Finally, let us remark that the algebras $\mathscr{A}_0(X_0)$ given by (2) are maximal subalgebras of $B(X)$ with respect to the property of being of square zero, and all such maximal subalgebras are of the form (2) (see [8]).

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L^2-Angles between one-dimensional tubes

by

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Abstract. Let $D_i \subset C^N$, $i = 1, 2$, be two domains with nonempty intersection, such that $D = D_1 \cup D_2$ is contained neither in $\bar{D}_1$ nor in $\bar{D}_2$. Denote by F_i, $i = 1, 2$, the closed subspace in $L^2(D)$ consisting of functions which are holomorphic in D_i and otherwise arbitrary. As indicated in [7], [8] the Bergman projection in D can be described in terms of the orthogonal projections P_i: $L^2(D) \to F_i$. In some cases, the relevent alternating projections procedure can be carried out by explicit analytic calculations [7], [9]. In this context it is natural to study the angle $\gamma \in [0, \pi/2]$ between the subspaces F_1 and F_2. We call it (for brevity) the L^2-angle between D_1 and D_2. It was shown in [5] that the L^2-angle between two halfplanes (bounded by parallel lines) is 0. In the present paper we are concerned with the more general situation when D_i, $i = 1, 2$, are arbitrary tubes in the complex plane. We determine the L^2-angle $\alpha(r)$ between the strip $\{0 < \operatorname{Re} z < 1\}$ and a halfplane $\{\operatorname{Re} z > r\}$ $(0 < r < 1)$, as well as the L^2-angle $\beta(r, s)$ between two strips $\{-\pi < \operatorname{Re} z < s\pi\}$, $\{r\pi < \operatorname{Re} z < \pi\}$ $(-1 < r < s < 1)$. It turns out that

$$\cos^2 \alpha(r) = r \quad \text{and} \quad \cos^2 \beta(r, s) = \frac{1-s}{1+s} \cdot \frac{1+r}{1-r}.$$

We include a brief presentation of the Genchev transform [2], [4] since it plays an important role in our considerations.

1. The abstract definition of an angle. Let F_1, F_2 be closed subspaces in an abstract Hilbert space $\mathscr{H}$. Assume that $F = F_1 \cap F_2$ is a proper subset in each of F_i, $i = 1, 2$. (This implies that each F_i contains a nonzero element orthogonal to F.)

Definition 1. The *angle* $\gamma \in [0, \pi/2]$ between F_1 and F_2 is defined by

$$\cos \gamma = \sup_{\substack{f_i \in F_i \setminus \{0\} \\ f_i \perp F}} \frac{|\langle f_1, f_2 \rangle|}{\|f_1\| \cdot \|f_2\|}. \tag{1}$$

We shall restate this definition in a way which is less symmetric, but more convenient for our purpose.

Lemma 1. *Denote by* P_2 *the orthogonal projection of* $\mathscr{H}$ *onto* F_2. *Then*

$$\cos \gamma = \sup_{\substack{f_1 \in F_1 \setminus \{0\} \\ f_1 \perp F}} \frac{\|P_2 f_1\|}{\|f_1\|}. \tag{2}$$

Proof. When $P_2(F_1) = F$ both (1) and (2) give the same value $\cos\gamma = 0$. In the opposite case there is $f_1 \in F_1$ admissible for (2) such that $P_2 f_1 \neq 0$ and in both (1) and (2) the supremum can be computed with the additional condition $P_2 f_1 \neq 0$. Assume that the pair f_1, f_2 (with $P_2 f_1 \neq 0$) is admissible in (1). By the Schwarz inequality

$$\frac{|\langle f_1, f_2\rangle|}{\|f_1\|\cdot\|f_2\|} = \frac{|\langle P_2 f_1, f_2\rangle|}{\|f_1\|\cdot\|f_2\|} \leqslant \frac{\|P_2 f_1\|}{\|f_1\|}$$

and (1) $\leqslant$ (2) follows. It remains to show the converse inequality. Assume that f_1 (with $P_2 f_1 \neq 0$) is admissible in (2). Then $P_2 f_1 \perp F$ and the pair f_1, f_2 where $f_2 = P_2 f_1$ is admissible in (1). Moreover,

$$\frac{|\langle f_1, f_2\rangle|}{\|f_1\|\cdot\|f_2\|} = \frac{\|P_2 f_1\|^2}{\|f_1\|\cdot\|P_2 f_1\|} = \frac{\|P_2 f_1\|}{\|f_1\|}.$$

Hence (2) $\leqslant$ (1). ∎

3. An angle in $L^2(D)$ related to alternating projections. We now consider the case when $\mathscr{H} = L^2(D)$, $D = D_1 \cup D_2$. To avoid indices we introduce the notation

$$A = D_1, \quad B = D_2, \quad T = D_1 \cap D_2, \quad f = f_1.$$

Let us recall that $f \in F_1$ (see the abstract) means that $f_A := f|_A$ is L^2-holomorphic and $f_{B\setminus T} := f|_{B\setminus T}$ is in L^2 (arbitrary). For $f \in F_1$ the restriction $f_B := f|_B$ is (in general) not holomorphic and its Bergman projection in B will be denoted by $\hat{f}$. It is well known [7, Th. 3] that $P_2\colon F_1 \to F_2$ is given by

$$P_2 f(z) = \begin{cases} f(z) & \text{for } z \in A\setminus T, \\ \hat{f}(z) & \text{for } z \in B. \end{cases} \tag{3}$$

Note that $F = L^2H(D)$ is the space of all functions which are L^2-holomorphic in D. Formula (2) takes the form

$$\cos^2\gamma = \sup_{\substack{f \in F_1\setminus\{0\} \\ f \perp L^2H(D)}} \frac{\|f\|_{A\setminus T}^2 + \|\hat{f}\|_B^2}{\|f\|_A^2 + \|f\|_{B\setminus T}^2}. \tag{4}$$

From this follows easily

THEOREM 1. *The L^2-angle between two domains $A, B \subset C^N$ is $\pi/2$ if and only if*

$$L^2H(A) = \{0\}, \quad L^2H(B) = \{0\}. \tag{5}$$

In particular, the L^2-angle between plane domains (if defined) is always smaller than $\pi/2$.

Proof. The condition is sufficient. Indeed, $f \in F_1$ implies that $f_A \in L^2H(A)$ and $\hat{f} \in L^2H(B)$, hence it follows from (5) that (4) has vanishing numerator. We conclude that $\cos^2\gamma = 0$, or equivalently that $\gamma = \pi/2$.

Conversely, assume that (5) does not hold. In view of symmetry it is enough to consider the case $L^2H(A) \neq \{0\}$. Take an arbitrary $u \in L^2H(A)\setminus\{0\}$ and extend it by 0 to D. The extended function $u \in L^2(D)$ is not holomorphic, therefore its Bergman projection $u^B \in L^2H(D)$ satisfies

$$\|u^B\|_A \leqslant \|u^B\|_D < \|u\|_D = \|u\|_A.$$

It follows that $f = u - u^B$ does not vanish identically on A, and is admissible for (4). Hence $\cos^2\gamma > 0$. ∎

For $N \neq 1$ the equality $\gamma = \pi/2$ can occur, as in the following

EXAMPLE 1. Let A and B be the exteriors of two closed disjoint balls in C^N, $N \neq 1$. Then condition (5) is satisfied.

