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Algebraic generation of $B(X)$ by two subalgebras with square zero

by

W. ŻELAZKO (Warszawa)

Abstract. Let X be a real or complex Banach space, $\dim X > 1$. The main result of this paper states that X is decomposable into a direct sum of two mutually isomorphic (closed) subspaces if and only if the algebra $B(X)$ is algebraically generated by two subalgebras of square zero, one of them being of dimension one. We also show that if X is a direct sum of n mutually isomorphic (closed) subspaces, then the algebra $B(X)$ is algebraically generated by two subalgebras of square zero. This extends our previous result [9] concerning the case when X is a Hilbert space.

This paper is a contribution towards solving the problem whether for every (real or complex) Banach space X the algebra $B(X)$ of all its continuous endomorphisms is generated by two of its abelian subalgebras. We say that the algebra $B(X)$ is *τ-generated* by its subset S if it coincides with the smallest τ-closed subalgebra of $B(X)$ containing S. Here τ denotes some topology on $B(X)$. When τ is the norm topology of $B(X)$ we simply say that S *generates* $B(X)$, when τ is the discrete topology we say that S *algebraically generates* $B(X)$. Thus S algebraically generates $B(X)$ if each operator T in $B(X)$ is a linear combination of finite products of elements of S. $B(X)$ is τ-generated by two abelian subalgebras $\mathscr{A}_1$ and $\mathscr{A}_2$ if it is τ-generated by the set $S = \mathscr{A}_1 \cup \mathscr{A}_2$.

For a subset S of $B(X)$ put $S^2 = \{T_1 T_2 \colon T_1, T_2 \in S\}$; thus a subalgebra $\mathscr{A} \subset B(X)$ of square zero is automatically commutative.

In [9] we have observed that for a real or complex Hilbert space H, $\dim H > 1$, the algebra $B(H)$ is algebraically generated by two of its subalgebras of square zero. More precisely, we have shown that if the dimension of H is even (all infinities are even), then there is a subalgebra $\mathscr{A}_0 \subset B(H)$ and an operator $V \in B(H)$, both of square zero, such that each operator T in $B(H)$ can be written in the form

$$T = R_1 + R_2 V + V R_3 + V R_4 V \tag{1}$$

with R_i in $\mathscr{A}_0$, $i = 1, 2, 3, 4$. Thus $B(H)$ is algebraically generated by two subalgebras of square zero, since clearly the one-dimensional subspace of $B(H)$ spanned by V is such a subalgebra.

In the case when the dimension of H is odd the algebra $B(H)$ is also algebraically generated by two subalgebras of square zero, but it cannot be claimed this time that one of them is of dimension one.

In the present note we observe that the formula (1) depends upon the decomposability of the Hilbert space into a direct sum of two isomorphic closed subspaces, rather than upon the Hilbert space structure. More interesting is the converse result stating that if for some Banach space X the algebra $B(X)$ is algebraically generated by a subalgebra and by an operator, both of square zero, then the space X is itself a "square", i.e. it can be written as a direct sum of two isomorphic closed subspaces. We also show that if X is an "nth power", $n > 1$, then $B(X)$ is algebraically generated by two subalgebras of square zero, and give some related propositions.

In the sequel we shall use the following simple fact (cf. [1], Ch. I, § 3, Corollary 4): If X_1 and X_2 are closed linear subspaces of a Banach space X such that $X_1 \cap X_2 = (0)$ and $\operatorname{span}(X_1 \cup X_2) = X$, then X is the (topological) direct sum of X_1 and X_2, i.e. each element x in X can be uniquely written in the form $x = x_1 + x_2$, $x_i \in X_i$, and the projections $P_i(x) = x_i$, $i = 1, 2$, are continuous linear operators. In this case we write $X = X_1 \oplus X_2$.

PROPOSITION 1. *Let X be a real or complex Banach space and suppose $X = X_1 \oplus X_2$ where X_1 and X_2 are isomorphic (i.e. linearly homeomorphic) closed linear subspaces of X. Then there is a subalgebra $\mathscr{A}_0 \subset B(X)$ with square zero and an operator V in $B(X)$, $V^2 = 0$, such that any element T in $B(X)$ can be written in the form* (1) *with R_i in $\mathscr{A}_0$.*

Proof. For a closed linear subspace $X_0 \subset X$ put

(2) $$\mathscr{A}_0(X_0) = \{T \in B(X) \colon \operatorname{im} T \subset X_0 \subset \ker T\}.$$

This is clearly a closed subalgebra of $B(X)$ with square zero.

