_{sk})$ and suppose that $\forall_{c>0}\ \exists_{c_1 \geqslant c}$ such that

$$\gamma_n^m B_c(N_n) \subset B_{c_1}(N_m). \tag{14}$$

The system $N_k = (N_{1k}, \ldots, N_{sk})$ is said to satisfy condition $(*)_1$ if

$$\forall_{\varepsilon>0}\ \exists_{n_0(\varepsilon)}\ \forall_{m>n\geqslant n_0(\varepsilon)}\quad \|(N_{pm}\gamma_n^m - \gamma_n^m N_{pn})f\| \leqslant \varepsilon(1+c)\|f\|,$$

for any $f \in B_c(N_n)$, $p = 1, \ldots, s$.

In what follows we shall restrict ourselves to commuting pairs ($s = 2$) of normal operators, but the results hold for any s.

THEOREM 4.5. *Let $H = \varinjlim_\gamma H_k$ and suppose we are given a sequence A_m, B_m of commuting normal operators in H_m. If γ_n^m satisfies* (14) *for any $m \geqslant n$ and (A_m, B_m) satisfy $(*)_1$, then*

$$\sigma(A, B) = \bigcap_{m=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(A_n, B_n)}, \tag{15}$$

where $A = \bar{A}_\infty$, $B = \bar{B}_\infty$.

Proof. Denote by σ_∞ the right-hand side of (15).

a) The inclusion $\sigma(A, B) \supset \sigma_\infty$ can be proved in the same way as the corresponding inclusion in the proof of Theorem 2.1 (note that A and B are normal by Corollary 4.4).

b) The opposite inclusion $\sigma_\infty \supset \sigma(A, B)$ may be proved as follows. If $\lambda = (\lambda_1, \lambda_2) \notin \sigma_\infty$, then for some $\delta > 0$ and n

$$\operatorname{dist}(\lambda, \sigma(A_m, B_m)) > \delta, \qquad \forall_{m\geqslant n}. \tag{16}$$

Now writing $A_{\lambda m} = \lambda_1 I - A_m$, $B_{\lambda m} = \lambda_2 I - B_m$ we see that $A_{\lambda m}^* A_{\lambda m} + B_{\lambda m}^* B_{\lambda m}$ is invertible, i.e. there exists $R_{\lambda m} \in L(H_m)$ such that

$$R_{\lambda m}(A_{\lambda m}^* A_{\lambda m} + B_{\lambda m}^* B_{\lambda m}) f = f \tag{17}$$

for f in the domain of $A_{\lambda m}^* A_{\lambda m} + B_{\lambda m}^* B_{\lambda m}$. Since A_m, B_m are commuting and normal,

$$\sigma(A_{\lambda m}^* A_{\lambda m} + B_{\lambda m}^* B_{\lambda m}) = \{|z_1|^2 + |z_2|^2 \colon (z_1, z_2) \in \sigma(A_{\lambda m}, B_{\lambda m})\}.$$

It follows that

$$\inf_{m\geqslant n} \inf\{|z_1|^2 + |z_2|^2 \colon (z_1, z_2) \in \sigma(A_{\lambda m}, B_{\lambda m})\} = \inf_{m\geqslant n} \operatorname{dist}(\lambda, \sigma(A_m, B_m)) \geqslant \delta$$

by (16). Thus

$$\|R_{\lambda m}\| \leqslant 1/\delta, \qquad \forall_{m\geqslant n}. \tag{+}$$

We claim that $\lambda \notin \sigma(A, B)$. In fact, by (17) and (+) we have

$$((A_{\lambda m}^* A_{\lambda m} + B_{\lambda m}^* B_{\lambda m}) f, f) \geqslant \delta \|f\|$$

so letting $m \to \infty$ we obtain the desired claim. ∎

Remark 4.6. The following reasoning shows that Theorem 4.5 cannot be extended further (even to bounded commuting subnormal pairs). In fact, by the result of [9] there exists a sequence of commuting bounded subnormal pairs T_k, W_k in certain H_k such that

$$\sigma\Big(\bigoplus_1^\infty T_k, \bigoplus_1^\infty W_k\Big) \neq \overline{\bigcup_1^\infty \sigma(T_k, W_k)}, \tag{α}$$

where $\sigma(T_k, W_k)$ stands for the Taylor joint spectrum. Obviously

$$\bigoplus_1^\infty H_k = \varinjlim_\gamma \bigoplus_1^s H_k,$$

where

$$\gamma_n^{n+1} \colon (x_1, \ldots, x_n) \to (x_1, \ldots, x_n, 0),$$

$$\gamma_n \colon (x_1, \ldots, x_n) \to (x_1, \ldots, x_n, 0, 0, \ldots).$$

Let $L_n = \bigoplus_1^n T_k$, $M_n = \bigoplus_1^n W_k$. Then $L_{n+1}\gamma_n^{n+1} - \gamma_n^{n+1} L_n = 0$, $M_{n+1}\gamma_n^{n+1} - \gamma_n^{n+1} M_n = 0$. If we had for $L = \bigoplus_1^\infty T_k$, $M = \bigoplus_1^\infty W_k$ the equality