For later use we shall need a slight modification of (4) given in the following

THEOREM 2. *Assume that* (5) *does not hold. Then*

$$\cos^2\gamma = \sup\left\{\frac{\|f\|_{A\setminus T}^2 + \|\hat{f}\|_B^2}{\|f\|_A^2 + \|f\|_{B\setminus T}^2}\colon f \in F_1\setminus\{0\},\ f \perp L^2H(D),\ f \text{ holomorphic in } \operatorname{int} B\setminus T\right\}. \tag{6}$$

Proof. Let h be any function admissible in (4) for which the expression under the supremum sign is positive. (Such functions exist in view of Theorem 1.) Let $f \in L^2(D)$ be the projection of h onto the subspace in $L^2(D)$ of all functions which are L^2-holomorphic in $\operatorname{int} B\setminus T$ and otherwise arbitrary. Then $f(z) = h(z)$ for $z \notin \operatorname{int} B\setminus T$. The assumption $f = 0$ implies that $\|h\|_{A\setminus T}^2 + \|\hat{h}\|_B^2 = 0$ in contradiction to the assumed property of h. Therefore $f \neq 0$ and f is admissible in (6). (This shows that there do exist functions admissible in (6).) Since the numerator corresponding to h in (4) is equal to the one corresponding to f in (6), and the denominator corresponding to h in (4) is not smaller than the one corresponding to f in (6) we conclude that (4) $\leqslant$ (6). The inequality (6) $\leqslant$ (4) is obvious. ∎

A biholomorphic mapping $\varphi\colon D \to \tilde{D}$ can be of help in computing γ in view of the following

THEOREM 3. *Assume that $\varphi\colon D \to \tilde{D}$ is biholomorphic. Then the L^2-angle between A and B is the same as the one between $\tilde{A} = \varphi(A)$ and $\tilde{B} = \varphi(B)$.*

Proof. It suffices to apply the unitary mapping $U_\varphi\colon L^2(\tilde{D}) \to L^2(D)$ given by

$$(U_\varphi f)(z) = f(\varphi(z))\,\varphi'(z).$$

3. The Genchev transform. We shall recall briefly some L^2-theorems of Paley–Wiener type due to T. Genchev [4] and M. Dzhrbashyan [2]. The following lemma should be compared with [4]:

LEMMA 2. *Let $D = \{\operatorname{Re} z \in J\}$ be the one-dimensional tube over an open interval $J \subset \mathbf{R}$ (in particular, one can take for J a halfline). For every $f \in L^2H(D)$ and every $x \in J$ the function $y \mapsto f(x+iy)$ belongs to $L^2(\mathbf{R})$.*

Proof. Fix $r > 0$ such that $(x-r, x+r) \subset J$. For every $y \in \mathbf{R}$ the function $f_y(z) = f(z+iy)$ is L^2-holomorphic in the variable $z = u+iv$ over the square $Q = (x-r, x+r)\times(-r, r)$. At the center of the square, f_y takes the value $f_y(x) = f(x+iy)$. Since Q contains the disc with center x and radius r, the Bergman theory [1] yields the inequality

$$|f_y(x)|^2 \leqslant (\pi r^2)^{-1}\|f_y\|_Q^2$$

which can be rewritten as

$$|f(x+iy)|^2 \leqslant (\pi r^2)^{-1} \int\limits_{v\in(-r,r)} \int\limits_{u\in(x-r,x+r)} |f(u+iv+iy)|^2\,du\,dv. \tag{7}$$

We shall now integrate both sides with respect to $y \in (-\infty, \infty)$. Since the Lebesgue measure is translation invariant we obtain

$$\int\limits_{y\in(-\infty,\infty)} |f(x+iy)|^2\,dy \leqslant \frac{1}{\pi r^2} \int\limits_{v\in(-r,r)} \int\limits_{u\in(x-r,x+r)} \int\limits_{y\in(-\infty,\infty)} |f(u+iv+iy)|^2\,dy\,du\,dv$$

$$= \frac{1}{\pi r^2} \int\limits_{v\in(-r,r)} \int\limits_{u\in(x-r,x+r)} \int\limits_{y\in(-\infty,\infty)} |f(u+iy)|^2\,dy\,du\,dv \leqslant \frac{2}{\pi r}\|f\|_D^2.$$

The right side is finite by assumption, hence the left side is finite. ∎

The (inverse) Fourier transform of $g(y) = f(x+iy)$ is given by (see G. Folland [3], p. 20)

$$\check{g}(t) = \int\limits_{-\infty}^{\infty} e^{2\pi i t y} f(x+iy)\,dy$$

and depends on $x \in J$ in a verv explicit way. In fact, $e^{2\pi tx}\check{g}(t)$ does not depend on x at all. The function

$$G_f(t) := e^{2\pi tx}\check{g}(t) = i^{-1} \lim_{E\to\infty} \int\limits_{x-iE}^{x+iE} e^{2\pi tz} f(z)\,dz \tag{8}$$

will be called the *Genchev transform* of $f \in L^2H(D)$. Note that G_f is completely determined by the values of f on one line $\operatorname{Re} z = x$. Therefore two functions $f_1 \in L^2H(D_1)$ and $f_2 \in L^2H(D_2)$ which agree on such a line have the same Genchev transforms.

In view of the Plancherel theorem, the L^2-norm of f over any tube $T = \{\alpha < \operatorname{Re} z < \beta\}$ $(T \subset D)$ can be easily expressed in terms of G_f. Namely,

$$\|f\|_T^2 = \int\limits_{-\infty}^{\infty} |G_f(t)|^2 w_T(t)\,dt, \qquad w_T(t) = \int\limits_{\alpha}^{\beta} e^{-4\pi tx}\,dx. \tag{9}$$

We can now state

THEOREM 4 (T. Genchev, M. Dzhrbashyan). *The correspondence $f \mapsto G_f$ defines a unitary mapping of $L^2H(D)$ onto $L^2(\mathbf{R}, w_D)$.*

Proof. The mapping is linear and isometric in view of (9). It remains to show that its image contains a dense subset of $L^2(\mathbf{R}, w_D)$. Indeed, it is easy to verify that every $h \in L^2(\mathbf{R}, w_D)$ bounded and vanishing outside a compact set can be written as G_f with $f \in L^2H(D)$ given by the (two-sided) Laplace transform of h,

$$f(z) = \int\limits_{-\infty}^{\infty} e^{-2\pi zt} h(t)\,dt.$$

Remark 1. An immediate calculation shows that:

1° For $J = (a, b)$, $w_D(t) = \dfrac{e^{-4\pi at} - e^{-4\pi bt}}{4\pi t}$, $t \in (-\infty, \infty)$.

2° For $J = (a, \infty)$, $w_D(t) = \dfrac{e^{-4\pi at}}{4\pi t}$, $t > 0$, and $w_D(t) \equiv \infty$, $t < 0$.

3° For $J = (-\infty, b)$, $w_D(t) = \dfrac{e^{-4\pi bt}}{-4\pi t}$, $t < 0$, and $w_D(t) \equiv \infty$, $t > 0$.

Note that $L^2(\mathbf{R}, w_D)$ in case 2° can be identified with $L^2(\mathbf{R}^+, w_D)$ and in case 3° with $L^2(\mathbf{R}^-, w_D)$.

The following three corollaries of Theorem 4 will be needed later. We omit the easy proofs (see also [9]).

COROLLARY 1. *Let D be a tube over a bounded interval $J = (a, b)$. Consider the following subspaces in $L^2H(D)$:*

$$L^2_-H(D) = \{f \in L^2H(D);\ G_f(t) = 0 \text{ for every } t > 0\},$$

$$L^2_+H(D) = \{f \in L^2H(D);\ G_f(t) = 0 \text{ for every } t < 0\}.$$

There is an orthogonal decomposition

$$L^2H(D) = L^2_-H(D) \oplus L^2_+H(D).$$

COROLLARY 2 (D as above). *The subspace $L^2_-H(D)$ consists of all $L^2(D)$-limits $f = \lim_{k\to\infty} f_k$, where every f_k is L^2-holomorphic in the halfplane $\{\operatorname{Re} z < b\}$.*

The subspace $L^2_+H(D)$ *consists of all* $L^2(D)$*-limits* $f=\lim_{k\to\infty}f_k$, *where every* f_k *is* L^2*-holomorphic in the halfplane* $\{\operatorname{Re} z>a\}$.

