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An isomorphic Banach–Stone theorem

by

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Abstract. For a given Banach space Y let $\lambda_0(Y)$ denote the infimum of the Banach–Mazur distances between the two-dimensional subspaces of Y and the two-dimensional L^1-space l_2^1.

It is shown that every Banach space X such that $\lambda_0(X')$ is greater than one satisfies the following isomorphic version of the classical Banach–Stone theorem: there is a $\delta > 0$ such that two locally compact Hausdorff spaces K and L are necessarily homeomorphic provided that there is an isomorphism T between $C_0(K, X)$ and $C_0(L, X)$ with $\|T\|\|T^{-1}\| \leqslant 1+\delta$.

This result properly includes all isomorphic Banach–Stone theorems now existing in the literature. It is obtained by means of the description of small-bound isomorphisms between certain L^1-direct sums of Banach spaces.

1. Introduction. For a locally compact Hausdorff space K and a Banach space X we denote by $C_0(K, X)$ the space of X-valued continuous functions on K which vanish at infinity, provided with the supremum norm. If X is the scalar field $\boldsymbol{K}$ (where $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{K} = \boldsymbol{C}$) this space is denoted by $C_0 K$.

The classical Banach–Stone theorem states that if $C_0 K$ and $C_0 L$ are isometrically isomorphic, then K and L are homeomorphic. Various authors, beginning with M. Jerison [10], have considered the problem of determining geometric properties of X which allow generalizations of this theorem to spaces of continuous vector-valued functions $C_0(K, X)$. A compilation of results of this nature may be found in [3].

A second kind of generalization deals with scalar-valued functions, but replaces isometries by isomorphisms T with $\|T\|\|T^{-1}\|$ close to one [1], [5], [6].

These two directions have been combined in [7] and [9] where it is shown that a small-bound theorem is obtainable for spaces of vector-valued functions $C_0(K, X)$ for certain X.

Here we prove a theorem which strictly contains the results in [7] and [9]. To state this theorem we need the following definition: for any Banach space Y we denote by $\lambda_0(Y)$ the number

$$\lambda_0(Y) := \inf d(Y_2, l_2^1),$$

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where Y_2 runs through all two-dimensional subspaces of Y and l_2^1 denotes the two-dimensional L^1-space (in the case $\dim Y = 1$ we set $\lambda_0(Y) := 2$); here $d(V, W)$ means, for Banach spaces V and W, the *Banach–Mazur distance* between V and W, i.e.

$$d(V, W) := \inf\{\|T\|\,\|T^{-1}\| \mid T\colon V \to W \text{ an isomorphism}\}.$$

We will prove that for every X such that $\lambda_0(X') > 1$ there is a $\delta > 0$ such that two locally compact Hausdorff spaces K and L are homeomorphic if there exists an isomorphism T from $C_0(K, X)$ onto $C_0(L, X)$ with $\|T\|\,\|T^{-1}\| \leqslant 1+\delta$.

It is not hard to see that uniformly convex spaces X or spaces which satisfy the condition in Jarosz's paper [9] have the property that $\lambda_0(X') > 1$ so that the theorems in [7] and [9] are special cases of the present results. We note that the condition $\lambda_0(X') > 1$ coincides in the real case with the fact that X is uniformly nonsquare (see [2] for definitions). In the complex case, however, $\lambda_0(X') > 1$ does not imply that X is uniformly nonsquare as the example of the complex l_2^1 shows (this space provides also an example of a space X with $\lambda_0(X') > 1$ to which the results of [7] and [9] do not apply).

In [4] related but distinct results involving small-bound isomorphisms for spaces $C_0(K, X)$ are obtained by considering certain variants of M-structure-theoretical notions. Here our results follow from the consideration of the L-structure in dual spaces. In Section 2 we first prove a theorem concerning small-bound isomorphisms between two Banach spaces which are L^1-direct sums in which all but possibly one of the summands U have $\lambda_0(U)$ uniformly bounded away from one. In Section 3 this is then applied to obtain theorems concerning isomorphisms between spaces of continuous vector-valued functions.

2. Small-bound isomorphisms between L^1-direct sums. We begin this section with the following easily verified

2.1. LEMMA. *Given a Banach space X suppose that there exist $x_1, x_2 \in X$ and a c with $0 \leqslant c < 1$ such that for all scalars a, b with $|a|+|b| = 1$ we have*

$$1 \leqslant \|ax_1 + bx_2\| \leqslant 1+c.$$

Then $\lambda_0(X) \leqslant 1+c$.

