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On a class of weighted function spaces and related pseudodifferential operators

by

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Abstract. The paper deals with weighted spaces of type B^s_{pq} and F^s_{pq} on a bounded domain in $\boldsymbol{R}^n$, including equivalent quasi-norms and lift properties. Related classes of weighted pseudo-differential operators are introduced and studied.

1. Introduction

The two scales $B^s_{pq}(\boldsymbol{R}^n)$ and $F^s_{pq}(\boldsymbol{R}^n)$ of quasi-Banach spaces on the euclidean n-space $\boldsymbol{R}^n$ cover many classical function spaces: Sobolev spaces, Bessel-potential spaces, Hölder–Zygmund spaces, Besov–Lipschitz spaces and Hardy spaces. The usual approach, currently, to these spaces, where $-\infty < s < \infty$, $0 < p \leqslant \infty$ (with $p < \infty$ in the case of the F-spaces), $0 < q \leqslant \infty$, is based on Fourier-analytic decomposition techniques which go back to J. Peetre. A systematic study of these spaces on this basis is given in [Tri2]. On the other hand, the well-known direct definitions of the Sobolev spaces, Hölder spaces etc. show the local-global nature of these spaces: local as far as the differentiability conditions are concerned, whereas global refers to the usual $L_p(\boldsymbol{R}^n)$-norms. Moreover, all the spaces $B^s_{pq}(\boldsymbol{R}^n)$ and $F^s_{pq}(\boldsymbol{R}^n)$ have this local-global nature. Unfortunately, the Fourier-analytic approach hides this fact completely. This causes a lot of trouble and makes some proofs (e.g. of the invariance of these spaces under diffeomorphic maps of $\boldsymbol{R}^n$ onto itself) difficult, unnatural and cumbersome. These shortcomings can be remedied by the approach given in [Tri3] which on the one hand preserves the advantages of the Fourier-analytic techniques but on the other hand allows local considerations in the above sense. On this basis we dealt with spaces of B^s_{pq}-F^s_{pq} type on complete Riemannian manifolds (with bounded geometry and positive injectivity radius) and on Lie groups ([Tri4, 5, 6]). Furthermore, based on [Tri3] we gave in [Tri7] a short and natural proof of the following result due to L. Päivärinta [Pä]: Recall that Hörmander's symbol class $S^{\mu}_{1,\delta}$ with $-\infty < \mu < \infty$, $0 \leqslant \delta < 1$, is the collection of all $a(x, \xi) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ with

$$|D^\beta_x D^\alpha_\xi a(x, \xi)| \leqslant C_{\alpha\beta}(1+|\xi|)^{\mu-|\alpha|+\delta|\beta|} \tag{1.1}$$

for all multi-indices α, β and all $x\in R^n$, $\xi\in R^n$. Then the corresponding pseudodifferential operator $a(x, D)$ maps $F^{s+\mu}_{pq}(R^n)$ continuously into $F^s_{pq}(R^n)$ and $B^{s+\mu}_{pq}(R^n)$ continuously into $B^s_{pq}(R^n)$, where s, p, q obey the above conditions.

The present paper follows this path. Its aim is twofold. First, we furnish a given bounded domain Ω in R^n with a Riemannian metric g such that the resulting manifold has the properties mentioned above. In this special situation, compared with what has been done in [Tri4, 5, 6] much more can be said about the corresponding spaces $F^s_{pq}(\Omega, g)$ and $B^s_{pq}(\Omega, g)$. This will be done in Section 2 of this paper where we occasionally use some recent results from Riemannian geometry. Secondly, in Section 3 we introduce some global classes of weighted pseudodifferential operators connected with (Ω, g). The main aim is to give a convincing definition and to prove a mapping theorem which is the counterpart of the above-mentioned result for the Hörmander class $S^{\mu}_{1,\delta}$ in R^n. Furthermore, we deal with algebraic properties of these classes and with parametrices of the corresponding elliptic pseudodifferential operators.

The paper is organized as follows. Sections 2 and 3 contain all definitions and results accompanied by the necessary remarks and comments. If the results can be proved by modifications of previous proofs we restrict ourselves to outlines and references. All longer proofs are shifted to Section 4. A peculiarity should be mentioned. It will be convenient for us, and it seems to be of independent interest, to use a characterization of some spaces $B^s_{pq}(R^n)$ and $F^s_{pq}(R^n)$ via variable differences. This is covered by [GT] as far as the spaces $B^s_{pq}(R^n)$ are concerned. For the spaces $F^s_{pq}(R^n)$ we formulate the corresponding assertion in the appendix of Section 2 and prove it in detail in 4.1.

2. Weighted spaces

2.1. Basic spaces on R^n. First of all we have to define the unweighted spaces $F^s_{pq}(R^n)$ and $B^s_{pq}(R^n)$ on the euclidean n-space R^n. We give a description which is especially well adapted to our later purposes and which is an outgrowth of [Tri3]. Let B be the unit ball in R^n. Let k_0 and k be functions defined on the real line such that both $k_0'(y) = k_0(|y|)$ and $k'(y) = k(|y|)$ with $y\in R^n$ are C^∞ functions on R^n with

$$\operatorname{supp} k_0' \subset B, \qquad \operatorname{supp} k' \subset B.$$

Let

$$(k')^\wedge(0) \neq 0, \qquad (k_0')^\wedge(y) \neq 0 \quad \text{for all } y\in R^n,$$

where $(k_0')^\wedge$ and $(k')^\wedge$ stand for the Fourier transforms of k_0' and k', respectively. If N is a natural number then we put

$$k_N' = \Big(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\Big)^N k'.$$

Of course, $k_N'(y) = k_N(|y|)$ with $y\in R^n$ is also rotation-invariant. If $N = 0, 1, 2, \dots$ then we introduce the means

$$(2.1) \qquad k_N(t, f)(x) = \int_{R^n} k_N(|y|) f(x+ty)\, dy, \qquad x\in R^n,\ t>0.$$

This is a local procedure which makes sense for any tempered distribution $f\in S'(R^n)$ (under appropriate interpretation). If b is a real number then we put $b_+ = \max(b, 0)$. Let

$$(2.2) \qquad \|h \mid L_p(R^n)\| = \Big(\int_{R^n} |h(x)|^p\, dx\Big)^{1/p}, \qquad 0<p\leqslant\infty,$$

with the usual modification if $p=\infty$. Finally, $(\cdot,\cdot)_{\theta,q}$ stands for the real interpolation method of quasi-Banach spaces (see [BL] or [Tri1]).

2.1.1. Definition. (i) Let k_N be the function defined above. Let $0<\varepsilon<\infty$, $0<r<\infty$ and $-\infty<s<\infty$. Let either $0<p<\infty$ and $0<q\leqslant\infty$, or $p=q=\infty$. Let N be a natural number with $2N>\max(s, n(1/p-1)_+)$. Then

$$(2.3) \qquad F^s_{pq}(R^n) = \{f \mid f\in S'(R^n),\ \|f \mid F^s_{pq}(R^n)\|^{k_0,k_N}_{\varepsilon,r}$$
$$= \|k_0(\varepsilon, f) \mid L_p(R^n)\| + \Big\|\Big(\int_0^r t^{-sq} |k_N(t,f)(\cdot)|^q\, dt/t\Big)^{1/q} \Big| L_p(R^n)\Big\| < \infty\}$$

(with the usual modification if $q=\infty$).

(ii) Let $-\infty<s_0<s<s_1<\infty$, $s=(1-\theta)s_0+\theta s_1$, $0<p\leqslant\infty$ and $0<q\leqslant\infty$. Then

$$(2.4) \qquad B^s_{pq}(R^n) = (F^{s_0}_{pp}(R^n), F^{s_1}_{pp}(R^n))_{\theta,q}.$$

2.1.2. Remark. As we have said the above definition is a consequence of [Tri3]. The preference given to F^s_{pq} over B^s_{pq} has its origin in the theory of these spaces on Riemannian manifolds and Lie groups ([Tri4, 5, 6]); it will also become clearer later in this paper.

2.1.3. Remark. The theory of these spaces has been developed in [Tri2]. They are all quasi-Banach spaces, independent of the functions k_0 and k, and of the numbers ε, r, and N (up to equivalent quasi-norms). They cover many classical function spaces on R^n (Sobolev, Bessel-potential, Hölder–Zygmund, Besov–Lipschitz, Hardy). In particular,

$$(2.5) \qquad W^m_p(R^n) = F^m_{p2}(R^n), \qquad 1<p<\infty,\ m=0, 1, 2, \dots,$$

are the usual Sobolev spaces.

2.2. Weight functions. We always assume that Ω stands for a bounded connected domain in $\boldsymbol{R}^n$ (no smoothness assumptions are required).

2.2.1. DEFINITION. An *admissible weight* is a positive C^∞ function $g(x)$ on Ω which satisfies the following conditions:

(i) There exists a positive number c such that

$$U_x = \{y \mid y \in \boldsymbol{R}^n, |y-x| < c/g(x)\} \subset \Omega \tag{2.6}$$

for every $x \in \Omega$.

(ii) There exist positive numbers c_1 and c_2 such that

$$c_1 g(x) \leqslant g(y) \leqslant c_2 g(x) \quad \text{if } x \in \Omega \text{ and } y \in U_x. \tag{2.7}$$

(iii) For every multi-index γ there exists a positive number c_γ with

$$|D^\gamma g(x)| \leqslant c_\gamma g^{1+|\gamma|}(x) \quad \text{for every } x \in \Omega. \tag{2.8}$$

2.2.2. Remark. By (2.6) we have

$$g(x) \geqslant c/d(x) \quad \text{with } d(x) = \operatorname{dist}(x, \partial\Omega). \tag{2.9}$$

Furthermore, by (2.7) and (2.8) the weight g is slowly varying.

2.2.3. EXAMPLE. There exists a C^∞ function $\varrho(x)$ on Ω with

$$cd(x) \leqslant \varrho(x) \leqslant Cd(x) \quad \text{for every } x \in \Omega,$$

where c and C are appropriate positive numbers (independent of $x \in \Omega$), such that

$$g(x) = \varrho^{-\varkappa}(x), \quad x \in \Omega,$$

is an admissible weight if and only if $\varkappa \geqslant 1$ (see [Tri1, 3.2.3] for some details). The necessary restriction $\varkappa \geqslant 1$ comes from (2.9).

2.2.4. LEMMA (Resolution of unity). *There exists a uniformly locally finite covering $\bigcup_{j=1}^\infty U_{x^j} = \Omega$ by balls of the type (2.6) and a resolution of unity $\{\varphi_j\}_{j=1}^\infty$ with the following properties:*

(i) *Every φ_j is a C^∞ function in Ω with*

$$\operatorname{supp} \varphi_j \subset U_{x^j}, \quad j = 1, 2, \ldots \tag{2.10}$$

(ii) *For every multi-index α there exists a positive number c_α with*

$$|D^\alpha \varphi_j(x)| \leqslant c_\alpha g^{|\alpha|}(x^j), \quad x \in \Omega \text{ and } j = 1, 2, \ldots \tag{2.11}$$

(iii) *We have*

$$\sum_{j=1}^\infty \varphi_j(x) = 1 \quad \text{for } x \in \Omega. \tag{2.12}$$

2.2.5. Remark. Recall that "uniformly locally finite" means that there exists a natural number L such that any fixed ball U_{x^j} has a nonempty intersection with at most L of the balls $\{U_{x^k}\}_{k=1}^\infty$. The existence of such a covering and of the corresponding resolution of unity can be proved by elementary arguments ([Tri1, 3.2.3]). On the other hand, after introducing normal geodesic coordinates the above resolution of unity coincides with the resolution of unity used in [Tri4, 5, 6] (see the next subsection).

2.3. The Riemannian background. As above, let Ω be a bounded connected domain in $\boldsymbol{R}^n$ equipped with an admissible weight g. We equip (Ω, g) with the Riemannian metric

$$ds^2 = g^2(y) \sum_{j=1}^n (dy_j)^2, \quad y = (y_1, \ldots, y_n) \in \Omega. \tag{2.13}$$

2.3.1. THEOREM. (Ω, g) *with the metric* (2.13) *is a connected complete Riemannian manifold with bounded geometry and positive injectivity radius.*

2.3.2. Remark. We explain briefly what is meant by this theorem and refer for more details to [Tri4]. A Riemannian manifold is complete if it is complete as a metric space. (By the Hopf–Rinow theorem this is equivalent to the assertion that any geodesic is infinitely extendable with respect to its arc length.) Let $\exp_x$ with $x \in \Omega$ be the exponential map from the tangent space $T_x\Omega$ (identified with $\boldsymbol{R}^n$) into Ω. If $r > 0$ is small then $\exp_x$ is a diffeomorphism from

$$B(r) = \{X \mid X \in \boldsymbol{R}^n, \|X\| < r\} \quad \text{onto} \quad \exp_x B(r), \tag{2.14}$$

where $\|X\|$ stands for the Riemannian metric. Let r_x be the supremum of all numbers r with this property. Then $r_0 = \inf r_x$ is called the *injectivity radius* where the infimum is taken over all $x \in \Omega$. Finally, *bounded geometry* means that the curvature tensor and all its covariant derivatives are bounded on Ω.