COROLLARY 3 (D as above). *Every* $f\in L^2_-H(D)$ *extends to a holomorphic function in* $\{\operatorname{Re} z<b\}$, *and for every* $c>0$

$$\|f\|^2_{D-c}\leqslant\|f\|^2_D.$$

Every $f\in L^2_+H(D)$ *extends to a holomorphic function in* $\{\operatorname{Re} z>a\}$, *and for every* $c>0$

$$\|f\|^2_{D+c}\leqslant\|f\|^2_D.$$

4. Operators related to the Bergman projection in a halfplane. In order to use formula (6) we need to know the Bergman projection in D for a class of piecewise L^2-holomorphic functions. Therefore first we address the following

PROBLEM 1. Let U be a subdomain in D. Describe explicitly the operator P_{UD}: $L^2H(U)\to L^2H(D)$, where $P_{UD}f$ is the Bergman projection of the trivial extension of f to D.

Note that P_{UD} maps the Bergman function $K_U(\cdot,p)$, $p\in U$, to $K_D(\cdot,p)$. Since the functions $K_U(\cdot,p)$, $p\in U$, are linearly dense in $L^2H(U)$, the above property is characteristic of P_{UD}.

LEMMA 3. *Assume that* $D=\{\operatorname{Re} z>0\}$ *and* $U=\{\operatorname{Re} z>r\}$ $(r>0)$. *Then* $P_{UD}f(z)=f(z+2r)$.

Proof. The operator defined by the above formula is linear and continuous. It also maps $K_U(\cdot,p)$ to $K_D(\cdot,p)$ in view of the well-known formulas

$$K_D(z,p)=\frac{1}{\pi(z+\bar p)^2},\qquad K_U(z,p)=\frac{1}{\pi(z+\bar p-2r)^2}. \tag{10}$$

LEMMA 4. *Assume that* $D=\{\operatorname{Re} z>0\}$ *and* $U=\{0<\operatorname{Re} z<r\}$. *Then* $P_{UD}f(z)=f^{(+)}(z)-f^{(+)}(z+2r)$, *where* $f=f^{(-)}+f^{(+)}$ *is the orthogonal decomposition of* $f\in L^2H(U)$ *from Corollary* 1.

Proof. In view of Corollaries 1–3 the above formula defines a continuous linear operator, and it suffices to show that it agrees with P_{UD} in two cases: 1° when f is L^2-holomorphic in $\{\operatorname{Re} z<r\}$, 2° when f is L^2-holomorphic in $\{\operatorname{Re} z>0\}$. (Indeed, the sum of classes 1° and 2° is linearly dense in $L^2H(U)$).

In case 1°, $f^{(+)}=0$. Since $K_D(\cdot,z)$ is L^2-holomorphic in the halfplane D its restriction to U is in $L^2_+H(U)$, hence orthogonal to $f^{(-)}$. This yields

$$P_{UD}f(z)=P_{UD}f^{(-)}(z)=\int_U f^{(-)}(w)\overline{K_D(w,z)}\,dm(w)=0$$

according to the statement. In case 2°, f extends to D and

$$P_{UD}f(z)=\int_D f(z)\overline{K_D(w,z)}\,dm(w)-\int_{D\setminus U} f(w)\overline{K_D(w,z)}\,dm(w).$$

The first integral equals $f(z)$ by the reproducing property of the Bergman function, and the second equals $f(z+2r)$ by Lemma 3. Since $f=f^{(+)}$ the statement follows. ∎

5. L^2-angle between a strip and a halfplane. In view of Theorem 3 it suffices to determine the L^2-angle $\alpha(r)$ between the strip A and the halfplane B given by

$$A=\{0<\operatorname{Re} z<1\},\qquad B=\{\operatorname{Re} z>r\},$$

where $0<r<1$. Both domains are illustrated in Fig. 1, where $T=A\cap B$.

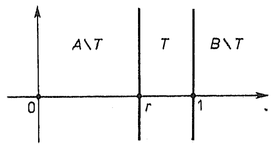


Fig. 1

First of all we shall determine all pairs $f_A\in L^2H(A)$, $f_{B\setminus T}\in L^2H(B\setminus\overline{T})$ which describe a function f, admissible in (6). The Bergman projection in $D=A\cup B$ of (the trivial extension of) f_A is $f_A^{(+)}(z)-f_A^{(+)}(z+2)$ according to Lemma 4. The Bergman projection of (the trivial extension of) $f_{B\setminus T}$ is $f_{B\setminus T}(z+2)$ according to Lemma 3. Therefore $f\perp L^2H(D)$ is equivalent to

$$f_{B\setminus T}(z+2)=f_A^{(+)}(z+2)-f_A^{(+)}(z). \tag{11}$$

It follows that $f_{B\setminus T}$ has to be defined by the formula

$$f_{B\setminus T}(z)=f_A^{(+)}(z)-f_A^{(+)}(z-2) \tag{12}$$

and that (12) defines a function in $L^2H(B\setminus\overline{T})$ if and only if $f_A^{(+)}$ is L^2-holomorphic in the strip $\{|\operatorname{Re} z|<1\}$. We also see from (12) that $f\neq 0$ if and only if $f_A\neq 0$.

Next we shall describe (for an admissible f) the Bergman projection $\hat f\in L^2H(B)$ of the (nonholomorphic) function f_B. The Bergman projection of $f_T:=f|_T$ is (according to Lemma 4) $f_A^{(+)}(z)-f_A^{(+)}(z+2(1-r))$. The Bergman projection of $f_{B\setminus T}$ is (according to Lemma 3 and (12)) $f_A^{(+)}(z+2(1-r))-f_A^{(+)}(z-2r)$. Consequently,

$$\hat f(z)=f_A^{(+)}(z)-f_A^{(+)}(z-2r). \tag{13}$$

Denote by $\Phi(t)$ the Genchev transform of $f_A^{(+)}$ and by $\Psi(t)$ the Genchev transform of $f_A^{(-)}$. From (12) and (13) it follows that

$$G_{f_{B\setminus T}}(t)=\Phi(t)(1-e^{2\cdot 2\pi t}),\qquad G_{\hat f}(t)=\Phi(t)(1-e^{2r\cdot 2\pi t}). \tag{14}$$

The expression under the supremum sign in (6) has the form

$$\frac{\|f_A^{(-)}\|^2_{A\setminus T}+\|f_A^{(+)}\|^2_{A\setminus T}+\|\hat f\|^2_B}{\|f_A^{(-)}\|^2_A+\|f_A^{(+)}\|^2_A+\|f\|^2_{B\setminus T}}. \tag{15}$$

The first terms in the numerator and denominator depend on Ψ and do not depend on Φ; the remaining terms depend on Φ, and do not depend on Ψ. It follows that

$$\cos^2\alpha(r) = \max\left(\sup_{\Psi}\frac{\|f_A^{(-)}\|_{A\setminus T}^2}{\|f_A^{(-)}\|_A^2},\ \sup_{\Phi}\frac{\|f_A^{(+)}\|_{A\setminus T}^2+\|\hat{f}\|_{\hat{B}}^2}{\|f_A^{(+)}\|_A^2+\|f\|_{\hat{B}\setminus T}^2}\right). \tag{16}$$

In view of Remark 1 the corresponding weights are given by

$$w_{A\setminus T}(t) = \frac{1-e^{-4\pi rt}}{4\pi t},\qquad t\in(-\infty,\infty),$$

$$w_A(t) = \frac{1-e^{-4\pi t}}{4\pi t},\qquad t\in(-\infty,\infty),$$

$$w_{\hat{B}}(t) = e^{-4\pi rt}/(4\pi t),\qquad t\in(0,\infty),$$

$$w_{B\setminus T}(t) = e^{-4\pi t}/(4\pi t),\qquad t\in(0,\infty).$$

Using (9) we find that

$$\sup_{\Psi}\frac{\|f_A^{(-)}\|_{A\setminus T}^2}{\|f_A^{(-)}\|_A^2} = \sup_{\Psi}\frac{\int_{-\infty}^{0}|\Psi(t)|^2\frac{1-e^{-4\pi rt}}{4\pi t}\,dt}{\int_{-\infty}^{0}|\Psi(t)|^2\frac{1-e^{-4\pi t}}{4\pi t}\,dt} = r. \tag{17}$$

Indeed, the ratio $(1-e^{-4\pi rt})/(1-e^{-4\pi t})$ is not greater than r, and converges to r as $t\nearrow 0$. (Further details are left to the reader).