Our principal tool in establishing the theorem of this section is the following

2.2. LEMMA. *Given Banach spaces U, V, and W, let S be a linear map from W to $U \oplus_1 V$ (the L^1-direct sum of U and V) and let E be the canonical projection of $U \oplus_1 V$ onto U. Let $e_1, e_2 \in W$ and let τ, ε_1, and ε_2 be nonnegative*

numbers such that $1-\tau-\varepsilon_1-\varepsilon_2 > 0$ and $(1+\tau)/(1-\tau-\varepsilon_1-\varepsilon_2) < 2$. Assume that:

(i) $(1+\tau)^{-1}\|w\| \leqslant \|Sw\| \leqslant (1+\tau)\|w\|$ *for all* $w \in W$.
(ii) $\|E \circ Se_i\| \geqslant 1-\varepsilon_1$ *for* $i = 1, 2$.
(iii) $(1-\varepsilon_2)(1+\tau) \leqslant \|ae_1 + be_2\| \leqslant 1$ *for all scalars a, b with* $|a|+|b| = 1$.

Then $\lambda_0(U) \leqslant (1+\tau)/(1-\tau-\varepsilon_1-\varepsilon_2)$.

Proof. We define $u_i := E \circ Se_i$ $(i = 1, 2)$, and we will show that

$$1-\tau-\varepsilon_1-\varepsilon_2 \leqslant \|au_1 + bu_2\| \leqslant 1+\tau$$

for all a, b with $|a|+|b| = 1$ so that our claim follows from Lemma 2.1.

Let such scalars a, b be given. Since E is norm-decreasing we have

$$\|au_1 + bu_2\| = \|E(aSe_1 + bSe_2)\| \leqslant \|aSe_1 + bSe_2\|$$
$$\leqslant (1+\tau)\|ae_1 + be_2\| \leqslant 1+\tau.$$

Next write, for $i = 1, 2$, $Se_i = u_i + v_i$, where $u_i \in U$, $v_i \in V$. We have $\|Se_i\| = \|u_i\| + \|v_i\| \leqslant 1+\tau$, and, by (ii), $\|u_i\| = \|E \circ Se_i\| \geqslant 1-\varepsilon_1$. We thus have $\|v_i\| \leqslant \varepsilon_1 + \tau$, and it follows that

$$\|au_1 + bu_2\| = \|aSe_1 + bSe_2 - (av_1 + bv_2)\|$$
$$\geqslant \frac{1}{1+\tau}\|ae_1 + be_2\| - \|av_1 + bv_2\|$$
$$\geqslant 1-\varepsilon_2-\varepsilon_1-\tau.$$

2.3. COROLLARY. (a) *If in the preceding lemma we replace* (iii) *by $\|ae_1 + be_2\| = 1$ (all a, b with $|a|+|b| = 1$), then we have*

$$\lambda_0(U) \leqslant (1+\tau)^2/(1-(1+\tau)(\varepsilon_1+\tau))$$

provided that this number is less than two and the denominator is positive.

(b) *If we assume that $\|ae_1 + be_2\| = 1$ for all a, b with $|a|+|b| = 1$ and replace* (ii) *by the assumption that $\|E \circ S((e_1+e_2)/2)\| \geqslant 1-\varepsilon$ then*

$$\lambda_0(U) \leqslant (1+\tau)^2/(1-(1+\tau)(2\varepsilon+2\tau))$$

provided that this expression is less than two and the denominator is positive.

Proof. (a) is an immediate consequence of 2.2 by defining ε_2 as $\varepsilon_2 := 1-1/(1+\tau)$, and for the proof of (b) one only has to note that the assumptions imply the inequalities $\|E \circ Se_i\| \geqslant 1-2\varepsilon-\tau$ $(i = 1, 2)$.

We now turn to the main theorem of this section which will be crucial for our Banach–Stone theorem: a description of small-bound isomorphisms between L^1-direct sums of spaces where all but possibly one summand are "bounded away from l_2^1" and this one summand behaves as an L^1-space over a measure space without atoms.