2.3.3. Remark. In our special situation all these things can be simplified. Instead of $\exp_x B(r)$ from (2.14) we use the (euclidean) balls U_x from (2.6). Furthermore, we replace $\exp_x^{-1}$ by the dilations

$$y \to Y = H_x(y) = g(x)(y-x) \quad \text{from } U_x \text{ onto } V_c = \{Y \mid Y \in \boldsymbol{R}^n, |Y| < c\}, \tag{2.15}$$

where c comes from (2.6). (Of course, H_x does not coincide with $\exp_x^{-1}$ in general.) We have

$$ds^2 = G_x^2(Y) \sum_{j=1}^n (dY_j)^2 \quad \text{with } G_x(Y) = g(y)/g(x). \tag{2.16}$$

From (2.7) and (2.8) it follows that

(2.17) $$G_x(Y) \geqslant C > 0, \quad |D_Y^\gamma G_x(Y)| \leqslant c_\gamma, \quad Y \in V_c,$$

where C and c_γ are independent of $Y \in V_c$ and $x \in \Omega$.

2.3.4. Proof of Theorem 2.3.1. The completeness of (Ω, g) follows from (2.6) and (2.13). The considerations in Remark 2.3.3 show that all geometric quantities e.g. the curvature tensor and its derivatives, can be estimated uniformly with respect to $x \in \Omega$. Hence, (Ω, g) has bounded geometry. Finally, the nontrivial fact that (Ω, g) has a positive injectivity radius follows now from Theorem 4.7 in [CGT].

2.3.5. *Our method.* The above theorem shows that the weighted spaces $F_{pq}^s(\Omega, g)$ and $B_{pq}^s(\Omega, g)$ treated in this paper and defined in 2.4 are special cases of the spaces considered in [Tri4, 5]. However, we prefer a more direct approach which yields much better results: reduction of the weighted spaces to the spaces $F_{pq}^s(\boldsymbol{R}^n)$ via Lemma 2.2.4. However, occasionally we shall use some technicalities elaborated in [Tri4]. Then we restrict ourselves to references, also in those cases where the results from [Tri4, 5] can be taken over without any changes (e.g. the lifting property from 2.7). Then (and only then) Theorem 2.3.1 will be needed. Furthermore, occasionally we need some deeper results from differential geometry, in particular in Section 3 which deals with pseudodifferential operators. Our main source here is [CGT].

2.4. Definitions and basic results. Recall that (Ω, g) stands for a bounded connected domain in $\boldsymbol{R}^n$ furnished with an admissible weight in the sense of Definition 2.2.1. Let $\varphi = \{\varphi_j\}_{j=1}^\infty$ be given by Lemma 2.2.4 and let H_x^{-1} be the inverse function of H_x from (2.15). Recall that $(\cdot, \cdot)_{\theta,q}$ denotes real interpolation.

2.4.1. Definition. (i) Let $-\infty < s < \infty$. Let either $0 < p < \infty$ and $0 < q \leqslant \infty$, or $p = q = \infty$. Then

(2.18) $$F_{pq}^s(\Omega, g) = \{f \mid f \in D'(\Omega),$$
$$\|f \mid F_{pq}^s(\Omega, g)\|^p = \Big(\sum_{j=1}^{\infty} \|(\varphi_j f) \circ H_{xj}^{-1} \mid F_{pq}^s(\boldsymbol{R}^n)\|^p\Big)^{1/p} < \infty\}$$

(with the usual modification if $p = q = \infty$).

(ii) Let $-\infty < s_0 < s < s_1 < \infty$ and $s = (1-\theta)s_0 + \theta s_1$. Let $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. Then

(2.19) $$B_{pq}^s(\Omega, g) = (F_{pp}^{s_0}(\Omega, g), F_{pp}^{s_1}(\Omega, g))_{\theta,q}.$$

2.4.2. Remark. As usual, $D'(\Omega)$ is the collection of all complex-valued distributions in Ω. Furthermore, $(\varphi_j f) \circ H_{xj}^{-1}$ in (2.18) is an element of $D'(V_c)$ (see (2.15)) extended by zero outside V_c, and hence it may be considered as an element of $S'(\boldsymbol{R}^n)$. Now (2.19) is the natural way to incorporate the spaces $B_{pq}^s(\Omega, g)$ in this theory, in contrast to the situation in $\boldsymbol{R}^n$, where (2.4) can be replaced by direct definitions of $B_{pq}^s(\boldsymbol{R}^n)$ without using $F_{pq}^s(\boldsymbol{R}^n)$ as a vehicle.

2.4.3. *Motivation.* We wish to shed some light on the construction (2.18). Let φ be a C^∞ function on $\boldsymbol{R}^n$ with compact support, and let

$$\sum_{j \in \boldsymbol{Z}^n} \varphi(x+j) = 1 \quad \text{if } x \in \boldsymbol{R}^n,$$

where $\boldsymbol{Z}^n$ stands for the lattice of all points in $\boldsymbol{R}^n$ with integer-valued components. Then

(2.20) $$\|f \mid F_{pq}^s(\boldsymbol{R}^n)\|^p \sim \sum_{j \in \boldsymbol{Z}^n} \|\varphi(\cdot + j) f \mid F_{pq}^s(\boldsymbol{R}^n)\|^p$$

(with the usual modification if $p = q = \infty$). But (2.20) with $B_{pq}^s(\boldsymbol{R}^n)$, $p \neq q$, is wrong. In other words, (2.18) looks reasonable, but not its counterpart with $B_{pq}^s(\boldsymbol{R}^n)$, $p \neq q$, instead of $F_{pq}^s(\boldsymbol{R}^n)$. Furthermore, $(\varphi_j f) \circ H_{xj}^{-1}$ is our simplified version of $(\varphi_j f) \circ \exp_{xj}^{-1}$ in the more general context of complete Riemannian manifolds (see [Tri4, p. 310]). What we really need is the fact that $\{\Phi_j(Y)\}_{j=1}^\infty$ with

(2.21) $$\Phi_j = \varphi_j \circ H_{xj}^{-1}, \quad j = 1, 2, \ldots,$$

is a bounded set in $C^\infty(\bar{V}_c)$ (see (2.11) and (2.15)). Finally, we mention that some of the spaces $F_{p2}^s(\Omega, g)$, $s \geqslant 0$, $1 < p < \infty$, are closely connected with weighted spaces of H_p^s type introduced in [Tri1, 3.2.3]. But we shall not stress this point in the sequel.

2.4.4. Proposition. (i) *$F_{pq}^s(\Omega, g)$ is a quasi-Banach space (Banach space if $p \geqslant 1$, $q \geqslant 1$), independent of the resolution of unity $\{\varphi_j\}_{j=1}^\infty$ (and the underlying decomposition $\Omega = \bigcup_{j=1}^\infty U_{xj}$).*

(ii) *$B_{pq}^s(\Omega, g)$ is a quasi-Banach space (Banach space if $p \geqslant 1$, $q \geqslant 1$), independent of the numbers s_0 and s_1.*

(iii) *For all $-\infty < s < \infty$ and $0 < p \leqslant \infty$, $F_{pp}^s(\Omega, g) = B_{pp}^s(\Omega, g)$.*

2.4.5. Proof of Proposition 2.4.4. The proof of (i) is based on the fact that $\{\Phi_j\}_{j=1}^\infty$ is a bounded set in $C^\infty(\bar{V}_c)$. Otherwise one can proceed as in [Tri4, Theorem 1(i) and its proof in 4.1]. The proof of (ii) and (iii) is based on the well-known techniques of interpolation theory. We refer to [Tri4, 4.6] for details.

2.4.6. Remark. In particular, the above proposition justifies the definitions given in 2.4.1.

2.4.7. *Sobolev spaces.* Let $0 < p \leqslant \infty$. Then $L_p(\Omega, g)$ is the collection of all complex-valued locally integrable functions f in Ω with

(2.22) $$\|f \mid L_p(\Omega, g)\| = \Big(\int_\Omega |f(x)|^p g^n(x)\, dx\Big)^{1/p} < \infty$$

(with the usual modification if $p=\infty$). Of course, $L_p(\Omega, g)$ is a quasi-Banach space (Banach space if $p \geqslant 1$). Let $1<p<\infty$ and $m=1, 2, \ldots$ Then by definition

$$(2.23)\qquad W_p^m(\Omega, g)=\{f \mid f\in D'(\Omega),$$
$$\|f \mid W_p^m(\Omega, g)\| = \Big(\sum_{|\alpha|\leqslant m} \|g^{-|\alpha|}D^\alpha f \mid L_p(\Omega, g)\|^p\Big)^{1/p}<\infty\}.$$

Let $W_p^0(\Omega, g)=L_p(\Omega, g)$.

2.4.8. THEOREM. *Let $1<p<\infty$ and $m=0, 1, 2, \ldots$ Then*

$$(2.24)\qquad F_{p2}^m(\Omega, g)=W_p^m(\Omega, g) \quad \textit{(with equivalent norms)}.$$

2.4.9. Proof of Theorem 2.4.8. On the one hand, we have (2.5). On the other hand, standard arguments yield

$$(2.25)\qquad \|f \mid W_p^m(\Omega, g)\| \sim \Big(\sum_{j=1}^{\infty} \|(\varphi_j f)\circ H_{x^j}^{-1} \mid W_p^m(\boldsymbol{R}^n)\|^p\Big)^{1/p}$$

(see (2.15) and (2.21)). Then (2.24) is a consequence of (2.5) and (2.18).

2.4.10. EXAMPLE. Let $d(x)$ be the distance function from (2.9). Then the weighted Sobolev spaces normed via

$$\Big(\sum_{|\alpha|\leqslant m} \|d^{\varkappa|\alpha|}D^\alpha f \mid L_p(\Omega, d^{-\varkappa})\|^p\Big)^{1/p}, \quad 1<p<\infty,\ \varkappa\geqslant 1,$$

are covered by our theory. This follows from 2.2.3 and Theorem 2.4.8. Spaces of this type attracted much attention (see the theorem and references in [Tri1, 3.2.6]).

2.5. Equivalent quasi-norms: means. We are looking for intrinsic descriptions of the spaces $F_{pq}^s(\Omega, g)$ and $B_{pq}^s(\Omega, g)$. Let k_N be the same as at the beginning of 2.1. The counterpart of (2.1) reads as follows:

$$(2.26)\qquad k_N^g(t, f)(x)=\int_{\boldsymbol{R}^n} k_N(|y|)\, f\big(x+g^{-1}(x)ty\big)\,dy, \quad x\in\Omega,\ 0<t<c,$$

where we may assume that c is the same as in (2.6). Recall that $k_N(|y|)$ is supported by the unit ball in $\boldsymbol{R}^n$. In particular, $k_N^g(t, f)(x)$ makes sense for any $f\in D'(\Omega)$ (under appropriate interpretation) and it is a C^∞ function in Ω.

2.5.1. THEOREM. *Let c be the same number as in (2.6).*

(i) *Let $-\infty<s<\infty$. Let either $0<p<\infty$ and $0<q\leqslant\infty$, or $p=q=\infty$. If the natural number N is sufficiently large and if $\varepsilon>0$ is sufficiently small then*

$$(2.27)\qquad \|k_0^g(\varepsilon, f) \mid L_p(\Omega, g)\| + \Big\|\Big(\int_0^c t^{-sq}|k_N^g(t, f)(\cdot)|^q\,dt/t\Big)^{1/q} \Big| L_p(\Omega, g)\Big\|$$

(modified if $q=\infty$) is an equivalent quasi-norm in $F_{pq}^s(\Omega, g)$.

(ii) *Let $-\infty<s<\infty$, $0<p\leqslant\infty$ and $0<q\leqslant\infty$. If N is sufficiently large and if $\varepsilon>0$ is sufficiently small then*

$$(2.28)\qquad \|k_0^g(\varepsilon, f) \mid L_p(\Omega, g)\| + \Big(\int_0^c t^{-sq}\|k_N^g(t, f) \mid L_p(\Omega, g)\|^q\,dt/t\Big)^{1/q}$$

(modified if $q=\infty$) is an equivalent quasi-norm in $B_{pq}^s(\Omega, g)$.