In a similar way we find that

$$\|f_A^{(+)}\|_{A\setminus T}^2 = \int_0^{\infty}|\Phi(t)|^2\frac{1-e^{-4\pi rt}}{4\pi t}\,dt,$$

$$\|f_A^{(+)}\|_A^2 = \int_0^{\infty}|\Phi(t)|^2\frac{1-e^{-4\pi t}}{4\pi t}\,dt,$$

and in view of (14)

$$\|\hat{f}\|_{\hat{B}}^2 = \int_0^{\infty}|\Phi(t)(1-e^{4\pi rt})|^2\frac{e^{-4\pi tr}}{4\pi t}\,dt = \int_0^{\infty}\frac{|\Phi(t)|^2}{4\pi t}(e^{-4\pi rt}-2+e^{4\pi rt})\,dt,$$

$$\|f\|_{\hat{B}\setminus T}^2 = \int_0^{\infty}|\Phi(t)(1-e^{4\pi t})|^2\frac{e^{-4\pi t}}{4\pi t}\,dt = \int_0^{\infty}\frac{|\Phi(t)|^2}{4\pi t}(e^{-4\pi t}-2+e^{4\pi t})\,dt.$$

It follows that

$$\sup_{\Phi}\frac{\|f_A^{(+)}\|_{A\setminus T}^2+\|\hat{f}\|_{\hat{B}}^2}{\|f_A^{(+)}\|_A^2+\|f\|_{\hat{B}\setminus T}^2} = \sup_{\Phi}\frac{\int_0^{\infty}\frac{|\Phi(t)|^2}{4\pi t}(e^{4\pi rt}-1)\,dt}{\int_0^{\infty}\frac{|\Phi(t)|^2}{4\pi t}(e^{4\pi t}-1)\,dt} = r. \tag{18}$$

From (16), (17), (18) follows immediately

THEOREM 5. *The L^2-angle $\alpha(r)$ between the strip $A = \{0 < \operatorname{Re} z < 1\}$ and the halfplane $B = \{\operatorname{Re} z > r\}$ is*

$$\alpha(r) = \arccos(r^{1/2}),\qquad 0 < r < 1.$$

Remark 2. It may be worth recalling that according to [5] the L^2-angle $\gamma(r)$ between the domains $A = \{|z| < 1\}$ and $B = \{|z| > r\}$ is

$$\gamma(r) = \arccos r,\qquad 0 < r < 1.$$

5. Operators related to the Bergman projection in a strip. We begin with further remarks about the Genchev transform.

LEMMA 5. *Assume that the mapping $\varphi(z) = kz+c$ ($k, c\in \mathbf{R}$, $k\neq 0$) transforms the tube $D = \{\operatorname{Re} z\in J\}$ onto the tube $\tilde{D} = \{\operatorname{Re} w\in \tilde{J}\}$. Let U_φ: $L^2H(\tilde{D})\to L^2H(D)$ be the canonical isometry*

$$U_\varphi f(z) = f(\varphi(z))\,\varphi'(z) = kf(kz+c). \tag{19}$$

If $h\in L^2(\mathbf{R}, w_D)$ is the Genchev transform of $U_\varphi f\in L^2H(D)$, then

$$I_\varphi h(t) = (\operatorname{sgn} k)\,e^{2\pi tc}h(kt) \tag{20}$$

is the Genchev transform of $f\in L^2H(\tilde{D})$. In particular, one has the commutative diagram

$$\begin{array}{ccc} L^2H(D) & \xleftarrow{\ U_\varphi\ } & L^2H(\tilde{D}) \\ \downarrow G & & \downarrow G \\ L^2(\mathbf{R}, w_D) & \xrightarrow{\ I_\varphi\ } & L^2(\mathbf{R}, w_{\tilde{D}}) \end{array}$$

which shows that I_φ: $L^2(\mathbf{R}, w_D)\to L^2(\mathbf{R}, w_{\tilde{D}})$ is a unitary mapping.

Proof. We shall compute the Genchev transform of f over the line $\operatorname{Re} w = \varphi(x)$ (which is the image under φ of the line $\operatorname{Re} z = x$). The natural orientation on these lines (with imaginary part as parameter) is preserved when $k > 0$ and reversed when $k < 0$. Formula (8) yields

$$G_f(t) = i^{-1}\operatorname{sgn} k\lim_{E\to\infty}\int_{\varphi(x-iE)}^{\varphi(x+iE)} e^{2\pi tw}f(w)\,dw = i^{-1}\operatorname{sgn} k\lim_{E\to\infty}\int_{x-iE}^{x+iE} e^{2\pi t\varphi(z)}U_\varphi f(z)\,dz$$

$$= i^{-1}\operatorname{sgn} k\,e^{2\pi tc}\lim_{E\to\infty}\int_{x-iE}^{x+iE} e^{2\pi(tk)z}U_\varphi f(z)\,dz = (\operatorname{sgn} k)\,e^{2\pi tc}h(kt).$$

The proof is complete.

Remark 3. When $k = 1$ the mapping $\varphi(z) = z+c$ is a translation and U_φ is the shift operator $(S_c f)(z) = f(z+c)$. In this case Lemma 5 says that

$$G_f(t) = e^{2\pi tc}G_{S_c f}(t). \tag{21}$$

The above formula was implicitly used in (14).

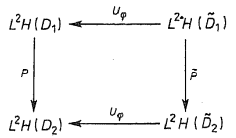
DEFINITION 2. Let P: $L^2H(D_1)\to L^2H(D_2)$ be a linear continuous operator. A measurable function μ: $(-\infty,\infty)\to C$ will be called a *multiplier* for P if

$$G_{Pf}(t)=\mu(t)G_f(t),\qquad t\in(-\infty,\infty),$$

for every $f\in L^2H(D_1)$.

The following simple corollary of Lemma 5 will be needed later.

COROLLARY 4. *Assume that $\varphi(z)=kz+c$ maps D_1 onto $\tilde D_1$ and D_2 onto $\tilde D_2$. Assume further that P: $L^2H(D_1)\to L^2H(D_2)$ is a linear continuous operator with multiplier $\mu(t)$. Consider the operator $\tilde P$: $L^2H(\tilde D_1)\to L^2H(\tilde D_2)$ uniquely defined by the following commutative diagram:*



Then $\tilde\mu(t)=\mu(kt)$ is a multiplier for $\tilde P$.

Proof. We want to find the Genchev transform of $\tilde Pf$ for $f\in L^2H(\tilde D_1)$. Since P has multiplier μ we have the equality

$$G_{PU_\varphi f}=\mu G_{U_\varphi f}\tag{22}$$

and by Lemma 5

$$\begin{aligned}G_{\tilde Pf}(t)&=(\operatorname{sgn}k)e^{2\pi tc}G_{U_\varphi\tilde Pf}(kt)=(\operatorname{sgn}k)e^{2\pi tc}G_{PU_\varphi f}(kt)\\&=(\operatorname{sgn}k)e^{2\pi tc}\mu(kt)\,G_{U_\varphi f}(kt)=\mu(kt)\,G_f(t).\end{aligned}\tag{23}$$

This shows that $\tilde\mu(t)=\mu(kt)$ is a multiplier for $\tilde P$. ∎

Let us now consider the standard strip $D=\{-\pi<\operatorname{Re}z<\pi\}$ divided by the line $\operatorname{Re}z=s\pi$ $(-1<s<1)$ (see Fig. 2).

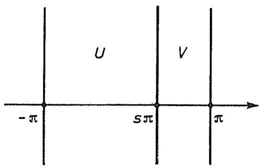


Fig. 2

Along with the strips $U=\{-\pi<\operatorname{Re}z<s\pi\}$, $V=\{s\pi<\operatorname{Re}z<\pi\}$ we shall study the operators P_{UD}: $L^2H(U)\to L^2H(D)$, P_{VD}: $L^2H(V)\to L^2H(D)$

defined in Section 4 (Problem 1). We are going to show that these operators have multipliers. This result will extend to nonstandard strips in view of Corollary 4.