Throughout the remainder of this section let $(V_i)_{i\in I}$ and $(W_j)_{j\in J}$ be families of nonzero Banach spaces with

$$\lambda_0 := \inf\{\lambda_0(V_i),\ \lambda_0(W_j) \mid i\in I,\ j\in J\} > 1.$$

Further, let V_∞, W_∞ be Banach spaces such that for every x with $\|x\| = 1$ there are x_1, x_2 satisfying $x = (x_1+x_2)/2$ and $\|ax_1+bx_2\| = 1$ for all scalars a, b with $|a|+|b| = 1$. Set

$$V := V_\infty \oplus_1 \big(\bigoplus_{i\in I}^1 V_i\big) \quad (= \text{the usual } L^1\text{-direct sum}),$$

$$W := W_\infty \oplus_1 \big(\bigoplus_{j\in J}^1 W_j\big),$$

and let $\hat{E}_i$, $\hat{E}_\infty$ (resp. E_j, E_∞) be the canonical projections of V onto V_i, V_∞ (resp. of W onto W_j, W_∞). We assume that $S\colon V\to W$ is an isomorphism such that for a fixed τ with $0<\tau\leqslant 1$ we have

$$\frac{1}{1+\tau}\|v\| \leqslant \|Sv\| \leqslant (1+\tau)\|v\| \quad \text{for every } v\in V.$$

The operator S^{-1} satisfies the same norm condition so that for each property established for S there is an analogous property for S^{-1}.

Clearly $\|S\|\,\|S^{-1}\| \leqslant (1+\tau)^2$, and it will be convenient to introduce a number δ defined by

$$1+\delta = (1+\tau)^2.$$

2.4. THEOREM. *Suppose that for a fixed $n\geqslant 3$ the number δ is sufficiently small that $1-n\delta$ is positive and that $(1+\delta)/(1-n\delta) < \lambda_0$. Then:*

(i) *For every $j\in J$ there is precisely one $i\in I$ with*

$$\|E_j\circ S|_{V_i}\| \geqslant 1/(1+\tau).$$

(ii) *For every $i\in I$ there is precisely one $j\in J$ with*

$$\|\hat{E}_i\circ S^{-1}|_{W_j}\| \geqslant 1/(1+\tau).$$

(iii) *Let $\varphi\colon J\to I$ (resp. $\psi\colon I\to J$) be defined by means of assertion* (i) *(resp.* (ii)*). Then $\varphi\circ\psi = \mathrm{Id}_I$ and $\psi\circ\varphi = \mathrm{Id}_J$.*

(iv) *With $i := \varphi(j)$ we have $\|E_j\circ Sv\| \geqslant (n-4)\delta/3$ for every $v\in V_i$ with $\|v\| = 1$ provided that not only $n\geqslant 3$ but $n>7$.*

Proof. The proof of the theorem will be established by means of a sequence of propositions.

2.5. PROPOSITION. *There is an $\eta_0>0$ such that $\|E_j\circ S|_{V_\infty}\| \leqslant 1/(1+\tau) - \eta_0$ for every $j\in J$.*

Proof. Define $\eta_0 := ((2n-4)\tau+(n-2)\tau^2)/(2+2\tau)$. We will prove our claim by showing that for every j_0 and every $v\in V_\infty$ with $\|v\| = 1$ such that $\|E_{j_0}\circ Sv\| \geqslant 1/(1+\tau)-\alpha$ for some $\alpha>0$ one necessarily has $\alpha\geqslant\eta_0$.

Let such j_0, v, α be given. We define ε by $1-\varepsilon := 1/(1+\tau)-\alpha$, and we apply Corollary 2.3(b) to the map

$$S|_{V_\infty}\colon V_\infty \to W_{j_0}\oplus_1\big(W_\infty\oplus_1\big(\bigoplus_{j\neq j_0}^1 W_j\big)\big)$$

and vectors e_1, e_2 such that $v = (e_1+e_2)/2$ and $\|ae_1+be_2\| = 1$ for all a, b with $|a|+|b| = 1$ (such e_1, e_2 exist by assumption). By the corollary it follows that

$$\lambda_0(W_{j_0}) \leqslant (1+\tau)^2/(1-(1+\tau)(2\varepsilon+2\tau)),$$

and hence

$$(1+\delta)/(1-n\delta) = (1+\tau)^2/(1-n(2\tau+\tau^2)) \leqslant \lambda_0 \leqslant \lambda_0(W_{j_0})$$
$$\leqslant (1+\tau)^2/(1-(1+\tau)(2\varepsilon+2\tau)).$$

It follows that $n(2\tau+\tau^2) \leqslant (1+\tau)(2\varepsilon+2\tau)$ which implies $\alpha\geqslant\eta_0$.

2.6. PROPOSITION. $\sup_{i\in I}\|E_j\circ S|_{V_i}\| \geqslant 1/(1+\tau)$ *for every $j\in J$.*

Proof. If this were not true there would exist an $\eta>0$ such that $\|E_j\circ S|_{V_i}\| \leqslant 1/(1+\tau)-\eta$ for every $i\in I\cup\{\infty\}$. Then, for any $w\in W_j$ with $\|w\| = 1$ we would obtain

$$1 = \|E_j w\| = \|E_j\circ S\circ S^{-1}w\| = \Big\|E_j\circ S\Big(\sum_{i\in I\cup\{\infty\}} \hat{E}_i\circ S^{-1}w\Big)\Big\|$$
$$\leqslant (1/(1+\tau)-\eta)\sum_{i\in I\cup\{\infty\}} \|\hat{E}_i\circ S^{-1}w\|$$
$$= (1/(1+\tau)-\eta)\|S^{-1}w\| \leqslant 1-\eta(1+\tau),$$

which is absurd.