2.5.2. Remark. Let M be the Riemannian manifold from Theorem 2.3.1. Let $F_{pq}^s(M)$ and $B_{pq}^s(M)$ be the spaces defined in [Tri4, 5]. Then we have

$$(2.29)\qquad F_{pq}^s(M)=F_{pq}^s(\Omega, g), \qquad B_{pq}^s(M)=B_{pq}^s(\Omega, g)$$

for all admissible values of s, p, q. This follows immediately from the above Definition 2.4.1 and Definition 3 in [Tri4]. Then the above theorem is the counterpart of Definition 2 and Theorem 2 in [Tri4]. In order to compare these two representations we rewrite (2.26) as

$$(2.30)\qquad k_N^g(t, f)(x) = \int_{\boldsymbol{R}^n} k_N\big(g(x)|y|\big)\, f(x+ty)\, g^n(x)\,dy$$
$$= \int_{T_x\Omega} k_N(\|Y\|)\, f(x+tY)\,dY, \quad x\in\Omega,$$

where $T_x\Omega$ is the tangent space (identified with $\boldsymbol{R}^n$) equipped with the Riemannian metric $\|Y\|=g(x)|y|$ and the Riemannian volume element $dY=g^n(x)\,dy$. The means (2.30) differ from the corresponding means in [Tri4, (21)] inasmuch as the rays $x+tY$, which have no invariant meaning for general manifolds, replace the geodesics $c(x, Y, t)$ with $c(x, Y, 0)=x$ and $(dc/dt)(x, Y, 0)=Y$.

2.5.3. Proof of Theorem 2.5.1. By (2.29) and (2.30) one can follow the arguments in [Tri4, 4.2, 4.5–4.8] where one has to replace the above geodesics $c(x, Y, t)$ by the rays $x+tY$. This even simplifies some calculations. But even in this simplified situation we have to use Theorem 2.3.1 and the lifting properties of the Laplace–Beltrami operator which will be described in 2.7. In this context we also refer to [Tri5, 3.8, 4.7, 4.8] where we proved these lifting properties in general (in contrast to [Tri4] where we needed an ugly extra condition).

2.5.4. Remark. As for the natural number N we have in both parts of Theorem 2.5.1 the estimate

$$(2.31)\qquad N>n(1/p-1)_+ + \max(s,\ 5+2n/p)$$

(see [Tri4], where we also gave an estimate for ε). Of course, (2.31) is not natural. Maybe one can improve this estimate somewhat in our more special situation. But it is very doubtful whether the natural estimate for N from Definition 2.1.1(i) can be obtained.

2.6. Equivalent quasi-norms: differences. Now we come to those parts of the theory of the spaces from Definition 2.4.1 where substantial improvements compared with [Tri4, 5] can be obtained. First we describe the situation in $\boldsymbol{R}^n$. Let $h \in \boldsymbol{R}^n$ and $m = 1, 2, \ldots$ Then

$$\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f\left(x + \frac{j}{m} h\right), \qquad x \in \boldsymbol{R}^n, \tag{2.32}$$

are the usual differences of functions. Let $c > 0$. If $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s > n(1/p-1)_+$ and $m > s$ then

$$\|f | L_p(\boldsymbol{R}^n)\| + \Big(\int_{|h| \leqslant c} |h|^{-sq} \|\Delta_h^m f | L_p(\boldsymbol{R}^n)\|^q dh/|h|^n\Big)^{1/q} \tag{2.33}$$

is an equivalent quasi-norm in $B_{pq}^s(\boldsymbol{R}^n)$ (modified if $q = \infty$). If $0 < p < \infty$, $0 < q \leqslant \infty$, $s > n \max(1/p, 1/q)$ and $m > s$ then

$$\|f | L_p(\boldsymbol{R}^n)\| + \Big\|\Big(\int_{|h| \leqslant c} |h|^{-sq} |\Delta_h^m f(\cdot)|^q dh/|h|^n\Big)^{1/q} \Big| L_p(\boldsymbol{R}^n)\Big\| \tag{2.34}$$

is an equivalent quasi-norm in $F_{pq}^s(\boldsymbol{R}^n)$ (modified if $q = \infty$). We refer to [Tri2, 2.5.10 and 2.5.12] and [Tri3, Theorems 5 and 6]. It is our aim to extend these assertions to the spaces $F_{pq}^s(\Omega, g)$ and $B_{pq}^s(\Omega, g)$.

2.6.1. *Differences.* Let again (Ω, g) be a bounded connected domain in $\boldsymbol{R}^n$ equipped with an admissible weight g. Let $h \in \boldsymbol{R}^n$ and $m = 1, 2, \ldots$ Then

$$\Delta_h^m [g] f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f\left(x + \frac{j}{m} h g^{-1}(x)\right) \tag{2.35}$$

are the differences of interest. If c is the same as in (2.6) then $\Delta_h^m [g] f(x)$ makes sense for any function f defined in Ω provided that $x \in \Omega$ and $|h| < c$.

2.6.2. Theorem. (i) *Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $s > n \max(1/p, 1/q)$. Let m be a natural number with $m > s$ and let $b > 0$ be small (in dependence on s, p, q, m). Then*

$$\|f | L_p(\Omega, g)\| + \Big\|\Big(\int_{|h| \leqslant b} |h|^{-sq} |\Delta_h^m [g] f(\cdot)|^q dh/|h|^n\Big)^{1/q} \Big| L_p(\Omega, g)\Big\| \tag{2.36}$$

(modified if $q = \infty$) is an equivalent quasi-norm in $F_{pq}^s(\Omega, g)$.

(ii) *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $s > n(1/p-1)_+$. Let m be a natural number with $m > s$ and let $b > 0$ be small (in dependence on s, p, q, m). Then*

$$\|f | L_p(\Omega, g)\| + \Big(\int_{|h| \leqslant b} |h|^{-sq} \|\Delta_h^m [g] f | L_p(\Omega, g)\|^q dh/|h|^n\Big)^{1/q} \tag{2.37}$$

(modified if $q = \infty$) is an equivalent quasi-norm in $B_{pq}^s(\Omega, g)$.

2.6.3. Remark. The proof of this theorem will be given in 4.2. The spaces $F_{\infty\infty}^s(\Omega, g)$ with $s > 0$ are covered by part (ii) (see Proposition 2.4.4(iii)). In this case we have

$$\begin{aligned} \|f | F_{\infty\infty}^s(\Omega, g)\| &\sim \sup_{x \in \Omega} |f(x)| + \sup_{x \in \Omega, |h| \leqslant c} |h|^{-s} |\Delta_h^m [g] f(x)| \\ &= \sup_{x \in \Omega} |f(x)| + \sup_{x \in \Omega, g(x)|h| \leqslant c} g^{-s}(x) |h|^{-s} |\Delta_h^m f(x)|. \end{aligned} \tag{2.38}$$

Hence $\mathscr{C}^s(\Omega, g) = F_{\infty\infty}^s(\Omega, g)$ with $s > 0$ are weighted Hölder–Zygmund spaces.

2.6.4. Remark. We compare Theorem 2.6.2 and Remark 2.6.3 with known results. Under the same hypotheses as in Theorem 2.6.2(i), the expression (2.36) with $L_p(\boldsymbol{R}^n)$ instead of $L_p(\Omega, g)$ and with Δ_h^m instead of $\Delta_h^m[g]$ is an equivalent quasi-norm in $F_{pq}^s(\boldsymbol{R}^n)$. Furthermore, under the same hypotheses as in Theorem 2.6.2(ii) and after the just-mentioned replacements, (2.37) is an equivalent quasi-norm in $B_{pq}^s(\boldsymbol{R}^n)$. We refer to [Tri2, 2.5.10 and 2.5.12]. On the other hand, we extended in [Tri5, 3.5] these results from $\boldsymbol{R}^n$ to complete Riemannian manifolds with bounded geometry and positive injectivity radius, where in general only an extension for the spaces F_{pq}^s can be expected. However, the restrictions for the above parameters s, p, q and m are far from being optimal. For example, we obtained the equivalent quasi-norm (2.38) only under the unnatural restriction $s > 4$ (see [Tri5, Theorem 3]).

2.7. Lifting properties. Let again $M = (\Omega, g)$ be the Riemannian manifold from Theorem 2.3.1. In this special case the Laplace–Beltrami operator Δ is given by

$$\Delta = g^{-n}(x) \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(g^{n-2} \frac{\partial}{\partial x_j}\right) \tag{2.39}$$

(see e.g. [Tri4, (36)] for the general case).

2.7.1. Theorem. *Let $0 < p \leqslant \infty$ (with $p < \infty$ in the case of the spaces $F_{pq}^s(\Omega, g)$), $0 < q \leqslant \infty$ and $-\infty < s < \infty$. There exists a nonnegative number $\varrho(\Omega, g, p, q)$ with the following property: If $\varrho > \varrho(\Omega, g, p, q)$ then $f \to \varrho^2 f - \Delta f$ yields an isomorphic map from $F_{pq}^s(\Omega, g)$ onto $F_{pq}^{s-2}(\Omega, g)$ and from $B_{pq}^s(\Omega, g)$ onto $B_{pq}^{s-2}(\Omega, g)$.*

2.7.2. Remark. This is a special case of Theorem 7 in [Tri5] (see also Theorem 6 in [Tri4]). We return to problems of this type in 3.5 where we prove a more general assertion.

Appendix

As we said the reduction of Theorem 2.6.2 to the corresponding results in [Tri5] is unsatisfactory. We reduce this assertion to its euclidean counterpart. But this causes some trouble. We overcome these difficulties with the help of variable differences. Let $\varepsilon(x, h)$ be a continuously differentiable map from

(A.1) $$E_c = \{(x, h) \mid x \in \mathbf{R}^n,\ h \in \mathbf{R}^n \text{ with } |h| \leqslant c\}$$

into $\mathbf{R}^n$, where $c > 0$ is given. Let

(A.2) $$|\varepsilon(x, h)| \leqslant \delta |h|, \quad \sum_{j=1}^{n} \left|\frac{\partial \varepsilon}{\partial x_j}(x, h)\right| \leqslant \varkappa, \quad (x, h) \in E_c,$$

for some $\delta > 0$ and $\varkappa > 0$. Let $x \in \mathbf{R}^n$, $y \in \mathbf{R}^n$ and $h \in \mathbf{R}^n$. Then

(A.3) $$\Delta^m_{h+\varepsilon(x,h)} f(y) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f\left(y + \frac{j}{m} h + \frac{j}{m} \varepsilon(x, h)\right)$$

with $y = x$ are the variable differences of interest (cf. (2.32) and (2.35)).

A.1. THEOREM. *Let $c > 0$ and let l be a real number. Let $\varepsilon(x, h)$ be a function as above.*

(i) *Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $s > n \max(1/p, 1/q)$. Let m be a natural number with $m > s$. If $\delta > 0$ and $\varkappa > 0$ in* (A.2) *are sufficiently small (in dependence on the given parameters) then*

(A.4) $$\|f \mid L_p(\mathbf{R}^n)\| + \Big\|\Big(\int_{|h| \leqslant c} |h|^{-sq} |\Delta^m_{h+\varepsilon(\cdot,h)} f(\cdot + lh)|^q \, dh/|h|^n\Big)^{1/q} \Big| L_p(\mathbf{R}^n)\Big\|$$

(modified if $q = \infty$) is an equivalent quasi-norm in $F^s_{pq}(\mathbf{R}^n)$.

(ii) *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $s > n(1/p - 1)_+$. Let m be a natural number with $m > s$. If $\delta > 0$ and $\varkappa > 0$ in* (A.2) *are sufficiently small (in dependence on the given parameters) then*

(A.5) $$\|f \mid L_p(\mathbf{R}^n)\| + \Big(\int_{|h| \leqslant c} |h|^{-sq} \|\Delta^m_{h+\varepsilon(\cdot,h)} f(\cdot + lh) \mid L_p(\mathbf{R}^n)\|^q \, dh/|h|^n\Big)^{1/q}$$

(modified if $q = \infty$) is an equivalent quasi-norm in $B^s_{pq}(\mathbf{R}^n)$.

A.2. Remark. Part (ii) of Theorem A.1 is essentially covered by [GT]. Part (i) will be proved in detail in 4.1. The case $l = 0$ is of independent interest. The extension to arbitrary real l will be useful later for technical reasons.

3. Weighted pseudodifferential operators

3.1. Differential operators. First we single out a class of differential operators which will later be covered by the more general class of pseudodifferential operators treated in this paper. We always assume that Ω stands for a bounded connected domain in $\mathbf{R}^n$ equipped with an admissible weight g in the sense of Definition 2.2.1. Let

(3.1) $$Af(x) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha f(x), \quad x \in \Omega,$$

where the coefficients $a_\alpha(x)$ are complex-valued C^∞ functions in Ω.

3.1.1. DEFINITION. Let m be a natural number. Then $\Sigma^m(\Omega, g)$ is the collection of all differential operators (3.1) such that for every multi-index γ there exists a positive number c_γ with

(3.2) $$|D^\gamma a_\alpha(x)| \leqslant c_\gamma g^{|\gamma| - |\alpha|}(x) \quad \text{for all } x \in \Omega,$$

where $|\alpha| \leqslant m$.

3.1.2. Remark. We have

(3.3) $$\Delta \in \Sigma^2(\Omega, g),$$

where Δ is the Laplace–Beltrami operator (2.39). This follows easily from (2.39) and (2.8).