The line $\operatorname{Re}z=s\pi$ determines two halfplanes $H_{s\pi-}$ and $H_{s\pi+}$:

$$H_{s\pi-}=\{\operatorname{Re}z<s\pi\},\qquad H_{s\pi+}=\{\operatorname{Re}z>s\pi\}.$$

The corresponding Bergman functions are

$$K_{H_{s\pi-}}(w,z)=\frac{1}{\pi(w+\bar z-2s\pi)^2},\qquad K_{H_{s\pi+}}(w,z)=\frac{1}{\pi(w+\bar z-2s\pi)^2}.$$

The Bergman function for the standard strip D can be obtained in the usual way, using the elementary mapping of D onto the unit disc. By the classical secans formula [6, p. 50] it can be written as

$$\begin{aligned}K_D(w,z)&=\frac{\cos^{-2}\dfrac{w+\bar z}{4}}{16\pi}\\&=\sum_{k=1}^{\infty}\frac{1}{\pi(w+\bar z-(2k-1)2\pi)^2}+\sum_{k=1}^{\infty}\frac{1}{\pi(w+\bar z+(2k-1)2\pi)^2}.\end{aligned}\tag{24}$$

For a fixed $z\in D$ the terms in both series are square integrable in the variable w over D since

$$\frac{1}{\pi(w+\bar z-(2k-1)2\pi)^2}=K_{H_{\pi-}}(w,z-(2k-2)2\pi)$$

with $\operatorname{Re}(z-(2k-2)2\pi)\leqslant\operatorname{Re}z<\pi$, and

$$\frac{1}{\pi(w+\bar z+(2k-1)2\pi)^2}=K_{H_{-\pi+}}(w,z+(2k-2)2\pi)$$

with $\operatorname{Re}(z+(2k-2)2\pi)\geqslant\operatorname{Re}z>-\pi$.

Moreover, the above formulas and Lemma 3 yield (for fixed $z\in D$)

$$\left\|\frac{1}{\pi(w+\bar z-(2k-1)2\pi)^2}\right\|_D\sim k^{-3/2},\qquad\left\|\frac{1}{\pi(w+\bar z+(2k-1)2\pi)^2}\right\|_D\sim k^{-3/2}.$$

It follows that both series in (24) are $L^2(D)$-convergent. Therefore (24) gives the orthogonal decomposition of Corollary 1 for the Bergman function $K_D(\cdot,z)$ in the standard strip D.

To study the operators P_{UD} and P_{UV} we shall rewrite slightly the terms in (24). To study P_{UD} we shall use the formula (with $z\in U$)

$$K_D(w,z)=\sum_{k=1}^{\infty}K_{H_{s\pi-}}(w,z-(2k-1-s)2\pi)+\sum_{k=1}^{\infty}K_{H_{s\pi+}}(w,z+(2k-2)2\pi).\tag{25}$$

(Note that $2k-1-s \geqslant 0$, hence $\operatorname{Re}(z-(2k-1-s)2\pi) < s\pi$.) To study P_{VD} we shall use the formula (with $z \in V$)

(26) $$K_D(w, z) = \sum_{k=1}^{\infty} K_{H_{\pi-}}(w, z-(2k-2)2\pi) + \sum_{k=1}^{\infty} K_{H_{s\pi+}}(w, z+(2k-1+s)2\pi).$$

(Note that $2k-1+s \geqslant 0$, hence $\operatorname{Re}(z+(2k-1+s)2\pi) > s\pi$.)

Consider $f \in L^2H(U)$ with $f = f^{(-)}+f^{(+)}$. Using (25) we find (for $z \in U$)

(27)
$$\begin{aligned} P_{UD}f^{(-)}(z) &= \int_U f^{(-)}(w) \sum_{k=1}^{\infty} \overline{K_{H_{s\pi-}}(w, z-(2k-1-s)2\pi)}\, dm(w) \\ &= \sum_{k=1}^{\infty} P_{UH_{s\pi-}} f^{(-)}(z-(2k-1-s)2\pi) \\ &= \sum_{k=1}^{\infty} [f^{(-)}(z-(2k-1-s)2\pi) - f^{(-)}(z-(2k-1-s)2\pi - 2\pi(s+1))] \\ &= \sum_{k=1}^{\infty} [f^{(-)}(z-(2k-1-s)2\pi) - f^{(-)}(z-(2k)2\pi)], \end{aligned}$$

(28)
$$\begin{aligned} P_{UD}f^{(+)}(z) &= \int_U f^{(+)}(w) \sum_{k=1}^{\infty} \overline{K_{H_{-\pi+}}(w, z+(2k-2)2\pi)}\, dm(w) \\ &= \sum_{k=1}^{\infty} P_{UH_{-\pi+}} f^{(+)}(z+(2k-2)2\pi) \\ &= \sum_{k=1}^{\infty} [f^{(+)}(z+(2k-2)2\pi) - f^{(+)}(z+(2k-2)2\pi + 2\pi(s+1))] \\ &= \sum_{k=1}^{\infty} [f^{(+)}(z+(2k-2)2\pi) - f^{(+)}(z+(2k-1+s)2\pi)]. \end{aligned}$$

Next let us consider $g \in L^2H(V)$ with $g = g^{(-)}+g^{(+)}$. Using (26) and proceeding as before we find (for $z \in V$)

(29)
$$\begin{aligned} P_{VD}g^{(-)}(z) &= \int_V g^{(-)}(w) \sum_{k=1}^{\infty} \overline{K_{H_{\pi-}}(w, z-(2k-2)2\pi)}\, dm(w) \\ &= \sum_{k=1}^{\infty} P_{VH_{\pi-}} g^{(-)}(z-(2k-2)2\pi) \\ &= \sum_{k=1}^{\infty} [g^{(-)}(z-(2k-2)2\pi) - g^{(-)}(z-(2k-2)2\pi - 2\pi(1-s))] \\ &= \sum_{k=1}^{\infty} [g^{(-)}(z-(2k-2)2\pi) - g^{(-)}(z-(2k-1-s)2\pi)], \end{aligned}$$

(30)
$$\begin{aligned} P_{VD}g^{(+)}(z) &= \int_V g^{(+)}(w) \sum_{k=1}^{\infty} \overline{K_{H_{s\pi+}}(w, z+(2k-1+s)2\pi)}\, dm(w) \\ &= \sum_{k=1}^{\infty} P_{VH_{s\pi+}} g^{(+)}(z+(2k-1+s)2\pi) \\ &= \sum_{k=1}^{\infty} [g^{(+)}(z+(2k-1+s)2\pi) - g^{(+)}(z+(2k-1+s)2\pi + 2\pi(1-s))] \\ &= \sum_{k=1}^{\infty} [g^{(+)}(z+(2k-1+s)2\pi) - g^{(+)}(z+(2k)2\pi)]. \end{aligned}$$

We can now prove the main theorem of this section.

THEOREM 6. *Assume that $s \in (-1, 1)$, $D = \{-\pi < \operatorname{Re} z < \pi\}$ and U, V are as in Fig. 2. Then:*

1° The operator P_{UD}: $L^2H(U) \to L^2H(D)$ has multiplier of the form

$$\mu(t) = \frac{1-q^{1+s}}{1-q^2}, \quad q = e^{-4\pi^2 t},\ t \neq 0.$$

2° The operator P_{VD}: $L^2H(V) \to L^2H(D)$ has multiplier of the form

$$\nu(t) = \frac{q^{1+s}-q^2}{1-q^2}, \quad q = e^{-4\pi^2 t},\ t \neq 0.$$

Proof. We shall consider only P_{UD} and statement 1°, since 2° can be obtained analogously.