2.7. PROPOSITION. *For every $j\in J$ there is precisely one $i\in I$ with* $\|E_j\circ S|_{V_i}\| \geqslant 1/(1+\tau)$ (which proves (i) — and by symmetry also (ii) — of the theorem).

Proof. Up to now we know that for every j_0 and every $\eta>0$ there is an $i\in I$ such that $\|E_{j_0}\circ S|_{V_i}\| \geqslant 1/(1+\tau)-\eta$.

Let $\eta>0$ be fixed but arbitrary and suppose that our condition is satisfied for $i = i_1$ and $i = i_2$, where $i_1\neq i_2$. Choose $e_k\in V_{i_k}$ with $\|e_k\| = 1$ and $\|E_j\circ Se_k\| \geqslant 1/(1+\tau)-\eta$ $(k = 1, 2)$. Let X be the linear span of e_1 and e_2 in V, and consider

$$S|_X\colon X\to W_{j_0}\oplus_1\big(W_\infty\oplus_1\big(\bigoplus_{j\neq j_0}^1 W_j\big)\big).$$

Part (a) of Corollary 2.3 (with $\varepsilon_1 := \eta+\tau/(1+\tau)$) implies that

$$\lambda_0(W_{j_0}) \leqslant (1+\tau)^2/\big(1-(1+\tau)(\tau+\eta+\tau/(1+\tau))\big)$$

so that

$$(1+\tau)^2/(1-n(2\tau+\tau^2)) < \lambda_0 \leqslant (1+\tau)^2/\big(1-(1+\tau)(\tau+\eta+\tau/(1+\tau))\big).$$

This is not possible for small η so that there is a uniquely determined i with $\|E_j \circ S|_{V_i}\| \geqslant 1/(1+\tau)-\eta$ for *all* $\eta > 0$ and consequently $\|E_j \circ S|_{V_i}\| \geqslant 1/(1+\tau)$.

2.8. Proposition. $\psi \circ \varphi = \mathrm{Id}_J$ *and* $\varphi \circ \psi = \mathrm{Id}_I$.

Proof. By symmetry it is only necessary to prove the first part of the assertion. Let $j \in J$ be given and set $i := \varphi(j)$, $j' := \psi(i)$. We assume that $j \neq j'$, and we will show that this assumption leads to a contradiction.

Given $\eta > 0$ arbitrary but fixed take $v_i \in V_i$ with $\|v_i\| = 1$ such that $\|E_j \circ Sv_i\| \geqslant 1/(1+\tau)-\eta$ and take $w_{j'} \in W_{j'}$ with $\|w_{j'}\| = 1$ such that $\|\hat{E}_i \circ S^{-1} w_{j'}\| \geqslant 1/(1+\tau)-\eta$. Set $e_2 := w_{j'}$ and $\tilde{e}_1 := E_j \circ Sv_i$. We are going to apply Lemma 2.2 to

$$S^{-1}|_X\colon\ X \to V_1 \oplus_1 \big(V_\infty \oplus_1 \bigoplus_{i'\neq i}^1 V_{i'}\big),$$

where $X := \operatorname{lin}\{e_1, e_2\}$ with $e_1 := \tilde{e}_1/(1+\tau)$.

(i) of Lemma 2.2 is satisfied by assumption and (ii) is true with ε_1 defined by $1-\varepsilon_1 = (1-\delta)/(1+\tau)-\eta$ (clearly $\|\hat{E}_i \circ S^{-1} e_2\| \geqslant 1-\varepsilon_1$ by construction, and to prove the corresponding inequality for e_1 one first has to note that $Sv_i = \tilde{e}_1 + r$ for an r with $\|r\| \leqslant 1+\tau-1/(1+\tau)+\eta = \eta+\delta/(1+\tau)$ so that

$$1 = \|v_i\| = \|\hat{E}_i \circ S^{-1} \circ Sv_i\| \leqslant \|\hat{E}_i \circ S^{-1} \tilde{e}_1\| + \|\hat{E}_i \circ S^{-1} r\|$$
$$\leqslant \|\hat{E}_i \circ S^{-1} \tilde{e}_1\| + \eta(1+\tau)+\delta$$

which gives $\|\hat{E}_i \circ S^{-1} e_1\| \geqslant 1-\varepsilon_1$).