3.1.3. PROPOSITION. *Let $A \in \Sigma^m(\Omega, g)$.*

(i) *Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $-\infty < s < \infty$. Then A yields a continuous map from $F^{s+m}_{pq}(\Omega, g)$ into $F^s_{pq}(\Omega, g)$.*

(ii) *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $-\infty < s < \infty$. Then A yields a continuous map from $B^{s+m}_{pq}(\Omega, g)$ into $B^s_{pq}(\Omega, g)$.*

3.1.4. Remark. This proposition is covered by Theorems 3.3.1 and 3.3.5 and so we shall not give a detailed separate proof. The shortest way to prove this assertion directly runs as follows. On the basis of (2.18) one decomposes and transforms the operator A. The result is a bounded set $\{A_j\}_{j=1}^\infty$ of differential operators on $\mathbf{R}^n$ which can be estimated uniformly with respect to j. Retransformation yields the desired result.

3.2. Preliminaries and definitions. Let $a(x, \xi) \in S^\mu_{1,\delta}$ with $-\infty < \mu < \infty$, $0 \leqslant \delta < 1$ (see the introduction, in particular (1.1)). Then the corresponding pseudodifferential operators are defined via

(3.4) $$a(x, D) f(x) = \int_{\mathbf{R}^n} e^{ix\xi} a(x, \xi) \hat{f}(\xi) \, d\xi,$$

where $\hat{f}$ stands for the Fourier transform of f. Let $\Psi^\mu_\delta(\mathbf{R}^n)$ be the collection of all these operators. We have the mapping properties mentioned in the introduction. One of the main aims of this paper is to find a natural counterpart of $\Psi^\mu_\delta(\mathbf{R}^n)$ for the manifold $M = (\Omega, g)$ from Theorem 2.3.1. Again the domain Ω and the weight g are as above.

3.2.1. *Basic assumptions.* Let A be a linear and continuous map from $D(\Omega)$ into $D'(\Omega)$. Then by Schwartz's kernel theorem A can be represented as

(3.5) $$Af(x) = \int_\Omega A(x, y) f(y) g^n(y) \, dy, \quad A(x, y) \in D'(\Omega \times \Omega)$$

(with appropriate interpretation as distributions; see e.g. [Tre, p. 10] or [Hö1, 5.2]). It is convenient for us to incorporate the factor $g^n(y)$ in (3.5) because $g^n(y)\,dy$ is the Riemannian volume element on $M = (\Omega, g)$. Let ω

$= \{(x, y) \mid (x, y) \in \Omega\times\Omega,\ x = y\}$ be the diagonal of $\Omega\times\Omega$. Then we assume

$$A(x, y) \in C^\infty(\Omega\times\Omega-\omega), \tag{3.6}$$

i.e. the kernel $A(x, y)$ is a C^∞ function off the diagonal in $\Omega\times\Omega$. This is reasonable because we are looking for a class of pseudodifferential operators (see [Tre, Theorem 2.1, p. 224] for the case of the classes $\Psi^\mu_\delta(\boldsymbol{R}^n)$).

3.2.2. *Desirable properties.* The classes $\Psi^\mu_\delta(\boldsymbol{R}^n)$ have some well-known properties. It is desirable that possible global classes of pseudodifferential operators on $M = (\Omega, g)$ have similar properties. We list some of them:

(i) The classes $\Sigma^m(\Omega, g)$ of differential operators from Definition 3.1.1 should be covered.

(ii) Let Δ be the Laplace–Beltrami operator (2.39) and let I be the identity. It is reasonable to expect that $(\varrho^2 I-\Delta)^\mu$ with μ real and $\varrho > 0$ is a pseudodifferential operator belonging to the global classes we are looking for.

(iii) It is desirable that compositions $A\circ B$ can be described within these global classes of pseudodifferential operators.

(iv) The parametrices of elliptic pseudodifferential operators of the treated global classes should belong to corresponding global classes.

(v) However, our main aim is to find global classes of pseudodifferential operators with mapping properties similar to those of the classes $\Psi^\mu_\delta(\boldsymbol{R}^n)$ (see the introduction).

3.2.3. *Geometrical preparations.* In order to provide a better understanding of the definition below some geometrical considerations seem to be useful. Let $M = (\Omega, g)$ be the Riemannian manifold from Theorem 2.3.1; however, what follows is true for any connected complete Riemannian manifold M with bounded geometry and positive injectivity radius. Let $d(x, y)$ be the Riemannian distance between $x\in\Omega$ and $y\in\Omega$, where in our case the Riemannian metric is given by (2.13). Let

$$B_x(r) = \{y \mid y\in\Omega,\ d(x, y) < r\} \tag{3.7}$$

be the geodesic ball centered at $x\in\Omega$ with radius $r > 0$. It seems to be a known fact that there exist positive numbers a and b such that

$$\operatorname{vol} B_x(r) \leqslant a e^{br} \tag{3.8}$$

for all $x\in\Omega$ and all $r > 0$. Of course, $\operatorname{vol} B_x(r)$ stands for the Riemannian volume of the ball (3.7). We outline a proof of (3.8). Assume by induction that the ball $B_x(cj)$ can be covered by e^{bcj} balls with Riemannian volume V, where j is a natural number and $c, V, b > 0$. If c and V are sufficiently small and if b is sufficiently large then $B_x(cj+c)$ can be covered by, say, $2e^{bcj}$ balls of volume V and hence by $e^{bc(j+1)}$ balls. This can be done uniformly with respect to $x\in\Omega$ and by euclidean arguments, because M has bounded geometry and a positive injectivity radius. This proves (3.8). Now, define

$$G = \inf b, \tag{3.9}$$

where the infimum is taken over all b for which there exists a with (3.8). We have in general $G > 0$. Even more, the well-known volumes of balls in simply connected Riemannian manifolds with constant negative sectional curvature show that any $G > 0$ can be expected ([CGT, p. 22]).

3.2.4. *Technical preparations.* Let $\Omega = \bigcup_{j=1}^\infty U_{x^j}$ be a covering as in Lemma 2.2.4 and let $\varphi = \{\varphi_j\}_{j=1}^\infty$ be the corresponding resolution of unity. Let $d_{jk} = d(U_{x^j}, U_{x^k})$ be the Riemannian distance of U_{x^j} and U_{x^k} measured via (2.13). Let A be given by (3.5), (3.6). We introduce

$$A_{jk} f(x) = \int_\Omega A_{jk}(x, y) f(y) g^n(y)\, dy \quad \text{with} \tag{3.10}$$

$$A_{jk}(x, y) = \varphi_j(x) A(x, y) \varphi_k(y),$$

again interpreted in the sense of distributions ([Hö1, 5.2]). Of course,

$$A = \sum_{j,k=1}^n A_{jk}. \tag{3.11}$$

Furthermore, $A_{jk}(x, y)$ are C^∞ kernels if $d_{jk} > 0$. Finally, we need the transformation of the operators A_{jk} to the normalized coordinates from (2.15), i.e.

$$A'_{jk} u = A_{jk}(u\circ H_{x^k})\circ H_{x^j}^{-1}, \qquad u\in D(\boldsymbol{R}^n), \tag{3.12}$$

which can be interpreted as a linear and continuous operator either from $D(V_c)$ into $D'(V_c)$ or from $D(\boldsymbol{R}^n)$ into $D'(\boldsymbol{R}^n)$ (see e.g. [Ta, II, § 5]). If we use (3.10) then we have

$$\begin{aligned} A'_{jk} u(X) &= \int_\Omega A_{jk}(H_{x^j}^{-1}(X), y)\, u(H_{x^k}(y))\, g^n(y)\, dy \\ &= \int_{\boldsymbol{R}^n} A_{jk}(H_{x^j}^{-1}(X), H_{x^k}^{-1}(Y))\, u(Y) \frac{g^n(H_{x^k}^{-1}(Y))}{g^n(x^k)}\, dY \\ &= \int_{\boldsymbol{R}^n} A'_{jk}(X, Y)\, u(Y)\, dY, \end{aligned} \tag{3.13}$$

where $u(Y)$ and $A'_{jk}(X, Y)$ are extended by zero outside V_c and $V_c\times V_c$, respectively. By (2.7), (2.8) and (2.15),

$$\left\{\frac{g^n(H_{x^k}^{-1}(Y))}{g^n(x^k)}\right\}_{k=1}^\infty$$

is a bounded set in $D(V_c)$ and in $D(\boldsymbol{R}^n)$ (after appropriate extensions from V_c to $\boldsymbol{R}^n$). In other words, except for these unimportant factors the kernels

(3.14) $$A'_{jk}(X, Y) \quad \text{and} \quad A_{jk}(H_{xj}^{-1}(X), H_{xk}^{-1}(Y))$$

coincide essentially in $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

3.2.5. Definition. Let $M = (\Omega, g)$ be the above manifold and let G be the number defined by (3.9). Let L, μ, δ be real numbers with

(3.15) $$L > G, \quad -\infty < \mu < \infty, \quad 0 \leqslant \delta < 1.$$

Then $\Psi_\delta^{\mu,L}(\Omega, g)$ is the class of all operators A given by (3.5), (3.6) with the following properties:

(i) For every couple of multi-indices (α, β) there exists a positive number $c_{\alpha\beta}$ with

(3.16) $$|D_x^\alpha D_y^\beta A_{jk}(x, y)| \leqslant c_{\alpha\beta}\, e^{-Ld_{jk}} g^{|\alpha|}(x) g^{|\beta|}(y)$$

for all $x \in \Omega$, $y \in \Omega$ and all natural numbers j, k with $d_{jk} \geqslant 1$.

(ii) $\{A'_{jk}\}_{j,k=1}^{\infty}$ is a bounded set in $\Psi_\delta^\mu(\boldsymbol{R}^n)$.

3.2.6. Remark. These are the classes of pseudodifferential operators we are looking for. From (3.6) and (3.10) it follows that (3.16) makes sense. By (2.7), (2.15), (3.13) and (3.14) we have

(3.17) $$|D_X^\alpha D_Y^\beta A'_{jk}(X, Y)| \leqslant c'_{\alpha\beta}\, e^{-Ld_{jk}}$$

provided that $d_{jk} \geqslant 1$. It follows that the subfamily $\{A'_{jk}$ with $d_{jk} \geqslant 1\}$ is a bounded set in $\Psi^{-\infty}(\boldsymbol{R}^n)$, and hence also in $\Psi_\delta^\mu(\boldsymbol{R}^n)$. In other words, part (ii) of Definition 3.2.5 is only for those j and k with $d_{jk} \leqslant 1$ of interest. In order to make clear what is meant by this requirement we put

(3.18) $$A'_{jk}(X, Y) = B_{jk}(X, X-Y).$$

Recall that $A'_{jk}(X, Y) \in D'(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ with support in $V_c \times V_c$ (see (2.15) and (3.13)). Then $B_{jk}(X, Z) \in D'(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ has a support in $V_c \times V_{2c}$. In particular, $B_{jk}(X, Z) \in S'(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ and $F_Z B_{jk}(X, \cdot)$ makes sense, where F_Z stands for the n-dimensional Fourier transform with respect to the second variable $Z \in \boldsymbol{R}^n$. We put (3.18) in (3.13), apply $F_Z^{-1} F_Z$ and obtain

(3.19) $$A'_{jk} u(X) = \int_{\boldsymbol{R}^n} e^{iX\xi} F_Z B_{jk}(X, \cdot)(\xi)\, \hat{u}(\xi)\, d\xi.$$

This is the canonical form of a pseudodifferential operator in $\boldsymbol{R}^n$ with symbol $F_Z B_{jk}(X, \cdot)(\xi)$ (see (3.4)). Hence Definition 3.2.5 (ii) can be reformulated as follows: For every couple (α, β) of multi-indices there exists a positive number $c_{\alpha\beta}$ such that

(3.20) $$|D_X^\beta D_\xi^\alpha F_Z B_{jk}(X, \cdot)(\xi)| \leqslant c_{\alpha\beta}(1+|\xi|)^{\mu-|\alpha|+\delta|\beta|}$$

for all $X \in \boldsymbol{R}^n$, $\xi \in \boldsymbol{R}^n$ and all natural numbers j and k.

3.3. Properties. We describe some properties of the classes $\Psi_\delta^{\mu,L}(\Omega, g)$ which cover more or less the desirable properties listed in 3.2.2. Recall that the class $\Sigma^m(\Omega, g)$ has been described in Definition 3.1.1, whereas Δ is the Laplace–Beltrami operator from (2.39). Furthermore, I stands for the identity, and G is defined in (3.9).