We need to show that

(31) $$G_{P_{UD}f}(t) = \frac{1-q^{1+s}}{1-q^2} G_f(t)$$

for every $f \in L^2H(U)$. Consider the orthogonal decomposition $f = f^{(-)}+f^{(+)}$ of Corollary 1. It obviously suffices to prove (31) in the following two cases:

Case I: $f = f^{(+)}$. The function $P_{UD}f$ is given on U by (28) (and this will suffice to determine its Genchev transform). The terms in (28) are shifts of f restricted to various strips contained in $H_{-\pi+}$. By Remark 3 the mth partial sum of (28) has Genchev transform equal to

(32)
$$\begin{aligned} &G_f(t)\Big(\sum_{k=1}^{m} e^{-2\pi t(2k-2)2\pi} - \sum_{k=1}^{m} e^{-2\pi t(2k-1+s)2\pi}\Big) \\ &= G_f(t)\Big(\sum_{k=1}^{m} (q^2)^{k-1} - q^{1+s}\sum_{k=1}^{m} (q^2)^{k-1}\Big) = G_f(t)(1-q^{1+s})\frac{1-(q^2)^m}{1-q^2}. \end{aligned}$$

Note that (32) vanishes for $t<0$. For $t>0$ we have $q \in (0,1)$ and it is easy to see that

$$(33)\qquad (1-q^{1+s})\frac{1-(q^2)^m}{1-q^2} \leqslant \frac{1+s}{2}, \quad q \in (0,1).$$

Moreover, when m goes to infinity (32) converges pointwise to $\mu(t)G_f(t)$. It follows from the Lebesgue dominated convergence theorem that (32) converges in $L^2(\boldsymbol{R}, w_U)$ to $\mu(t)G_f(t)$. By Theorem 4 the series (28) is $L^2(U)$-convergent, the Genchev transform of its sum $P_{UD}f$ is $\mu(t)G_f(t)$, as claimed.

Case II: $f = f^{(-)}$. The function $P_{UD}f$ is given on U by (27) (and this will suffice to determine its Genchev transform). The terms in (27) are shifts of f restricted to various strips contained in $H_{s\pi-}$. By Remark 3 the mth partial sum of (27) has Genchev transform equal to

$$(34)\quad G_f(t)\Big(\sum_{k=1}^{m} e^{2\pi t(2k-1-s)2\pi} - \sum_{k=1}^{m} e^{2\pi t(2k)2\pi}\Big) = G_f(t)\frac{1-q^{1+s}}{1-q^2}\left(1-\left(\frac{1}{q^2}\right)^m\right).$$

Note that (34) vanishes for $t>0$. For $t<0$ we have $q \in (1,\infty)$ and it is easy to see that the function

$$(35)\qquad \frac{1-q^{1+s}}{1-q^2}\left(1-\left(\frac{1}{q^2}\right)^m\right)$$

is bounded by a constant independent of m (the first factor vanishes at infinity). As m goes to infinity the sequence (34) converges pointwise to $\mu(t)G_f(t)$. By the Lebesgue dominated convergence theorem, (34) converges in $L^2(\boldsymbol{R}, w_U)$ to $\mu(t)G_f(t)$. By Theorem 4 the Genchev transform of $P_{UD}f$ is $\mu(t)G_f(t)$, as claimed. ∎

We shall now treat the standard strip $D = \{-\pi < \operatorname{Re} z < \pi\}$ as the union of two strips $A = \{-\pi < \operatorname{Re} z < s\pi\}$ and $B = \{r\pi < \operatorname{Re} z < \pi\}$ $(-1 < r < s < 1)$. The strips A and B intersect along the strip $T = \{r\pi < \operatorname{Re} z < s\pi\}$. This is shown in Fig. 3.

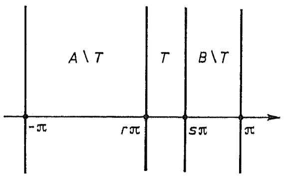


Fig. 3

It is easy to verify that the standard strip D is transformed onto the strip B by the mapping $w = \varphi(z)$, where

$$(36)\qquad \varphi(z) = \frac{1-r}{2}z + \frac{1+r}{2}\pi,$$

and that the line $\operatorname{Re} z = \varrho\pi$ $(\varrho = (2s-r-1)/(1-r))$ is transformed by (36) onto the line $\operatorname{Re} w = s\pi$. From Theorem 6 and Corollary 4 (with $D_1 = D \cap H_{\varrho-}$ or $D \cap H_{\varrho+}$, $D_2 = D$, and $\tilde{D}_1 = T$ or V, $\tilde{D}_2 = B$) follows

COROLLARY 5. *Assume that $-1 < r < s < 1$ and that A, B, T and $D = A \cup B$ are as in Fig. 3. Write $V = \operatorname{int} B \setminus T$. Then*

1° *The operator P_{TB}: $L^2H(T) \to L^2H(B)$ has multiplier*

$$\tilde{\mu}(t) = \frac{1-q^{s-r}}{1-q^{1-r}}, \quad q = e^{-4\pi^2 t},\ t \neq 0.$$

2° *The operator P_{VB}: $L^2H(V) \to L^2H(B)$ has multiplier*

$$\tilde{\nu}(t) = \frac{q^{s-r}-q^{1-r}}{1-q^{1-r}}, \quad q = e^{-4\pi^2 t},\ t \neq 0.$$

Proof. In view of Theorem 6 the operators P_{D_1D} and P_{D_2D} have multipliers

$$\mu(t) = \frac{1-q^{1+\varrho}}{1-q^2} = \frac{1-(q^2)^{(s-r)/(1-r)}}{1-q^2},$$

$$\nu(t) = \frac{q^{1+\varrho}-q^2}{1-q^2} = \frac{(q^2)^{(s-r)/(1-r)}-q^2}{1-q^2}.$$

We shall apply Corollary 4 with the mapping (36). Since $k = (1-r)/2$ we find that

$$q(kt) = e^{-4\pi^2(1-r)t/2} = q(t)^{(1-r)/2}, \quad q(kt)^2 = q(t)^{1-r}.$$

Therefore

$$\tilde{\mu}(t) = \mu(kt) = \frac{1-(q^{1-r})^{(s-r)/(1-r)}}{1-q^{1-r}} = \frac{1-q^{s-r}}{1-q^{1-r}},$$

$$\tilde{\nu}(t) = \nu(kt) = \frac{(q^{1-r})^{(s-r)/(1-r)}-q^{1-r}}{1-q^{1-r}} = \frac{q^{s-r}-q^{1-r}}{1-q^{1-r}}.$$

The proof is complete.

6. L^2-angle between two strips. We shall determine the L^2-angle between the strips $A = \{-\pi < \operatorname{Re} z < s\pi\}$ and $B = \{r\pi < \operatorname{Re} z < \pi\}$, where $-1 < r < s < 1$ (see Fig. 3). Since the boundary of $T = A \cap B$ has plane measure zero we can replace $B \setminus T$ by $V = \operatorname{int} B \setminus T$ in formula (6). Our first task is to characterize the functions $f \in L^2(D)$ which are admissible in (6), or equivalently to characterize the admissible pairs $f_A \in L^2H(A)$, $f_V \in L^2H(V)$. The Genchev transform of the Bergman projection of f can be easily computed from Theorem 6 (note that $A = U$). It is equal to

$$(37)\qquad \frac{1-q^{1+s}}{1-q^2}G_{f_A} + \frac{q^{1+s}-q^2}{1-q^2}G_{f_V}.$$

Therefore $f \perp L^2 H(D)$ implies that

$$G_{f_V}(t) = \frac{q^{1+s}-1}{q^{1+s}-q^2} G_{f_A}(t). \tag{38}$$

We see that if f is admissible in (6) then $f_A \neq 0$, and f_V is given by (38). Setting $h = G_{f_A}$ we have the following