$$\sigma(L, M) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{s=n}^{\infty} \sigma(L_s, M_s)},$$

then we would have

$$\sigma(L, M) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{s=n}^{\infty} \sigma(L_s, M_s)} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=1}^{\infty} \sigma(T_k, W_k)} = \overline{\bigcup_{k=1}^{\infty} \sigma(T_k, W_k)},$$

and this contradicts (α).

References

[1] K. Clancey, *Seminormal Operators*, Lecture Notes in Math. 742, Springer, 1979.
[2] R. L. Dobrushin and R. A. Minlos, *Polynomials of linear random functions*, Uspekhi Mat. Nauk 32 (2) (194) (1977), 67–122 (in Russian).
[3] A. V. Marchenko, *Selfadjoint differential operators with an infinite number of independent variables*, Mat. Sb. 96 (2) (1975), 276–293 (in Russian).
[4] W. Mlak, *Introduction to the Theory of Hilbert Spaces*, PWN, Warszawa 1982 (in Polish).
[5] —, *Operators induced by change of Gaussian variables*, Ann. Polon. Math., to appear.
[6] A. Pokrzywa, *On continuity of spectra in norm ideals*, Linear Algebra Appl. 69 (1985), 121–130.
[7] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness*, Academic Press, New York 1975.
[8] M. Rosenblum, *On a theorem of Fuglede and Putnam*, J. London Math. Soc. 33 (1958), 376–377.
[9] J. K. Rudol, *The spectrum of orthogonal sums of subnormal pairs*, preprint, 1985.
[10] Yu. S. Samoilenko, *Spectral Theory of Systems of Selfadjoint Operators*, Naukova Dumka, Kiev 1984.
[11] J. Stochel and F. Szafraniec, *Bounded vectors and formally normal operators*, Operator Theory: Adv. Appl. 11 (1983), 363–370.
[12] Y. Yamasaki, *Kolmogorov's extension theorem for infinite measures*, Publ. RIMS Kyoto Univ. 10 (1975), 381–411.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
ODDZIAŁ W KRAKOWIE
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
KRAKÓW BRANCH
Solskiego 30, 31-027 Kraków, Poland

Received June 27, 1986 (2184)
Revised version May 4, 1987

STUDIA MATHEMATICA, T. XC. (1988)

Groups of isometries on operator algebras

by

STEEN PEDERSEN (Aarhus)

Abstract. Let ϱ be a C_0-group of isometries on a unital C^*-algebra A. If $u(t) = \varrho(t)1$ and $\alpha(t)a = u(t)^*\varrho(t)a$, then $\varrho(t)a = u(t)\alpha(t)a$, α is a C_0-group of $*$-automorphisms on A and u is a unitary 1-cocycle. We study this decomposition of ϱ; as a consequence we obtain a classification of the generators of C_0-groups of isometries on A.

Introduction. In [18] Kadison proved that an isometry of a unital C^*-algebra A onto itself can be decomposed into a C^*-homomorphism followed by multiplication by a unitary. We study the consequences of applying this decomposition to a strongly continuous isometric representation ϱ of a topological group on A. We prove that the C^*-homomorphic part of ϱ is a strongly continuous group of $*$-automorphisms and that ϱ is norm-continuous if A is a von Neumann algebra. We establish conditions, global as well as local, which are satisfied by ϱ if and only if it is a group of $*$-automorphisms.

Using perturbation theory for $*$-automorphism groups we prove that if ϱ is a one-parameter group of isometries on A with generator δ, then there exist (γ, v, h), where γ is the generator of a one-parameter group of $*$-automorphisms on A, v is a unitary in A and h is a selfadjoint element of A, such that $\mathscr{D}(\delta) = v^*\mathscr{D}(\gamma)$ and

$$\delta(a) = v^*\gamma(va) + iv^*hva$$

for a in $\mathscr{D}(\delta)$. Using this we give local and global conditions equivalent to the fact that the unitary part of ϱ is a group.

In the next part of the paper we specialize to the case where ϱ is a one-parameter group. We observe that in some representations of A, $\varrho(t)a = u(t)av(t)$, where u and v are strongly continuous unitary groups. We study the generators of (semi-) groups of this form.

This study was motivated by applications to quantum mechanics (e.g. [15], [22], [25]) and partially inspired by the corresponding problems for a one-parameter semigroup on a Hilbert space if each element of the semigroup is polar decomposed [11], [12].