3.3.1. Theorem. (i) *Let m be a natural number. Then*

(3.21) $$\Sigma^m(\Omega, g) \subset \Psi_0^{m,L}(\Omega, g) \quad \textit{for every } L > G.$$

(ii) *Let μ and ϱ be real numbers with $\varrho > G$. Then*

(3.22) $$(\varrho^2 I - \Delta)^{\mu/2} \in \Psi_0^{\mu,\varrho}(\Omega, g).$$

3.3.2. Remark. We prove this theorem in 4.3. Part (i) is not difficult and the corresponding Schwartz kernel in the sense of (3.5) has a support in the diagonal ω of $\Omega \times \Omega$. Part (ii) is deeper but essentially covered by [CGT]. If $\mu = 2m$ is even then $(\varrho^2 I - \Delta)^m \in \Sigma^{2m}(\Omega, g)$. This follows from (3.3) and improves (3.22) (see (3.21)).

3.3.3. Theorem. *Let $A \in \Psi_{\delta_A}^{\mu_A,L_A}(\Omega, g)$ and $B \in \Psi_{\delta_B}^{\mu_B,L_B}(\Omega, g)$, where μ_A and μ_B are real numbers,*

(3.23) $$L_A > G, \quad L_B > G, \quad 0 \leqslant \delta_A < 1, \quad 0 \leqslant \delta_B < 1.$$

Let $\mu_C = \mu_A + \mu_B$, $\delta_C = \max(\delta_A, \delta_B)$ and

(3.24) $$\frac{1}{L_C} = \frac{1}{L_A} + \frac{1}{L_B} \quad \textit{with } L_C > G.$$

Then

(3.25) $$C = A \circ B \in \Psi_{\delta_C}^{\mu_C,L_C}(\Omega, g).$$

3.3.4. Remark. We prove this theorem in 4.4. Here we shall need essentially $L_A > G$ and $L_B > G$. This explains the restriction $L > G$ in (3.15). Otherwise, Definition 3.2.5 makes sense for any $L > 0$ and, after this extension, (3.22) holds for all $\varrho > 0$.

3.3.5. Theorem. *Let $-\infty < s < \infty$, $-\infty < \mu < \infty$ and $0 \leqslant \delta < 1$.*

(i) *Let $0 < p < \infty$, $0 < q \leqslant \infty$ and*

(3.26) $$A \in \Psi_\delta^{\mu,L}(\Omega, g) \quad \textit{with } L > G \max(1, 1/p, 1/q).$$

Then A yields a continuous map from $F_{pq}^{s+\mu}(\Omega, g)$ into $F_{pq}^s(\Omega, g)$.

(ii) *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $A \in \Psi_\delta^{\mu,L}(\Omega, g)$ with $L > G \max(1, 1/p)$. Then A yields a continuous map from $B_{pq}^{s+\mu}(\Omega, g)$ into $B_{pq}^s(\Omega, g)$.*

3.3.6. Remark. We prove this theorem in 4.5. The restriction for L in (3.26) is essential. This assertion is one of the main results of this paper. It is the extension of the mapping theorem mentioned in the introduction.

3.4. Elliptic operators. Let A be an elliptic pseudodifferential operator in Ω. Then a pseudodifferential operator P in Ω is called a *parametrix* for A if

$$A \circ P - I \in \Psi^{-\infty}(\Omega), \qquad P \circ A - I \in \Psi^{-\infty}(\Omega), \tag{3.27}$$

where I is the identity (see e.g. [Ta, III, § 1] or [Hö2, Theorem 18.1.24] for details). This is a local notion. It is our aim to find a global counterpart. For this purpose we complement Definition 3.2.5. Let

$$\Psi^{-\infty,L}(\Omega, g) = \bigcap_{\mu<0} \Psi_0^{\mu,L}(\Omega, g) = \bigcap_{\mu<0} \Psi_\delta^{\mu,L}(\Omega, g), \qquad 0 \leqslant \delta < 1, \tag{3.28}$$

where the first equality is a definition and the second is an assertion which follows easily from (3.20). Furthermore, we write $A \in \Psi_\delta^{\mu,\infty}(\Omega, g)$ if $A \in \Psi_\delta^{\mu,L}(\Omega, g)$ for some L and

$$A_{jk}(x, y) = 0 \quad \text{if } d_{jk} \geqslant 1 \tag{3.29}$$

(see (3.16)). Roughly speaking, A belongs to $\Psi_\delta^{\mu,\infty}(\Omega, g)$ if part (ii) of Definition 3.2.5 holds and if part (i) is replaced by the assumption that the support of the kernel $A(x, y)$ from (3.5) is contained in a tube around the diagonal of $\Omega \times \Omega$.

3.4.1. DEFINITION. Let $A \in \Psi_\delta^{\mu,L}(\Omega, g)$. Let $a'_x(Y, \xi)$ be the symbol of A in the normalized coordinates $(Y, \xi) \in V_c \times \boldsymbol{R}^n$ from (2.15) where $x \in \Omega$. Then A is called (Ω, g)*-elliptic* if there exist positive numbers c and C such that

$$|a'_x(0, \xi)| \geqslant c(1+|\xi|)^\mu \quad \text{for all } \xi \in \boldsymbol{R}^n \text{ with } |\xi| \geqslant C \tag{3.30}$$

and for all $x \in \Omega$.

3.4.2. Remark. This is the global version of the usual ellipticity condition (cf. e.g. [Ta, III, § 1]).

3.4.3. THEOREM. *Let $A \in \Psi_\delta^{\mu,L}(\Omega, g)$ be (Ω, g)-elliptic. Then there exists a parametrix $P \in \Psi_\delta^{-\mu,\infty}(\Omega, g)$ with*

$$A \circ P - I \in \Psi^{-\infty,L}(\Omega, g), \qquad P \circ A - I \in \Psi^{-\infty,L}(\Omega, g). \tag{3.31}$$

3.4.4. Remark. We prove this theorem in 4.6; it is more or less a simple consequence of known (local) results for pseudodifferential operators and the above technique.

3.5. Fractional lifting properties. In 2.7 we mentioned a lifting property which was simply a special case of a more general assertion for connected Riemannian manifolds with bounded geometry and positive injectivity radius. In our situation we can extend Theorem 2.7.1 to the fractional case.

3.5.1. THEOREM. *Let $0 < p \leqslant \infty$ (with $p < \infty$ in the case of the spaces $F^s_{pq}(\Omega, g)$), $0 < q \leqslant \infty$ and $-\infty < s < \infty$. There exists a nonnegative number $\varrho(\Omega, g, p, q)$ with the following property: If $\varrho > \varrho(\Omega, g, p, q)$ and if μ is a real number then $f \to (\varrho^2 I - \Delta)^{\mu/2} f$ yields an isomorphic map from $F^{s+\mu}_{pq}(\Omega, g)$ onto $F^s_{pq}(\Omega, g)$ and from $B^{s+\mu}_{pq}(\Omega, g)$ onto $B^s_{pq}(\Omega, g)$.*

3.5.2. Proof of Theorem 3.5.1. We have (3.22). Hence we can apply Theorem 3.3.5 provided that ϱ is large enough. We need this assertion both for μ and $-\mu$. The rest follows from Theorem 3.3.3.

4. Proofs

4.1. Proof of Theorem A.1. First we prove the theorem in the appendix of Section 2.

4.1.1. If $l = 0$ in (A.5) then part (ii) of this theorem is completely covered by Theorem 1 in [GT]. Let l be an arbitrary real number. Then one has to replace $\Delta^m_{h+\varepsilon} f(x)$ on p. 425 in [GT] by $\Delta^m_{h+\varepsilon(x,h)} f(x+lh)$ and use the translation-invariance of the Lebesgue measure in $\boldsymbol{R}^n$. The rest remains unchanged and one obtains part (ii) for all real l.

4.1.2. In order to prove part (i) we have to modify the method from [GT] essentially. First we recall the usual Fourier-analytic definition of $F^s_{pq}(\boldsymbol{R}^n)$. Let $\varphi(\xi)$ be a C^∞ function in $\boldsymbol{R}^n$ with

$$\operatorname{supp} \varphi \subset \{\eta \mid |\eta| \leqslant 2\}, \qquad \varphi(\xi) = 1 \quad \text{if} \quad |\xi| \leqslant 1. \tag{4.1}$$

Then we have the resolution of unity

$$1 = \sum_{j=0}^{\infty} \varphi_j(\xi) \quad \text{with } \varphi_j(\xi) = \varphi(2^{-j}\xi) - \varphi(2^{-j+1}\xi) \tag{4.2}$$

if $j = 1, 2, \ldots$ and $\xi \in \boldsymbol{R}^n$, where we put $\varphi_0 = \varphi$. Let

$$\varphi_j(D) f(x) = (F^{-1} \varphi_j Ff)(x) = \int_{\boldsymbol{R}^n} e^{ix\xi} \varphi_j(\xi) \hat{f}(\xi)\, d\xi \tag{4.3}$$

be the usual decomposition of $f \in S'(\boldsymbol{R}^n)$ in entire analytic functions, $j = 0, 1, 2, \ldots$, where F (as well as $\hat{f}$) and F^{-1} stand for the Fourier transform and its inverse, respectively. Then $F^s_{pq}(\boldsymbol{R}^n)$ can be defined as the collection of all $f \in S'(\boldsymbol{R}^n)$ such that

$$\|f \mid F^s_{pq}(\boldsymbol{R}^n)\|^\varphi = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq} |\varphi_j(D) f(\cdot)|^q\Big)^{1/q} \Big| L_p(\boldsymbol{R}^n)\Big\| < \infty \tag{4.4}$$

(with the usual modification if $q = \infty$), where the parameters s, p, q are as in Definition 2.1.1(i). We refer to [Tri2, 2.3.1] where this definition was taken as the starting point.

4.1.3. Let $m = m_1 + m_2$ be a natural number where m_1 and m_2 are nonnegative integers. Let $h \in \boldsymbol{R}^n$ with $2^{-j} \leqslant |h| \leqslant 2^{-j+1}$. Our first aim is to estimate

$$\Delta_\varepsilon^{m_1}\Delta_h^{m_2} f(x+lh) = \sum_{k=-\infty}^{\infty} \Delta_\varepsilon^{m_1}\Delta_h^{m_2}\varphi_{j+k}(D) f(x+lh) = \sum_{k=-\infty}^{K}\ldots + \sum_{k=K+1}^{\infty}\ldots \tag{4.5}$$

with $\varphi_r = 0$ if $r < 0$ and $\varepsilon = \varepsilon(x, h)$. The natural number K will be specified later. Let $k \leqslant K$. By (A.2) and elementary calculations we have

$$|\Delta_\varepsilon^{m_1}\Delta_h^{m_2}\varphi_{j+k}(D) f(x+lh)| \leqslant c_1 \delta^{m_1} 2^{-jm} \sup_{|x-y|\leqslant c_2 2^{-j}} \sum_{|\alpha|=m} |D^\alpha \varphi_{j+k}(D) f(y)|, \tag{4.6}$$

where c_1 and c_2 are positive numbers independent of j, k and δ. By (4.3) we have

$$D^\alpha \varphi_{j+k}(D) f(y) = 2^{(j+k)|\alpha|}\varphi_{j+k,\alpha}(D) f(y) \quad \text{with} \quad \varphi_{r,\alpha}(\xi) = 2^{-r|\alpha|}\xi^\alpha\varphi_r(\xi),$$

see (4.2) (with the obvious modification if $r = 0$). We introduce the maximal function

$$\varphi_{j,\alpha}^* f(x) = \sup_{y\in \boldsymbol{R}^n} \frac{|\varphi_{j,\alpha}(D) f(y)|}{1+|2^j(x-y)|^a}, \tag{4.7}$$

where $a > 0$ will be specified later. Then (4.6) yields

$$|\Delta_\varepsilon^{m_1}\Delta_h^{m_2}\varphi_{j+k}(D) f(x+lh)| \leqslant c\delta^{m_1} 2^{-jm} 2^{(j+k)m}(1+2^{ka}) \sum_{|\alpha|=m}\varphi_{j+k,\alpha}^*(x) \leqslant c\delta^{m_1} 2^{km} 2^{Ka} \sum_{|\alpha|=m}\varphi_{j+k,\alpha}^*(x). \tag{4.8}$$

Let $k > K$. Then we have

$$|\Delta_\varepsilon^{m_1}\Delta_h^{m_2}\varphi_{j+k}(D) f(x+lh)| \leqslant c 2^{ka}\varphi_{j+k}^* f(x), \tag{4.9}$$

where $\varphi_r^* f$ is defined similarly to (4.7).