COROLLARY 6. *The functions $f \in L^2(D)$ admissible in (6) are in a one-to-one correspondence with the functions $h \in L^2(\mathbf{R}, w_A)\setminus\{0\}$ which satisfy the condition*

$$h\frac{q^{1+s}-1}{q^{1+s}-q^2} \in L^2(\mathbf{R}, w_V). \tag{39}$$

This correspondence is given by

$$G_{f_A} = h, \qquad G_{f_V} = h\frac{q^{1+s}-1}{q^{1+s}-q^2}. \tag{40}$$

The above result implies immediately

COROLLARY 7. *Every bounded measurable function which vanishes outside a compact subset of the real line corresponds to a unique function $f \in L^2(D)$ admissible in (6).*

We now assume that an element $h \in L^2(\mathbf{R}, w_A)\setminus\{0\}$ corresponds to an admissible function f, and proceed to determine the value of the expression under the supremum sign in (6). We start with writing down the corresponding weights:

$$\begin{aligned}
w_{A\setminus T}(t) &= \frac{e^{4\pi^2 t}-e^{-4\pi^2 rt}}{4\pi t} = \frac{q^{-1}-q^r}{4\pi t},\\
w_A(t) &= \frac{e^{4\pi^2 t}-e^{-4\pi^2 st}}{4\pi t} = \frac{q^{-1}-q^s}{4\pi t},\\
w_B(t) &= \frac{e^{-4\pi^2 rt}-e^{-4\pi^2 t}}{4\pi t} = \frac{q^r-q}{4\pi t},\\
w_V(t) &= \frac{e^{-4\pi^2 st}-e^{-4\pi^2 t}}{4\pi t} = \frac{q^s-q}{4\pi t}.
\end{aligned} \tag{41}$$

The Genchev transform of $\hat{f}$ is found using (40) and Corollary 5 as follows:

$$\begin{aligned}
G_{\hat{f}}(t) &= h(t)\tilde{\mu}(t) + h(t)\frac{q^{1+s}-1}{q^{1+s}-q^2}\tilde{\nu}(t)\\
&= h\left(\frac{1-q^{s-r}}{1-q^{1-r}} + \frac{q^{1+s}-1}{q^{s+1}-q^2}\cdot\frac{q^{-1-r}(q^{s+1}-q^2)}{1-q^{1-r}}\right)\\
&= \frac{h}{1-q^{1-r}}(1-q^{s-r}+q^{s-r}-q^{-1-r}) = h\frac{q^{1+r}-1}{q^{1+r}(1-q^{1-r})},
\end{aligned}$$

and Theorem 4 yields

$$\|\hat{f}\|_B^2 = \int_{-\infty}^{\infty} |h|^2 \left(\frac{q^{1+r}-1}{q^{1+r}(1-q^{1-r})}\right)^2 w_B\,dt. \tag{42}$$

Using (40) we find in a similar way

$$\|f\|_V^2 = \int_{-\infty}^{\infty} |h|^2 \left(\frac{q^{1+s}-1}{q^{1+s}-q^2}\right)^2 w_V\,dt. \tag{43}$$

Since $G_f = h$ we also have

$$\|f\|_{A\setminus T}^2 = \int_{-\infty}^{\infty} |h|^2 w_{A\setminus T}\,dt, \qquad \|f\|_A^2 = \int_{-\infty}^{\infty} |h|^2 w_A\,dt. \tag{44}$$

Combining (42)–(44) with (41) we see that the expression under the supremum sign in (6) is

$$\frac{\displaystyle\int_{-\infty}^{\infty} \frac{|h|^2}{4\pi t}\left(q^{-1}-q^r+\left(\frac{1-q^{1+r}}{q^{1+r}(1-q^{1-r})}\right)^2 q^r(1-q^{1-r})\right)dt}{\displaystyle\int_{-\infty}^{\infty} \frac{|h|^2}{4\pi t}\left(q^{-1}-q^s+\left(\frac{1-q^{1+s}}{q^{1+s}(1-q^{1-s})}\right)^2 q^s(1-q^{1-s})\right)dt}. \tag{45}$$

Note that

$$\begin{aligned}
&q^{-1}-q^r+\left(\frac{1-q^{1+r}}{q^{1+r}(1-q^{1-r})}\right)^2 q^r(1-q^{1-r}) \qquad (46)\\
&\qquad = q^{-1}(1-q^{1+r}) + \frac{(1-q^{1+r})^2}{1-q^{1-r}} q^{-2-r}\\
&\qquad = \frac{q^{-1}(1-q^{1+r})}{1-q^{1-r}}(1-q^{1-r}+(1-q^{1+r})q^{-1-r}) = \frac{q^{-2-r}(1-q^2)(1-q^{1+r})}{1-q^{1-r}}.
\end{aligned}$$

Denote by $F_{r,s}(q)$ the ratio of the (positive) integrands in (45). The following result is obvious.

LEMMA 6. *Assume that $-1 < r < s < 1$. Then the function*

$$F_{r,s}(q) = q^{s-r}\frac{1-q^{1+r}}{1-q^{1-r}}\cdot\frac{1-q^{1-s}}{1-q^{1+s}} = \frac{(q^{s-1}-1)(1-q^{1+r})}{(1-q^{1+s})(q^{r-1}-1)}, \qquad q > 0, \tag{47}$$

satisfies

$$\lim_{q\to 1} F_{r,s}(q) = \frac{1-s}{1+s}\cdot\frac{1+r}{1-r}, \tag{48}$$

$$\lim_{q\to 0} F_{r,s}(q) = 0, \qquad \lim_{q\to\infty} F_{r,s}(q) = 0. \tag{49}$$

It is now evident that $F_{r,s}(q)$ assumes its largest value at some point $q_0 = e^{-4\pi^2 t_0}$. This yields

THEOREM 7. *The L^2-angle $\beta(r, s)$ between the strips $A = \{-\pi < \operatorname{Re} z < s\pi\}$ and $B = \{r\pi < \operatorname{Re} z < \pi\}$ is given by*

$$\beta(r, s) = \arccos\big(\max_{q>0} F_{r,s}(q)\big)^{1/2} \quad (-1 < r < s < 1), \tag{50}$$

where $F_{r,s}(q)$ is defined by (47).

Proof. From (6) and (45) it is clear that $\cos^2\beta(r, s)$ is not greater than $F_{r,s}(q_0)$. On the other hand, for every $n = 1, 2, \ldots$ the characteristic function

$$h_n(t) = \chi_{[q_0-1/n, q_0+1/n]}(t), \quad t \in (-\infty, \infty), \tag{51}$$

corresponds (in view of Corollary 7) to an admissible function $f_n \in L^2(D)$. Substituting $h = h_n$ in (45) for $n = 1, 2, \ldots$ we see that the limit of (45) as n goes to infinity is $F_{r,s}(q_0)$. Hence $\cos^2\beta(r, s)$ is not smaller than $F_{r,s}(q_0)$. This completes the proof.

7. Further properties of $F_{r,s}(q)$, $q > 0$. From (47) it follows immediately that

$$F_{r,s}(q) = G_s(q)/G_r(q) \tag{52}$$

where

$$G_s(q) = \frac{q^s - q}{1 - q^{1+s}}. \tag{53}$$

The study of (52) will be reduced to that of (53). Note that $G_s(q)$ has interesting symmetries:

$$G_{-s}(q) = \frac{q^{-s} - q}{1 - q^{1-s}} = \frac{1 - q^{1+s}}{q^s - q} = G_s(q)^{-1}, \tag{54}$$

$$G_s\left(\frac{1}{q}\right) = \frac{q^{-s} - q^{-1}}{1 - q^{-1-s}} = \frac{q - q^s}{q^{1+s} - 1} = G_s(q). \tag{55}$$

We would now like to study the graph of $G_s(q)$. The case $G_0(q) \equiv 1$ is obvious. In view of the symmetry (54) we may therefore assume that $s \in (0, 1)$.

LEMMA 7. *Assume that* $s \in (0, 1)$. *Then*

$$\lim_{q\to 0} G_s(q) = 0, \quad \lim_{q\to 1} G_s(q) = \frac{1-s}{1+s}, \quad \lim_{q\to\infty} G_s(q) = 0,$$

$$\lim_{q\to 1} G_s'(q) = \infty, \quad \lim_{q\to 1} G_s'(q) = 0, \quad \lim_{q\to\infty} G_s'(q) = 0.$$

We omit the easy proof.