Finally, (iii) is valid with ε_2 defined by $(1-\varepsilon_2)(1+\tau) = 1/(1+\delta)-\eta/(1+\tau)$, and we conclude that

$$(1+\delta)/(1-n\delta) < \lambda_0(V_i) \leqslant (1+\tau)/(1-\tau-\varepsilon_1-\varepsilon_2).$$

But for $\eta = 0$ the right-hand side reduces to $(1+\delta)^2/(1-2\delta-2\delta^2)$, an expression which is strictly less than $(1+\delta)/(1-n\delta)$. Therefore we would obtain a contradiction for sufficiently small $\eta > 0$ which means that our assumption $j \neq j'$ is not possible.

2.9. Proposition. *Let* $j \in J$, $i := \varphi(j)$, *and let* $v \in V_i$ *be any element with* $\|v\| = 1$. *Let* $p > 1$ *be any number such that*

$$n\delta \geqslant \alpha_p := p\delta\big(1/(1+\tau)+1+\tau\big)+\delta/(1+\delta)+\delta.$$

Then $\|E_j \circ Sv\| > p\delta$.

In particular, $p = (n-4)/3$ *is admissible if* $n > 7$ *which settles part* (iv) *of our theorem.*

Proof. Set $M := \|E_j \circ Sv\|$ and choose $v_1 \in V_i$ with $\|v_1\| = 1$ and $\|E_j \circ Sv_1\| > 1/(1+\tau)-\eta$ ($\eta > 0$ fixed but arbitrary).

We define $\tilde{e}_1 := E_j \circ Sv_1$, $\tilde{e}_2 := (\mathrm{Id}-E_j)Sv$, and we will apply Lemma 2.2 with $e_k := \tilde{e}_k/(1+\tau)$ ($k = 1, 2$) and

$$S^{-1}|_X\colon\ X \to V_1 \oplus_1 \big(V_\infty \oplus_1 \bigoplus_{i'\neq i}^1 V_{i'}\big),$$

where $X := \operatorname{lin}\{e_1, e_2\}$. By using the same elementary techniques as before one obtains:

1. $\|\hat{E}_i \circ S^{-1} e_1\| \geqslant (1-\delta)/(1+\tau)-\eta$,
   $\|\hat{E}_i \circ S^{-1} e_2\| \geqslant 1/(1+\tau)-M$.
2. $1 \geqslant \|e_1\| \geqslant 1/(1+\delta)-\eta/(1+\tau)$,
   $1 \geqslant \|e_2\| \geqslant 1/(1+\delta)-M/(1+\tau)$.

We have to show that $M > p\delta$. Suppose that $M \leqslant p\delta$. Since $p > 1$ and $p\delta > 0$ we may suppose that $\eta > 0$ is so small that "1." and "2." yield:

1′. $\|\hat{E}_i \circ S^{-1} e_k\| \geqslant 1/(1+\tau)-p\delta$.

2′. $1 \geqslant \|e_k\| \geqslant 1/(1+\delta)-p\delta/(1+\tau)$ for $k = 1, 2$.

Thus (i) and (ii) of 2.2 are satisfied if we define ε_1, ε_2 by

$$1-\varepsilon_1 = 1/(1+\tau)-p\delta,$$
$$(1-\varepsilon_2)(1+\tau) = 1/(1+\delta)-p\delta/(1+\tau)$$

(to prove that "2′." implies (iii) one has to note that e_1, e_2 lie in different L^1-summands so that $\|ae_1+be_2\| = |a|\,\|e_1\|+|b|\,\|e_2\|$).

Lemma 2.2 provides us with the inequality

$$(1+\delta)/(1-n\delta) < \lambda_0 \leqslant \lambda_0(V_i) \leqslant (1+\tau)/(1-\tau-\varepsilon_1-\varepsilon_2)$$

which yields the contradiction $n\delta < \alpha_p$.

Thus the first part of the proposition is proved. A very rough estimation using δ, $\tau \leqslant 1$ shows that $p = (n-4)/3$ is admissible.

Our comparatively abstract theorem allows some conclusions about the behaviour of certain *vector measures* under small-bound isomorphisms, a fact which will be crucial for our applications to isomorphic Banach–Stone theorems:

2.10. Corollary. *Let* X *and* Y *be Banach spaces,* K *and* L *locally compact Hausdorff spaces, and*

$$S\colon\ C_0(L, Y)' \to C_0(K, X)'$$

an isomorphism with

$$\frac{1}{1+\tau}\|\mu\| \leqslant \|S\mu\| \leqslant (1+\tau)\|\mu\|$$

for some $\tau > 0$ *and every* μ. *Suppose further that* $\lambda_0 := \min\{\lambda_0(X'), \lambda_0(Y')\} > 1$ *and that* τ *is sufficiently small that* $(1+\delta)/(1-3\delta) < \lambda_0$, *where* $\delta := \tau+\tau^2$.