4.1.4. Now we estimate

$$I(x) = \Big(\int_{|h|\leqslant 2^{-L}} |h|^{-sq}|\Delta_{\varepsilon(x,h)}^{m_1}\Delta_h^{m_2} f(x+lh)|^q\, dh/|h|^n\Big)^{1/q} \tag{4.10}$$

where L is an integer. Let $0 < q \leqslant 1$. By (4.8) and (4.9) we have

$$\begin{aligned} I^q(x) &\leqslant \sum_{j=L}^{\infty} 2^{jsq} \sup_{2^{-j}\leqslant|h|\leqslant 2^{-j+1}} |\Delta_\varepsilon^{m_1}\Delta_h^{m_2} f(x+lh)|^q \\ &\leqslant \sum_{j=L}^{\infty} 2^{jsq} \sum_{k=-\infty}^{\infty} \sup_{2^{-j}\leqslant|h|\leqslant 2^{-j+1}} |\Delta_\varepsilon^{m_1}\Delta_h^{m_2}\varphi_{j+k}(D) f(x+lh)|^q \\ &\leqslant c\sum_{j=L}^{\infty} 2^{jsq} \sum_{k=-\infty}^{K} \delta^{m_1 q} 2^{kmq} 2^{Kaq} \sum_{|\alpha|=m} \varphi_{j+k,\alpha}^{*q} f(x) \\ &\quad + c\sum_{j=L}^{\infty} 2^{jsq} \sum_{k=K+1}^{\infty} 2^{kaq}\varphi_{j+k}^{*q} f(x) \\ &\leqslant c' \sum_{r=0}^{\infty} 2^{rsq}\varphi_r^{*q} f(x) \sum_{k=K+1}^{\infty} 2^{k(a-s)q} \\ &\quad + c' \sum_{|\alpha|=m}\sum_{r=0}^{\infty} 2^{rsq}\varphi_{r,\alpha}^{*q} f(x) \sum_{k=-\infty}^{K} \delta^{m_1 q} 2^{Kaq} 2^{k(m-s)q}. \end{aligned} \tag{4.11}$$

Let $a > n\max(1/p, 1/q)$ in (4.7) and in the related maximal functions. Then we have the maximal inequality

$$\Big\|\Big(\sum_{r=0}^{\infty} 2^{rsq}\varphi_{r,\alpha}^{*q} f(\cdot)\Big)^{1/q} \Big| L_p(\boldsymbol{R}^n)\Big\| \leqslant c\|f \mid F_{pq}^s(\boldsymbol{R}^n)\| \tag{4.12}$$

(see [Tri2, 2.3.6]) and the corresponding assertion with φ_r^* instead of $\varphi_{r,\alpha}^*$. By assumption we may assume $m > s > a$. Now we choose first K large and afterwards δ small. If $m_1 \geqslant 1$ then the factors on the right-hand side of (4.11) are small. Hence, if $m_1 \geqslant 1$ then we obtain from (4.10)–(4.12)

$$\|I \mid L_p(\boldsymbol{R}^n)\| \leqslant \eta\|f \mid F_{pq}^s(\boldsymbol{R}^n)\| \tag{4.13}$$

where $\eta > 0$ is at our disposal (of course δ depends on η etc.). If $1 \leqslant q \leqslant \infty$ then small technical modifications yield the same result. Hence (4.13) with (4.10) holds for all admissible s, p, q.

4.1.5. Now we finish the proof of part (i) of Theorem A.1 as in [GT]. Let again $\varepsilon = \varepsilon(x, h)$. Then

$$\begin{aligned} \Delta_{h+\varepsilon}^m f(y) &= F^{-1}(e^{i\xi(h+\varepsilon)}-1)^m Ff(y) \\ &= F^{-1}[e^{i\xi h}-1+e^{i\xi h}(e^{i\xi\varepsilon}-1)]^m Ff(y) \\ &= \Delta_h^m f(y) + \sum_{m_1=1}^{m} c_{m_1}\Delta_\varepsilon^{m_1}\Delta_h^{m-m_1} f(y+m_1 h). \end{aligned} \tag{4.14}$$

We choose $y = x+lh$. Now (4.14) and (4.13) yield

$$(4.15)\quad \Big\|\Big(\int_{|h|\leqslant c} |h|^{-sq}|\Delta^m_{h+\varepsilon(\cdot,h)} f(\cdot+lh)|^q\, dh/|h|^n\Big)^{1/q} \,\Big|\, L_p(\boldsymbol{R}^n)\Big\|$$
$$\leqslant c'\Big\|\Big(\int_{|h|\leqslant c} |h|^{-sq}|\Delta^m_h f(\cdot+lh)|^q\, dh/|h|^n\Big)^{1/q} \,\Big|\, L_p(\boldsymbol{R}^n)\Big\| + \eta\, \|f \mid F^s_{pq}(\boldsymbol{R}^n)\|,$$

where we assumed (without restriction of generality) $q < \infty$. Next we use the fact that

$$(4.16)\quad \|f \mid L_p(\boldsymbol{R}^n)\| + \Big\|\Big(\int_{|h|\leqslant c} |h|^{-sq}|\Delta^m_h f(\cdot+lh)|^q\, dh/|h|^n\Big)^{1/q} \,\Big|\, L_p(\boldsymbol{R}^n)\Big\|$$

is an equivalent quasi-norm in $F^s_{pq}(\boldsymbol{R}^n)$. If $l = 0$ then this assertion is covered by [Tri2, 2.5.10] and the corresponding assertions in [Tri3] (see also (2.34)). The approach in [Tri3] is based on

$$F\Delta^m_h f(\cdot) = (e^{i\xi h}-1)^m Ff.$$

On the other hand,

$$F\Delta^m_h f(\cdot+lh) = e^{il\xi h}(e^{i\xi h}-1)^m Ff.$$

But the multiplication by $e^{il\xi h}$ has no influence on the considerations in [Tri3]. This shows that (4.16) is an equivalent quasi-norm in $F^s_{pq}(\boldsymbol{R}^n)$ not only for $l = 0$ but for arbitrary real l. Then it follows from (4.15) that the expression in (A.4) can be estimated from above by $c\|f \mid F^s_{pq}(\boldsymbol{R}^n)\|$.

4.1.6. Let A be the expression in (A.4). Then it follows from (4.16), (4.14) with $y = x+lh$ and (4.13) that

$$\|f \mid F^s_{pq}(\boldsymbol{R}^n)\| \leqslant A + \eta\, \|f \mid F^s_{pq}(\boldsymbol{R}^n)\|,$$

where η is at our disposal. This proves the reverse inequality.

4.2. Proof of Theorem 2.6.2. We reduce the proof of Theorem 2.6.2 to Theorem A.1.

4.2.1. We prove part (i) in several steps and begin with a preparation. Let $\Phi = \Phi(Y)$ be one of the functions Φ_j from (2.21). Similarly, let $F(Y) = f\circ H^{-1}_{x^j}(Y)$ for the same j and $Y \in V_c$. In order to calculate $\Delta^m_h[g]\,\varphi_j f(y)$ from (2.35) we need

$$(4.17)\quad \varphi_j f\Big(y+\frac{j}{m}\frac{h}{g(y)}\Big) = \varphi_j f\Big(x^j+\frac{Y}{g(x^j)}+\frac{j}{m}\frac{h}{g(x^j)}+\frac{j}{m}\frac{\varepsilon(Y,h)}{g(x^j)}\Big)$$
$$= \Phi F\Big(Y+\frac{j}{m}h+\frac{j}{m}\varepsilon(Y,h)\Big),$$

where

$$(4.18)\quad \varepsilon(Y,h) = \Big(\frac{g(x^j)}{g(y)}-1\Big)h \quad \text{with } Y = g(x^j)(y-x^j).$$

In particular,

$$(4.19)\quad \Delta^m_h[g]\,\varphi_j f(y) = \Delta^m_{h+\varepsilon(Y,h)}\, \Phi F(Y).$$

By (2.7) and (2.8) we have

$$(4.20)\quad |\varepsilon(Y,h)| \leqslant \frac{|h|}{g(y)}|g(x^j)-g(y)| \leqslant d\,|y-x^j|\, g(x^j)\,|h|$$

for some number d independent of j. We may assume, without restriction of generality, that

$$(4.21)\quad \operatorname{supp}\Phi \subset V_{\mu c} \quad (\text{independent of } j)$$

where c is the same as in (2.6) and $1 > \mu > 0$ is at our disposal. This follows from a small modification of the resolution of unity described in Lemma 2.2.4. Hence we have by (4.21) and (2.15)

$$(4.22)\quad |\varepsilon(Y,h)| \leqslant \delta\,|h|, \quad Y \in \operatorname{supp}\Phi,$$

where $\delta > 0$ is at our disposal (in dependence on μ). Furthermore, we obtain from (4.18) and (2.8)

$$(4.23)\quad \Big|\frac{\partial\varepsilon(Y,h)}{\partial Y_k}\Big| = |h|\,\Big|g^{-2}(y)\frac{\partial g}{\partial y_k}\cdot\frac{g(x^j)}{g(x^j)}\Big| \leqslant d\,|h|$$

for some $d > 0$. In other words, if both $\mu > 0$ and b with $|h| \leqslant b$ are sufficiently small then (A.2) is satisfied. Hence we can apply Theorem A.1.

4.2.2. By (2.18), (A.4) and the previous step we have

$$(4.24)\quad \|f \mid F^s_{pq}(\Omega,g)\|^p \sim \sum_{j=1}^{\infty} \|\varphi_j f\circ H^{-1}_{x^j} \mid L_p(\boldsymbol{R}^n)\|^p$$
$$+ \sum_{j=1}^{\infty} \Big\|\Big(\int_{|h|\leqslant b} |h|^{-sq}|\Delta^m_{h+\varepsilon(\cdot,h)}\, \varphi_j f\circ H^{-1}_{x^j}(\cdot)|^q\, dh/|h|^n\Big)^{1/q} \,\Big|\, L_p(\boldsymbol{R}^n)\Big\|^p$$

(with the usual modification if $q = \infty$ or $p = \infty$) with $m > s$. By (4.19) this is near to the desired (2.36). But we must get rid of the φ_j's. We assume temporarily that Φ and F are as in step 4.2.1. We use the formula

$$(4.25)\quad \Delta^m_h(\Phi F)(Y) = \sum_{l=0}^{m} c_{l,m}\,\Delta^l_h F(Y)\,\Delta^{m-l}_h \Phi(Y+lh)$$

which can be proved by induction on m. First we assume additionally $m \geqslant 2s$, i.e. $m-[s] = s+\varkappa$ with $\varkappa \geqslant 0$. Then (4.25) yields

$$|\Delta_h^m(\Phi F)(Y)| \leqslant c_1 |h|^{s+\varkappa} \sup_{|Z-Y|\leqslant c_2} |F(Z)| + c_3 \sum_{l=[s]+1}^{m} |\Delta_h^l F(Y)|\,|h|^{m-l}. \tag{4.26}$$

We replace h in (4.25) and (4.26) by $h+\varepsilon(Y, h)$ and identify ΦF with $\varphi_j f \circ H_{x^j}^{-1}$, where we now write Φ_j instead of Φ and F_j instead of F. Then we have by (4.24)

$$\begin{aligned}\|f \mid F_{pq}^s(\Omega, g)\|^p \leqslant{} & C \sum_{j=1}^{\infty} \|\Phi_j F_j \mid L_p(\boldsymbol{R}^n)\|^p \\ & + C \sum_{j=1}^{\infty} \|\sup_{|Z-\cdot|\leqslant c_2} |\Phi_j(Z) F_j(Z)| \mid L_p(\boldsymbol{R}^n)\|^p \\ & + C \sum_{l=[s]+1}^{m-1} \sum_{j=1}^{\infty} \Big\|\Big(\int_{|h|\leqslant b} |h|^{-(s-m+l)q} \\ & \qquad \times |\Delta_{h+\varepsilon(\cdot,h)}^{l} \Phi_j F_j(\cdot)|^q\, dh/|h|^n\Big)^{1/q} \Big| L_p(\boldsymbol{R}^n)\Big\|^p \\ & + C \sum_{j=1}^{\infty} \Big\|\Big(\int_{|h|\leqslant b} |h|^{-sq} |\Delta_{h+\varepsilon(\cdot,h)}^{m} F_j(\cdot)|^q\, dh/|h|^n\Big)^{1/q} \Big| L_p(V_{\mu c})\Big\|^p,\end{aligned} \tag{4.27}$$

where we may assume that $V_{\mu c}$ is the same as in (2.15) and (4.21). Let

$$s > \sigma > n \max(1/p, 1/q). \tag{4.28}$$

Then we have

$$\begin{aligned}\|\sup_{|Z-\cdot|\leqslant c_2} |\Phi_j F_j(Z)| \mid L_p(\boldsymbol{R}^n)\| & \leqslant c\, \|\Phi_j F_j \mid F_{pq}^{\sigma}(\boldsymbol{R}^n)\| \\ & \leqslant \lambda \|\Phi_j F_j \mid F_{pq}^{s}(\boldsymbol{R}^n)\| + c_\lambda \|\Phi_j F_j \mid L_p(\boldsymbol{R}^n)\|,\end{aligned} \tag{4.29}$$

where $\lambda > 0$ is at our disposal. The first inequality comes from [Tri2, 2.5.9, end of the proof of Corollary 1]. The second is a known multiplicative inequality which can also be derived easily from (4.16). By the same argument the third terms in (4.27) can also be estimated from above by the right-hand side of (4.29). As for the last terms in (4.27) we use (4.19) with F_j instead of $\Phi F = \Phi_j F_j$. Now (4.27) yields

$$\begin{aligned}\|f \mid F_{pq}^s(\Omega, g)\|^p \leqslant{} & \tfrac{1}{2} \|f \mid F_{pq}^s(\Omega, g)\|^p + c \sum_{j=1}^{\infty} \|\Phi_j F_j \mid L_p(\boldsymbol{R}^n)\|^p \\ & + c \Big\|\Big(\int_{|h|\leqslant b} |h|^{-sq} |\Delta_h^m[g] f(\cdot)|^q\, dh/|h|^n\Big)^{1/q} \Big| L_p(\Omega, g)\Big\|^p.\end{aligned} \tag{4.30}$$

This proves that $\|f \mid F_{pq}^s(\Omega, g)\|$ can be estimated from above by the quasi-norm in (2.36).