We would like to show that for $s \in (0, 1)$ the function $G_s(q)$ is increasing in $(0, 1)$ and decreasing in $(1, \infty)$. In view of the symmetry (55) it suffices to consider the interval $(0, 1)$. Also it suffices to consider the case when $s = k/n$ is rational. After the monotonic change of variable $q = p^n$, $p \in (0, 1)$, the problem is reduced to the monotonicity of the function

$$W(p) = \frac{p^k - p^n}{1 - p^{n+k}}. \tag{56}$$

Introduce $m = n + k - 1$; then (56) can be rewritten as

$$W(p) = \frac{p^k + p^{k+1} + \ldots + p^{m-k}}{1 + p + \ldots + p^{m-1} + p^m}. \tag{57}$$

LEMMA 8. *The function* (57) *is increasing for* $p \in (0, 1)$.

Proof. We shall show that $W'(p) \geqslant 0$. Note that $W(p) = u/(u+v)$ with $u = p^k + p^{k+1} + \ldots + p^m$. Therefore

$$W'(p) = \frac{u'v - uv'}{(u+v)^2}. \tag{58}$$

The main idea is to introduce three variable indices

$$i \in \{0, \ldots, k-1\}, \quad j \in \{k, \ldots, m-k\}, \quad l \in \{m-k+1, \ldots, m\}.$$

We observe that the differences $j - i$ (with repetitions) are the same as the differences $l - j$ (in the sense that there is a one-to-one correspondence $(j, i) \to (j', l)$ such that $j - i = l - j'$). Furthermore, the numerator in (58) is equal to

$$\begin{aligned} \text{Num } W'(p) &= \Big(\sum_j p^j\Big)'\Big(\sum_i p^i + \sum_l p^l\Big) - \Big(\sum_j p^j\Big)\Big(\sum_i p^i + \sum_l p^l\Big)' \\ &= \sum_{j,i} j p^{j+i-1} + \sum_{j,l} j p^{j+l-1} - \sum_{j,i} i p^{j+i-1} - \sum_{j,l} l p^{j+l-1} \\ &= \sum_{j,i} (j-i)\, p^{j+i-1} - \sum_{j,l} (l-j)\, p^{j+l-1}. \end{aligned} \tag{59}$$

Consider a pair of indices j, i and a pair of indices j', l such that $j - i = l - j'$. Note that

$$(j-i)p^{j+i-1} - (l-j')p^{l+j'-1} = (j-i)(p^{j+i-1} - p^{j'+l-1}). \tag{60}$$

The factor $j - i$ is positive; moreover,

$$(j' + l - 1) - (j + i - 1) = (l - j + i) + l - j - i = 2(l - j) > 0.$$

Since $p \in (0, 1)$ we see that (60) is positive. Therefore (59) is a sum of positive terms, and $W(p)$ is increasing. ∎

As noticed above Lemma 8 yields

COROLLARY 8. *For* $s \in (0, 1)$ *the function*

$$G_s(q) = \frac{q^s - q}{1 - q^{1+s}}$$

is increasing in the variable $q \in (0, 1)$.

We now pass to the function $F_{r,s}(q)$. It has similar symmetries to those of $G_s(q)$, namely

$$F_{-s,-r}(q) = \frac{G_{-r}(q)}{G_{-s}(q)} = \frac{G_r(q)^{-1}}{G_s(q)^{-1}} = \frac{G_s(q)}{G_r(q)} = F_{r,s}(q), \tag{61}$$

$$F_{r,s}\left(\frac{1}{q}\right) = \frac{G_s(1/q)}{G_r(1/q)} = \frac{G_s(q)}{G_r(q)} = F_{r,s}(q). \tag{62}$$

THEOREM 8. *For $-1 < r < s < 1$ the function $F_{r,s}(q)$ is increasing in $(0, 1)$ and decreasing in $(1, \infty)$. In particular,*

$$\max_{q>0} F_{r,s}(q) = F_{r,s}(1) = \frac{1-s}{1+s}\cdot\frac{1+r}{1-r}. \tag{63}$$

Proof. With no loss of generality we may assume that $s \in (0, 1)$. Indeed, if this is not the case then $-r > 0$ and in view of (61) we may study the function $F_{-s,-r}(q)$. The case when $s \in (0, 1)$ and $r \in (-1, 0)$ is easy, since then

$$F_{r,s}(q) = G_s(q)G_{-r}(q)$$

and by Corollary 8 both factors on the right are increasing in $(0, 1)$ and decreasing in $(1, \infty)$. The case when $s \in (0, 1)$ and $r = 0$ reduces to Corollary 8. It remains to consider the case when $0 < r < s < 1$.

In view of the symmetry (62) it suffices to show that $F_{r,s}(q)$ is increasing for $q \in (0, 1)$. We may also assume with no loss of generality that $r = j/n$, $s = k/n$ are rational numbers and $k \equiv j \pmod 2$. After the monotonic change of variable $q = p^n$, $p \in (0, 1)$, we have to prove that $U(p)$ is increasing, where

$$\begin{aligned} U(p) &= \frac{p^k - p^n}{1-p^{k+n}} : \frac{p^j - p^n}{1-p^{j+n}} \\ &= \frac{p^k + \ldots + p^{n-1}}{1+p+\ldots+p^{n+k-1}} : \frac{p^j + \ldots + p^{n-1}}{1+p+\ldots+p^{n+j-1}} \\ &= \frac{p^{k-j} + \ldots + p^{n-j-1}}{1+p+\ldots+p^{n-j-1}} \cdot \frac{1+p+\ldots+p^{n+j-1}}{1+p+\ldots+p^{n+k-1}}. \end{aligned} \tag{64}$$

By assumption $k-j = 2l$ is a positive even integer. On the right side of (64) we multiply the second factor, and divide the first, by p^l. Hence

$$U(p) = \frac{p^{k-j-l} + \ldots + p^{n-j-1-l}}{1+p+\ldots+p^{n-j-1}} \cdot \frac{p^l + p^{l+1} + \ldots + p^{n+j-1+l}}{1+p+\ldots+p^{n+k-1}}. \tag{65}$$

Note that $(n-j-1)-(n-j-1-l) = l = k-j-l$, and that $(n+k-1)-(n+j-1+l) = l$. Thus both factors in (65) are of the form (57), hence are increasing for $p \in (0, 1)$. This shows that $U(p)$ is increasing, as a product of two positive increasing functions. ∎

The above theorem and Theorem 7 yield our main result

THEOREM 9. *The L^2-angle $\beta(r, s)$ between the strips $A = \{-\pi < \operatorname{Re} z < s\pi\}$ and $B = \{r\pi < \operatorname{Re} z < \pi\}$ is given by*

$$\beta(r, s) = \arccos\left(\frac{1-s}{1+s}\cdot\frac{1+r}{1-r}\right)^{1/2} \quad (-1 < r < s < 1). \tag{66}$$

Remark 4. The idea used in the proof of the monotonicity of $F_{r,s}(q)$ $(0 < r < s < 1)$ in the interval $(0, 1)$ leads to the following direct representation of $F_{r,s}(q)$ as a product of two increasing functions:

$$F_{r,s}(q) = G_{c/(2-d)}(q^{1-d/2})G_{c/(2+d)}(q^{1+d/2}) \quad (0 < r < s < 1), \tag{67}$$

where $c = s-r$, $d = s+r$. (Hint: consider rational parameters $r = j/n$, $s = k/n$ such that $j \equiv k \pmod 2$). As $r \to 0$ this yields another formula

$$G_s(q) = G_{s/(2-s)}(q^{1-s/2})G_{s/(2+s)}(q^{1+s/2}) \quad (0 < s < 1) \tag{68}$$

which can be verified by immediate calculations.

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Added in proof. In addition to [2], [4] one should consult

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