Then for every $k \in K$ *there is precisely one* $l \in L$ *such that*

$$\sup_{\|y'\| \leqslant 1} \|[S(y'\mu_l)](\{k\})\| \geqslant 1/(1+\tau)$$

(μ_l = *Dirac measure at* l).

Proof. We write

$$C_0(L, Y)' = W_\infty \oplus_1 \big(\textstyle\bigoplus^1_{l\in L} Y_l'\big),$$

where $Y_l' := \{y'\mu_l \mid y' \in Y'\} \cong Y'$, and W_∞ denotes the set of measures μ for which $\mu(\{l\}) = 0$ for every l. Similarly, $C_0(K, X)'$ is decomposed as $V_\infty \oplus_1 \big(\bigoplus^1_{k\in K} X_k'\big)$. It is clear from well-known facts about vector measures (see e.g. [8], p. 174, or [11] or [12], p. 192) that such a decomposition is possible and that the W_∞, V_∞ behave as is necessary to apply our theorem.

Note. A similar assertion can be proved for small-bound isomorphisms between spaces of the form

$$\prod^\infty_{\varrho=1,\ldots,r} C_0(K_\varrho, X_\varrho),$$

where $\min_\varrho \lambda_0(X_\varrho) > 1$.

3. Applications to isomorphic Banach–Stone theorems. Now suppose that we are given an isomorphism T mapping $C_0(K, X)$ onto $C_0(L, Y)$. Then $S := T'$ maps $C_0(L, Y)'$ isomorphically onto $C_0(K, X)'$. We decompose these spaces as in the proof of Corollary 2.10, and we assume that

$$\frac{1}{1+\tau}\|f\| \leqslant \|Tf\| \leqslant (1+\tau)\|f\| \quad (\text{all } f),$$

and consequently

$$\frac{1}{1+\tau}\|\mu\| \leqslant \|S\mu\| \leqslant (1+\tau)\|\mu\| \quad (\text{all } \mu),$$

where τ is such that $(1+\tau)^2 = \|T\|\,\|T^{-1}\|$ (this is no restriction of generality since we only have to pass—if necessary—from T to a suitable multiple).

We want to apply the results of the preceding section and therefore we assume that $\lambda_0 := \min\{\lambda_0(X'), \lambda_0(Y')\} > 1$ and that τ is sufficiently small (for a precise statement see Theorem 3.4 below).

Theorem 2.4 provides us with mappings $\varphi\colon K \to L$, $\psi\colon L \to K$ such that $\varphi\circ\psi = \mathrm{Id}_L$, $\psi\circ\varphi = \mathrm{Id}_K$ and:

1. $\sup_{\|y'\|=1} \|[T'(y'\mu_l)](\{k\})\| \geqslant 1/(1+\tau) \quad (l \in L,\ k := \psi(l)).$
2. $\sup_{\|x'\|=1} \|[T'^{-1}(x'\mu_k)](\{l\})\| \geqslant 1/(1+\tau) \quad (k \in K,\ l := \varphi(k)).$
3. For $l \in L$ and $y' \in Y'$ with $\|y'\| = 1$ we have
$$\|[T'(y'\mu_l)](\{k\})\| \geqslant (n-4)\delta/3 \quad (k := \psi(l)).$$
4. For $k \in K$ and $x' \in X'$ with $\|x'\| = 1$ we have
$$\|[T'^{-1}(x'\mu_k)](\{l\})\| \geqslant (n-4)\delta/3 \quad (l := \psi(k)).$$

Here, as usual, $1+\delta = (1+\tau)^2$, and $n > 7$ is a fixed number (where n can be chosen arbitrarily, but the range of admissible τ will depend on n: we have to assume that $(1+\delta)/(1-n\delta) < \lambda_0$).

Having obtained mappings between topological spaces it is natural to ask whether they are continuous. We will prove that our *φ and ψ are in fact continuous* provided that $n \geqslant 10$. (The proof is surprisingly complicated, in marked contrast to similar situations in connection with earlier Banach–Stone theorems where continuity is usually obtained by routine arguments.)