4.2.3. We have to remove the restriction $m \geqslant 2s$ and assume now $m > s$. The critical point is (4.26). The l-summands with $m-l \geqslant s$ and $l = m$ cause no trouble and can be treated as above. As for the remaining summands of interest we have

$$m > l > s-m+l > 0, \tag{4.31}$$

where now terms $\Delta_h^l F(Y)$ with $l \leqslant s$ cannot be excluded. In other words, we have to modify the considerations for the third terms on the right-hand side of (4.27). Recall the formula

$$\Delta_h^l F(Y) = 2^{-l} \Delta_{2h}^l F(Y) + \sum a_k \Delta_h^{l+1} F(Y+kh), \tag{4.32}$$

where $\sum$ is a finite sum and the a_k's are some constants ([Tri2, 2.5.9, formula (44)]). We apply (4.32) to the third terms on the right-hand side of (4.27) with $h+\varepsilon(Y, h)$ instead of h and $\Phi_j F_j$ instead of F. Then we have

$$\begin{aligned}&\Big\|\Big(\int_{|h|\leqslant b} |h|^{-(s-m+l)q} |\Delta_{h+\varepsilon(\cdot,h)}^{l} \Phi_j F_j(\cdot)|^q\, dh/|h|^n\Big)^{1/q} \Big| L_p(\boldsymbol{R}^n)\Big\| \\ &\quad \leqslant 2^{-l} 2^{s-m+l} \Big\|\Big(\int_{|h|\leqslant 2b} |h|^{-(s-m+l)q} \\ &\qquad \times |\Delta_{h+\varepsilon(\cdot,h)}^{l} \Phi_j F_j(\cdot)|^q\, dh/|h|^n\Big)^{1/q} \Big| L_p(\boldsymbol{R}^n)\Big\| + \ldots,\end{aligned} \tag{4.33}$$

where $+\ldots$ indicates the terms with $\Delta_{h+\varepsilon(Y,h)}^{l+1} \Phi_j F_j(Y+kh+k\varepsilon)$ (see (4.32)). Of course, $2^{-l} 2^{s-m+l} < 1$. Furthermore, we split $|h| \leqslant 2b$ in the first term on the right-hand side of (4.33) into $|h| \leqslant b$ and $b \leqslant |h| \leqslant 2b$. The latter part can be estimated as in the second term on the right-hand side of (4.27). Altogether, we can estimate the left-hand side of (4.33) by the terms $+\ldots$ and the just-mentioned additional term. We repeat this procedure and arrive at terms of the type $\Delta_{h+\varepsilon(Y,h)}^{m} \Phi_j F_j(Y+kh+k\varepsilon)$. However, these terms can be expressed as

$$\Delta_{k\varepsilon} \Delta_{h+\varepsilon}^{m} \Phi_j F_j(Y+kh) + \Delta_{h+\varepsilon}^{m} \Phi_j F_j(Y+kh). \tag{4.34}$$

For the terms resulting from the first term in (4.34) we have (4.13), (4.10), and those resulting from the second term in (4.34) are covered by (A.4) with $s-\varkappa$ for some $\varkappa > 0$ instead of s. This shows that also in this case the third terms in (4.27) can be estimated from above by the right-hand side of (4.29), where λ is at our disposal. The rest is covered by the previous step. Hence $\|f \mid F_{pq}^s(\Omega, g)\|$ can be estimated from above by the quasi-norm in (2.36).

4.2.4. The reverse direction is not so complicated. Let $\{\varphi_j\}$ be the resolution of unity from Lemma 2.2.4. Then the pth power of the quasi-norm in (2.36) can be estimated from above by (a constant times)

$$\begin{aligned}&\sum_{j=1}^{\infty} \|\varphi_j f \mid L_p(\Omega, g)\|^p \\ &\quad + \sum_{j=1}^{\infty} \Big\|\Big(\int_{|h|\leqslant b} |h|^{-sq} |\Delta_h^m[g]\, \varphi_j f(\cdot)|^q\, dh/|h|^n\Big)^{1/q} \Big| L_p(\Omega, g)\Big\|^p.\end{aligned} \tag{4.35}$$

We use (4.19) and (A.4). Then it follows that (4.35) is equivalent to $\|f\,|\,F^s_{pq}(\Omega, g)\|^p$. But this is just the desired expression. The proof of part (i) is complete.

4.2.5. We prove part (ii) and reduce (2.37) to (A.5). How to reduce the spaces $B^s_{pq}(\Omega, g)$ from (2.19) to the spaces $B^s_{pq}(\boldsymbol{R}^n)$ via the resolution of unity $\{\varphi_j\}$ from Lemma 2.2.4 has been treated in [Tri4, 4.7] even in a more general case. It turns out that one can use similar arguments to those in the case of the spaces $F^s_{pq}(\Omega, g)$, where now (A.4) must be replaced by (A.5). We omit the details, but one point should be mentioned explicitly which is different. Now s is restricted by $s > n(1/p-1)_+$ and the counterpart of (4.28) is for small admissible s simply not true. However, in the case of the spaces $B^s_{pq}(\boldsymbol{R}^n)$ it is sufficient to estimate (4.25) by

$$|\Delta^m_h(\Phi F)(Y)| \leqslant c_1 |h|^{s+\varkappa} \sum F(Y+kh) + c_2 \sum_{l=[s]+1}^{m} |\Delta^l_h F(Y)|\,|h|^{m-l}, \tag{4.36}$$

where $\sum$ is a finite sum. In the case of the spaces $B^s_{pq}(\boldsymbol{R}^n)$ one has first the integration $\|\Phi F(\cdot+kh)\,|\,L_p(\boldsymbol{R}^n)\|$, where kh does not play any role. But in the above arguments one has to deal with the more complicated situation

$$\big\|\Phi F(\cdot+kh+k\varepsilon(\cdot, h))\,\big|\,L_p(\boldsymbol{R}^n)\big\|. \tag{4.37}$$

It turns out that this quasi-norm is equivalent to $\|\Phi F\,|\,L_p(\boldsymbol{R}^n)\|$. Here one has to use the assumption about $\partial\varepsilon/\partial x_j$ from (A.2). We discussed this point in detail in [GT]. With these modifications in mind and based on the techniques from [Tri4, 4.7] one can carry over the above arguments for the spaces F^s_{pq} to the spaces B^s_{pq}. As a result we obtain part (ii).

4.3. Proof of Theorem 3.3.1.

4.3.1. We prove part (i). Let

$$Af(x) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha f(x), \qquad x \in \Omega, \tag{4.38}$$

be the differential operator from Definition 3.1.1. Then A can be represented by (3.5) with

$$A(x, y) = \sum_{|\alpha| \leqslant m} b_\alpha(x) D^\alpha_y \delta_x(y), \qquad (x, y) \in \Omega \times \Omega, \tag{4.39}$$

where the $b_\alpha(x)$'s are appropriate C^∞ coefficients and $\delta_x(y)$ is the δ-distribution in $\boldsymbol{R}^n$ equipped with y-coordinates, with x as the singular point. In particular, $A_{jk}(x, y) = 0$ if $d_{jk} \geqslant 1$, and (3.16) is satisfied for every L. Let A'_{jk} be given by (3.10) and (3.12). In order to check property (ii) from Definition 3.2.5 we may assume $j = k$. Then

$$A'_{jj}u(X) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\varphi_j(x)(D^\alpha_x \varphi_j f)(x), \qquad X = g(x^j)(x-x^j), \tag{4.40}$$

with $f(x) = u(X)$ (see (2.15)). Then we have

$$A'_{jj}u(X) = \sum_{|\alpha| \leqslant m} A^j_\alpha(X) D^\alpha_X u(X), \tag{4.41}$$

where the coefficients $A^j_\alpha(X)$ are linear combinations of

$$a_\beta(Xg^{-1}(x^j)+x^j)\,g^{|\beta|}(x^j) \tag{4.42}$$

multiplied with smooth functions supported near the origin. By (3.2), $\{A^j_\alpha(X)\}$ is a bounded set in $C^\infty(\boldsymbol{R}^n)$. Hence part (ii) of Definition 3.2.5 with $\mu = m$ and $\delta = 0$ is satisfied (see also (3.19), (3.20)).

4.3.2. We prove part (ii). By Theorem 2.3.1 we can apply the corresponding assertions from [CGT]. Now let $A = (\varrho^2 I - \Delta)^{\mu/2}$ and let A_{jk} and A'_{jk} be the same operators as in 3.2.4. Let $d_{jk} \leqslant 1$. Then we can apply Theorem 3.3 in [CGT] with $f(\lambda) = (\varrho^2+\lambda^2)^{\mu/2}$ (in the notation used there). It follows that $\{A'_{jk}$ with $d_{jk} \leqslant 1\}$ is a bounded set of pseudodifferential operators in $\Psi^\mu_0(\boldsymbol{R}^n)$ (see (3.22) and part (ii) of Definition 3.2.5). Let $d_{jk} \geqslant 1$. We wish to prove (3.16) with $L = \varrho$. Let again $f(z) = (\varrho^2+z^2)^{\mu/2}$. Then (3.16) with $|\alpha| = |\beta| = 0$ follows from [CGT, Definition 3.1, (3.44) and (3.45)]. In order to extend this assertion to the derivatives of $A_{jk}(x, y)$ one has to replace $f(z)$ above by $f(z) = (1+z^k)(\varrho^2+z^2)^{\mu/2}$ with $k = 1, 2, \ldots$ Then it follows again from [CGT] and the symmetry of $A_{jk}(x, y)$ with respect to x and y that (3.16) holds for all α and β. This completes the proof of (3.22).

4.3.3. Remark. The restriction $\varrho > G$ in Theorem 3.3.1(ii) comes from (3.15) and has nothing to do with the results from [CGT] which we used and which work for all $\varrho > 0$.

4.4. Proof of Theorem 3.3.3.

4.4.1. We use the decomposition (3.11), i.e.

$$A = \sum_{j,k=1}^{\infty} A_{jk}, \qquad B = \sum_{r,s=1}^{\infty} B_{rs}. \tag{4.43}$$

Then we have

$$C = A \circ B = \sum_{j,k,r,s} A_{jk} \circ B_{rs} = \sum_{j,s=1}^{\infty} C_{js} \tag{4.44}$$

with

$$C_{js} = \sum_{k=1}^{\infty} A_{jk} \circ B_{ks} + \ldots \tag{4.45}$$

where $+\dots$ indicates similar terms with $\operatorname{supp}\varphi_k \cap \operatorname{supp}\varphi_l \neq \emptyset$. It is clear that we may assume that the C_{js}'s are the pseudodifferential operators in the sense of the decomposition (3.10) with C instead of A. Furthermore, we may restrict our attention to the terms explicitly written down in (4.45). We have, at least formally,

$$(4.46)\qquad C_{js}f(x) = \int_\Omega \sum_{k=1}^{\infty} A_{jk}(x, y) \int_\Omega B_{kj}(y, z) f(z) g^n(z) g^n(y)\, dz\, dy = \int_\Omega C_{js}(x, z) f(z) g^n(z)\, dz$$

with

$$(4.47)\qquad C_{js}(x, z) = \sum_{k=1}^{\infty} \int_\Omega A_{jk}(x, y) B_{ks}(y, z) g^n(y)\, dy.$$

In the normalized form, based on (3.12) and (2.15) we have

$$(4.48)\qquad C'_{js}(X, Z) = \sum_{k=1}^{\infty} \int_{V_c} A'_{jk}(X, Y) B'_{ks}(Y, Z)\, dY.$$

If both $d_{jk} \geqslant 1$ and $d_{ks} \geqslant 1$ then the corresponding integrals make sense in the usual way. Otherwise the above calculations must be understood as compositions of the corresponding pseudodifferential operators.