It suffices to prove continuity for one of the maps, say φ. First we show that $\|(Tf)(\varphi(k))\|$ is "not too small" if $f(k)$ is large:

3.1. Proposition. *Let* $k \in K$ *and* $f \in C_0(K, X)$ *be given such that* $\|f\| = \|f(k)\| = 1$. *Then*

$$\|(Tf)(\varphi(k))\| \geqslant 1+\tau-3/(n-4).$$

Proof. Let k and f be given. We choose an $x' \in X'$ with $\|x'\| = 1$ and $x'(f(k)) = 1$. With $l := \varphi(k)$ and $y' := T'^{-1}(x'\mu_k)(\{l\})$ we have $\|y'\| \geqslant (n-4)\delta/3$ by "4.". Write $T'^{-1}(x'\mu_k) = y'\mu_l + \mu$ where $\mu(\{l\}) = 0$. Then $\|\mu\| \leqslant 1+\tau-\|y'\|$ since $\|T'^{-1}(x'\mu_k)\| \leqslant 1+\tau$ and $\|y'\mu_l+\mu\| = \|y'\|+\|\mu\|$. It follows that

$$\begin{aligned}1 = x'(f(k)) &= (x'\mu_k)(f) = (x'\mu_k)(T^{-1}\circ Tf) = [T'^{-1}(x'\mu_k)](Tf)\\ &= (y'\mu_l)(Tf)+\mu(Tf) \leqslant |y'((Tf)(l))| + \|\mu\|(1+\tau).\end{aligned}$$

Hence

$$\begin{aligned}\|y'\|\,\|(Tf)(l)\| \geqslant |y'((Tf)(l))| &\geqslant 1-(1+\tau)\big((1+\tau)-\|y'\|\big)\\ &= (1+\tau)\|y'\|-\delta,\end{aligned}$$

and dividing by $\|y'\|$ we obtain the inequality claimed.

3.2. Proposition. *For* $k \in K$, $\eta_0 > 0$, *and* $f \in C_0(K, X)$ *such that* $\|f\| \leqslant 1$ *and* $\|f(k)\| > 1-\eta_0$ *we have*

$$\|(Tf)(\varphi(k))\| \geqslant (1+\tau)(1-\eta_0)-3/(n-4).$$

Proof. Choose $\tilde{f} \in C_0(K, X)$ such that $\|\tilde{f}\| = \|\tilde{f}(k)\| = 1$, $\|f-\tilde{f}\| \leqslant \eta_0$; then $\|Tf-T\tilde{f}\| \leqslant \eta_0(1+\tau)$ and $\|(T\tilde{f})(\varphi(k))\| \geqslant 1+\tau-3/(n-4)$ so that $\|(Tf)(\varphi(k))\| \geqslant (1+\tau)(1-\eta_0)-3/(n-4)$.

3.3. Proposition. φ *is continuous provided that* $n \geqslant 10$.

Proof. Let $k_0 \in K$ be given and N a neighbourhood of $l_0 := \varphi(k_0)$. Fix an arbitrary $\eta > 0$. By "1." above there is a $y' \in Y'$ with $\|y'\| = 1$ such that $x' := [T'(y'\mu_{l_0})](\{k_0\})$ has norm at least $1/(1+\tau)-\eta$. By writing $T'(y'\mu_{l_0})$ as $x'\mu_{k_0}+\mu$ with $\mu(\{k_0\}) = 0$ it follows that

$$\|\mu\| \leqslant 1+\tau-1/(1+\tau)+\eta = \delta/(1+\tau)+\eta.$$

Now choose a function $g \in C_0(L, Y)$ such that $\|g\| = \|g(l_0)\| = 1$ and $g(l) = 0$ for $l \notin N$. Similarly to the proof of Proposition 3.1 we conclude that

$$|x'(T^{-1}g(k_0))| \geqslant 1-\eta-(1+\tau)(\delta/(1+\tau)+\eta)$$

so that, since $\|x'\| \leqslant 1+\tau$,

$$\|(T^{-1}g)(k_0)\| \geqslant (1-2\eta-\delta-\tau\eta)/(1+\tau).$$

Choose a neighbourhood $\tilde{N}$ of k_0 such that

$$\|(T^{-1}g)(k)\| > (1-3\eta-\delta-\tau\eta)/(1+\tau) =: (1+\tau)(1-\eta_0)$$

on $\tilde{N}$. Then it follows from 3.2 by considering $f := (1+\tau)^{-1}T^{-1}g$ that

$$\begin{aligned}\|g(\varphi(k))\| &= (1+\tau)\|(Tf)(\varphi(k))\| \geqslant (1+\tau)^2(1-\eta_0)-3(1+\tau)/(n-4)\\ &= 1-3\eta-\tau\eta-\delta-3(1+\tau)/(n-4) =: \alpha(\eta)\end{aligned}$$

for these k. But $\eta \mapsto \alpha(\eta)$ is a continuous function which assumes the value $1-\delta-3(1+\tau)/(n-4)$ at zero. This number is greater than zero since $n \geqslant 10$ (as is shown by a rough estimation) so that we have $\alpha(\eta) > 0$ for some $\eta > 0$. This implies that $\varphi(k) \in N$ whenever $k \in \tilde{N}$ so that φ is continuous at k_0.

Summing up we have shown the following

3.4. Theorem. *Let* X *and* Y *be Banach spaces such that* $\lambda_0 := \min\{\lambda_0(X'), \lambda_0(Y')\} > 1$. *Further, let* K *and* L *be locally compact Hausdorff spaces such that there is an isomorphism*

$$T\colon C_0(K, X) \to C_0(L, Y).$$

Then K *and* L *are homeomorphic provided that*

$$0 < \frac{1+\delta}{1-10\delta} < \lambda_0,$$

where δ *is defined by* $1+\delta = \|T\|\,\|T^{-1}\|$.

Note. Similarly it can be shown that the existence of small-bound isomorphisms between spaces

$$\prod_{\varrho=1,\ldots,r}^{\infty} C_0(K_\varrho, X_\varrho) \quad \text{and} \quad \prod_{\sigma=1,\ldots,s}^{\infty} C_0(L_\sigma, Y_\sigma)$$

is possible only if $K_1 \dot\cup \ldots \dot\cup K_r$ is homeomorphic to $L_1 \dot\cup \ldots \dot\cup L_s$ when we assume that

$$\min\{\lambda_0(X') \mid X = X_\varrho,\ X = Y_\sigma\} > 1.$$

As a corollary we can state the announced isomorphic Banach–Stone theorem:

3.5. Theorem. *Let* X *be a Banach space such that* $\lambda_0(X') > 1$. *Then there is a* $\delta > 0$ *such that two locally compact Hausdorff spaces* K *and* L *are necessarily homeomorphic whenever there is an isomorphism* T *between* $C_0(K, X)$ *and* $C_0(L, X)$ *with* $\|T\|\,\|T^{-1}\| \leqslant 1+\delta$ (i.e., in the terminology of [4], X has the *isomorphic Banach–Stone property*).

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The translation invariant uniform approximation property for compact groups

by

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Abstract. The translation invariant analogue in commutative harmonic analysis of the uniform approximation property of Banach spaces was introduced by M. Bożejko and A. Pełczyński (1978). This paper contains noncommutative analogues of their results.

§0. Introduction. Recall (see [1], [5]) that a Banach space X is said to have the *uniform bounded approximation property*, abbreviated ubap, if there exist a $k \geqslant 1$ and a positive sequence $q(m)$ such that given a finite-dimensional subspace $E \subset X$, there exists an operator $T\colon X \to X$ satisfying the following conditions:

(i) $T(x) = x$ for $x \in E$.

(ii) $\|T\| \leqslant k$.

(iii) $\dim T(X) \leqslant q(\dim E)$.

It is known ([4], [5]) that L^p-spaces, $C(K)$-spaces and reflexive Orlicz spaces have the ubap. The definition can be modified for function spaces on compact groups by, roughly speaking, assuming that X, E and T are translation invariant. In this way we obtain the translation invariant analogue of the uniform bounded approximation property introduced in [1] where the case of compact Abelian groups is considered.

§1. Preliminaries. In the sequel G is a compact group, Σ its dual object called also the *hypergroup*, λ the normalized Haar measure on G.

For $g \in G$ the left translation operator is defined by $(l_g f)(x) = f(g^{-1}x)$ and the right translation operator by $(r_g f)(x) = f(xg)$ for f λ-measurable and $x \in G$. A vector space X of λ-equivalence classes of λ-measurable functions is *left translation invariant* if $l_g X \subset X$, *right translation invariant* if $r_g X \subset X$ and *conjugate translation invariant* if $l_g r_g X \subset X$ for all $g \in G$. We call X *translation invariant* if it is both left and right translation invariant.

By $L^p(G)$ $(p \geqslant 1)$ we denote as usual the Banach space of the λ-equivalence classes of λ-measurable functions f with the norm $\|f\|_p = \left(\int |f(x)|^p \, d\lambda(x)\right)^{1/p}$. For $f, g \in L^1(G)$ we define the convolution $f * g \in L^1(G)$ by

$$(f * g)(x) = \int_G f(y^{-1}x) g(y) \, d\lambda(y).$$