4.4.2. Let $d_{js} \geqslant 1$. Then we split $\sum_{k=1}^{\infty}$ in (4.48) as

$$(4.49)\qquad C'_{js}(X, Z) = \sum_k{}' \dots + \sum_k{}'' \dots$$

where $\sum_k'$ collects all k with $\min(d_{jk}, d_{ks}) \geqslant 1$ and $\sum_k''$ the remaining k's. Then $\sum_k''$ is a finite sum (the number of terms can be estimated from above independently of the parameters) and the typical term is

$$(4.50)\qquad \int_{V_c} A'_{jj}(X, Y) B'_{js}(Y, Z)\, dY.$$

However, first we estimate $\sum_k'$. By (3.17) we have

$$(4.51)\qquad \Big|\sum_k{}'\Big| \leqslant c \sum_{k=1}^{\infty} e^{-L_A d_{jk} - L_B d_{ks}}$$

where c stands for an appropriate constant not necessarily the same as in (4.48) or (4.50). Let $L_A d_{jk} \geqslant L_B d_{ks}$. Then

$$(4.52)\qquad L_C d_{js} \leqslant L_C(d_{jk} + d_{ks} + c) \leqslant L_C L_A d_{jk}\left(\frac{1}{L_A} + \frac{1}{L_B}\right) + c' = L_A d_{jk} + c'.$$

We have a similar estimate if $L_B d_{ks} \geqslant L_A d_{jk}$. Consequently,

$$(4.53)\qquad \Big|\sum_k{}'\Big| \leqslant c e^{-L_C d_{js}} \sum_{k=1}^{\infty} \left(e^{-L_B d_{ks}} + e^{-L_A d_{jk}}\right).$$

Let $B_{x^s}(r)$ be the ball centered at x^s with geodesic radius r (see (3.7)). Then (3.8) and (3.9) yield

$$(4.54)\qquad \sum_{k=1}^{\infty} e^{-L_B d_{ks}} \leqslant c \sum_{l=1}^{\infty} e^{-L_B l} \operatorname{vol} B_{x^s}(l) \leqslant c' \sum_{l=1}^{\infty} e^{-L_B l + (G+\varepsilon) l}$$

for any $\varepsilon > 0$. Because $L_B > G$ this sum converges. Similarly for the second sum in (4.53). Then we have the desired estimate

$$(4.55)\qquad \Big|\sum_k{}'\Big| \leqslant c e^{-L_C d_{js}}.$$

In order to estimate $\sum_k''$ we must handle compositions of the type $A'_{jj} \circ B'_{js}$ with $d_{js} \geqslant 1$ (see (4.50)). By (3.17),

$$(4.56)\qquad \{e^{L_B d_{js}} B'_{js}\}_{j,s} \text{ is a bounded set in } \Psi^{-\infty}(\boldsymbol{R}^n).$$

Furthermore, $\{A'_{jj}\}$ is a bounded set in $\Psi^{\mu_A}_{\delta_A}(\boldsymbol{R}^n)$ (see part (ii) of Definition 3.2.5). By [Ta, II, Theorem 4.4],

$$(4.57)\qquad \{e^{L_B d_{js}} A'_{jj} \circ B'_{js}\}_{j,s} \text{ is a bounded set in } \Psi^{-\infty}(\boldsymbol{R}^n).$$

The corresponding Schwartz kernels are C^∞ functions ([Tre, I, Corollary 2.3]). Because $L_C \leqslant L_B$ we have (4.55) with $\sum_k''$ instead of $\sum_k'$. Altogether, we have

$$(4.58)\qquad |C'_{js}(X, Z)| \leqslant c e^{-L_C d_{js}}.$$

Let α and β be arbitrary multi-indices. Then $D_X^\alpha D_Z^\beta C'_{js}(X, Z)$ is given by (4.48) with the kernel $D_X^\alpha A'_{jk}(X, Y) D_Z^\beta B'_{ks}(Y, Z)$. On the other hand, we have (3.17) and we can repeat the above arguments without any changes. We obtain

$$(4.59)\qquad |D_X^\alpha D_Z^\beta C'_{js}(X, Z)| \leqslant c_{\alpha\beta} e^{-L_C d_{js}}.$$

Retransformation yields the counterpart of (3.16).

4.4.3. Let $d_{js} \leqslant 1$. The typical case is $j = s$. We again use the splitting (4.49), now with $j = s$. The terms with $d_{jk} \geqslant 1$ can be treated in the same way as above. The typical remaining term is $A'_{jj} \circ B'_{jj}$. However, by part (ii) of Definition 3.2.5 and the composition formula for pseudodifferential operators ([Ta, II, Theorem 4.4]),

$$\{A'_{jj} \circ B'_{jj}\}_j \text{ is a bounded set in } \Psi^{\mu_A + \mu_B}_{\max(\delta_A, \delta_B)}(\boldsymbol{R}^n).$$

The proof is complete.

4.4.4. Remark. We used essentially $L_A > G$ and $L_B > G$ (see (4.54)), but not $L_C > G$.

4.5. Proof of Theorem 3.3.5.

4.5.1. The proof of part (ii) follows immediately from part (i) and the interpolation property (see (2.19)). Here we shift the case $p = q = \infty$ from (ii) to (i) (see Proposition 2.4.4(iii)).

4.5.2. We prove part (i) including the case $p = q = \infty$. Let A be the pseudodifferential operator from (3.26). By (2.18) we have

$$\|Af \mid F^s_{pq}(\Omega, g)\|^p = \sum_{j=1}^{\infty} \|(\varphi_j Af) \circ H^{-1}_{xj} \mid F^s_{pq}(R^n)\|^p \tag{4.60}$$

(with the usual modification if $p = q = \infty$). We decompose A by (3.11) and (3.12). Let $\min(p, q) > 1$. Then we have

$$\|\varphi_j Af \circ H^{-1}_{xj} \mid F^s_{pq}(R^n)\| \leqslant c \sum_k \|A'_{jk}(\varphi_k f \circ H^{-1}_{xk}) \mid F^s_{pq}(R^n)\| + \ldots, \tag{4.61}$$

where $+\ldots$ indicates similar terms with neighbouring φ_l's (with respect to φ_j and φ_k) which can be treated in the same way. Now we use

$$\|A'_{jk} U \mid F^s_{pq}(R^n)\| \leqslant c e^{-d_{jk}L} \|U \mid F^{s+\mu}_{pq}(R^n)\| \tag{4.62}$$

(see [Pä] or [Tri7]), where c is independent of j, k and L. The estimate for the norm of the operator comes from (3.17). We put (4.62) in (4.61) and obtain

$$\begin{aligned} &\|\varphi_j Af \circ H^{-1}_{xj} \mid F^s_{pq}(R^n)\| \\ &\quad \leqslant c \Big(\sum_k \|\varphi_k f \circ H^{-1}_{xk} \mid F^{s+\mu}_{pq}(R^n)\|^p e^{-d_{jk}L}\Big)^{1/p} \Big(\sum_k e^{-d_{jk}L}\Big)^{1/p'} + \ldots \end{aligned} \tag{4.63}$$

with $1/p + 1/p' = 1$. Because $L > G$ the last sum converges (see (4.54)). We take the pth power of (4.63) and sum over j. Then (4.60) yields the desired estimate

$$\|Af \mid F^s_{pq}(\Omega, g)\| \leqslant c \|f \mid F^{s+\mu}_{pq}(\Omega, g)\|. \tag{4.64}$$

Let $\varrho = \min(p, q) \leqslant 1$. Then we replace (4.61) by

$$\|\varphi_j Af \circ H^{-1}_{xj} \mid F^s_{pq}(R^n)\|^{\varrho} \leqslant c \sum_k \|A'_{jk}(\varphi_k f \circ H^{-1}_{xk}) \mid F^s_{pq}(R^n)\|^{\varrho} + \ldots \tag{4.65}$$

(see (2.3)). Let $\varrho = p$. Then it follows from (4.65) and (4.62) that

$$\|\varphi_j Af \circ H^{-1}_{xj} \mid F^s_{pq}(R^n)\|^p \leqslant c \sum_k \|\varphi_k f \circ H^{-1}_{xk} \mid F^{s+\mu}_{pq}(R^n)\|^p e^{-pd_{jk}L} + \ldots \tag{4.66}$$

We have $pL > G$. Hence summation of (4.66) over j yields (4.64) by the same argument as above. Finally, let $\varrho = q$. Then (4.65) and (4.64) yield

$$\begin{aligned} &\|\varphi_j Af \circ H^{-1}_{xj} \mid F^s_{pq}(R^n)\|^q \\ &\quad \leqslant c \sum_k \|\varphi_k f \circ H^{-1}_{xk} \mid F^{s+\mu}_{pq}(R^n)\|^q e^{-qLd_{jk}} + \ldots \\ &\quad \leqslant c \Big(\sum_k \|\varphi_k f \circ H^{-1}_{xk} \mid F^{s+\mu}_{pq}(R^n)\|^p e^{-qLd_{jk}}\Big)^{q/p} \Big(\sum_k e^{-qLd_{jk}}\Big)^{1-q/p} + \ldots \end{aligned} \tag{4.67}$$

We have $qL > G$, hence the last sum converges. We take the (p/q)th power of (4.67) and sum over j. Then we obtain (4.64).

4.6. Proof of Theorem 3.4.3.

4.6.1. First we decompose A into

$$A = A_1 + A_2 = \sum_{j,k}{}' A_{jk} + \sum_{j,k}{}'' A_{jk} \tag{4.68}$$

where $\sum'_{j,k}$ is the sum over all j and k with $d_{jk} \leqslant 1$, and $\sum''_{j,k}$ collects the remaining A_{jk}'s (see (3.11)). Then A_1 is properly supported and A_2 is an operator with C^∞ kernel (see also [Hö2, Proposition 18.1.22]). Let $P \in \Psi^{-\mu,\infty}_{\delta}(\Omega, g)$. Then we claim that

$$A_2 \circ P \in \Psi^{-\infty,L}(\Omega, g), \qquad P \circ A_2 \in \Psi^{-\infty,L}(\Omega, g). \tag{4.69}$$

We use (4.44) and (4.45) with A_2 and P instead of A and B. In our case the sum in (4.45) reduces essentially to $A_{js} \circ P_{ss}$ where $d_{js} \geqslant 1$ (and similar terms). The rest is covered by step 4.4.2 (see in particular (4.56)).

4.6.2. By the previous step we have to find an operator $P \in \Psi^{-\mu,\infty}_{\delta}(\Omega, g)$ with

$$A_1 \circ P - I \in \Psi^{-\infty,L}(\Omega, g), \qquad P \circ A_1 - I \in \Psi^{-\infty,L}(\Omega, g).$$

However, locally (and uniformly with respect to j and k) such an assertion is covered by [Ta, III, Theorem 1.3]. How to glue together these local parametrices may be found in [Tre, I, 5, Appendix]. The proof is complete.

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Extensions and selections of maps with decomposable values

by

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Abstract. Let X be a separable metric space and E a Banach space. Let μ be a nonatomic probability measure on a measurable space T, and let $L^1 = L^1(T; E)$ be the Banach space of μ-integrable functions u: $T \to E$. A subset K of L^1 is *decomposable* if, for any μ-measurable set $A \subseteq T$ and all $u, v \in K$, one has $u \cdot \chi_A + v \cdot \chi_{T\setminus A} \in K$. Using the property of decomposability as a substitute for convexity, the analogues of three theorems by Dugundji, Cellina and Michael are proved.

1) A continuous map f from a closed set $Y \subseteq X$ into a decomposable subset K of L^1 can be continuously extended to a map $\tilde{f}$: $X \to K$.

2) An upper semicontinuous multivalued map F: $X \to 2^{L^1}$ with decomposable values has a continuous ε-approximate selection, for any $\varepsilon > 0$.

3) A lower semicontinuous multifunction G: $X \to 2^{L^1}$ with closed decomposable values admits a continuous selection.

The compactness assumption on X, which appears in previous papers, is here never used. From 1) it follows that, if $L^1(T; E)$ is separable, then any closed decomposable subset $K \subseteq L^1$ is a retract of the whole space, hence it has the compact fixed point property.

1. Introduction. Consider a measure space $(T, \mathscr{F}, \mu)$, where $\mathscr{F}$ is a σ-algebra of subsets of T and μ is a nonatomic probability measure on $\mathscr{F}$. If E is a Banach space, let $L^1(T; E)$ be the Banach space of all functions u: $T \to E$ which are Bochner μ-integrable [17]. According to [10], a subset $K \subseteq L^1(T; E)$ is *decomposable* if, for every measurable set $A \in \mathscr{F}$,

$$u \cdot \chi_A + v \cdot \chi_{T\setminus A} \in K \qquad \forall u, v \in K. \tag{1.1}$$

In several cases, the property of decomposability is a good substitute for convexity [15]. Three classical theorems, which make use of a convexity assumption, will be considered here.

THEOREM I (Dugundji [6, p. 188]). *Let A be a closed subset of a metric space X and let K be a convex subset of a Banach space Z. Then every continuous map f: $A \to K$ has a continuous extension $\tilde{f}$: $X \to K$.*

THEOREM II (Cellina [2, p. 84]). *Let X be a metric space and Z a Banach space. Let F: $X \to 2^Z$ be an upper semicontinuous map with convex values. Then, for every $\varepsilon > 0$, F admits a continuous ε-approximate selection,*