Let L be a linear homeomorphism of X_1 onto X_2 and put

$$Vx = \begin{cases} Lx & \text{for } x \in X_1, \\ 0 & \text{for } x \in X_2. \end{cases}$$

This is a well-defined element of $B(X)$ and $V^2 = 0$.

We also put

$$Ux = \begin{cases} 0 & \text{for } x \in X_1, \\ L^{-1}x & \text{for } x \in X_2. \end{cases}$$

Setting $\mathscr{A}_0 = \mathscr{A}_0(X_1)$ we have clearly $U \in \mathscr{A}_0$, and also

(3) $$UTU \in \mathscr{A}_0$$

for an arbitrary T in $B(X)$. This follows immediately from $\operatorname{im} UTU \subset \operatorname{im} U$ and $\ker UTU \supset \ker U$. The decomposition $X = X_1 \oplus X_2$ implies $I = P_1 + P_2$ where I is the identity operator and P_i is the corresponding projection of X

onto X_i, $i = 1, 2$. Using the obvious relations $P_1 = UV$ and $P_2 = VU$ we can write an arbitrary operator T in $B(X)$ in the form

(4) $$T = (UV+VU)T(VU+UV) = UVTVU + UVTUV + VUTVU + VUTUV.$$

Setting $R_1 = UVTVU$, $R_2 = UVTU$, $R_3 = UTVU$ and $R_4 = UTU$ we see by (3) that all operators R_i are in $\mathscr{A}_0$, and (1) follows.

Remark 2. If we put $Q = U + V$, then in (1) we can replace V by Q (because U is in $\mathscr{A}_0$ and $\mathscr{A}_0^2 = (0)$). The operator Q satisfies

$$Q^2 = U^2 + UV + VU + V^2 = I.$$

In the case when X is an even-dimensional Hilbert space we can take as V a partial isometry with initial space X_1 and final space $X_2 = X_1^{\perp}$. In this case we have $U = V^*$ and Q is a hermitian operator (cf. [9]) generating a two-dimensional subalgebra of $B(X)$. Thus in this case $B(X)$ can be generated by a subalgebra of square zero and a selfadjoint commutative subalgebra. We do not know whether $B(H)$ can be generated by two selfadjoint commutative subalgebras.

Somewhat surprising is the fact that the converse of Proposition 1 also holds true. To formulate the result we need the following notation. For a nonvoid subset $S \subset B(X)$ put

$$\ker S = \bigcap \{\ker T\colon T \in S\}, \qquad \operatorname{im} S = \operatorname{span}\left(\bigcup \{\operatorname{im} T\colon T \in S\}\right).$$

Thus $\ker S$ is a closed linear subspace of X and $\operatorname{im} S$ is a linear, but not necessarily closed subspace of X.

PROPOSITION 3. *Suppose that for a real or complex Banach space X the algebra $B(X)$ is algebraically generated by a subalgebra $\mathscr{A}_0$ and an operator V in $B(X)$, both of square zero. Then:*

(i) *There is a direct sum decomposition*

$$X = X_1 \oplus X_2,$$

where $X_1 = \ker \mathscr{A}_0$ and $X_2 = \ker V$.

(ii) *The closed subspaces X_1 and X_2 are isomorphic and V maps homeomorphically X_1 onto X_2. In particular, $X_2 = \operatorname{im} V$.*

(iii) $\operatorname{im} \mathscr{A}_0 = \ker \mathscr{A}_0$.

Proof. In order to prove (i) we have to show

$$\text{(a) } \operatorname{span}(X_1 \cup X_2) = X, \qquad \text{(b) } X_1 \cap X_2 = (0).$$

Observe first that setting $Y_1 = \operatorname{im} \mathscr{A}_0$ and $Y_2 = \operatorname{im} V$ we have

(5) $$Y_1 \subset X_1 \quad \text{and} \quad Y_2 \subset X_2.$$

The first relation follows from the fact that $\operatorname{im} T_1 \subset \ker T_2$ for all T_1 and T_2 in $\mathscr{A}_0$, and the second is straightforward.

Since $\mathscr{A}_0$ and V algebraically generate $B(X)$ we can write the identity operator I in the form

$$I = VT_0 + \sum_{i=1}^{n} R_i T_i$$

with $R_i \in \mathscr{A}_0$ and $T_i \in B(X)$. The image of the left-hand operator, which is X, equals the image of the right-hand operator, which is contained in $\operatorname{span}(Y_1 \cup Y_2)$. Thus $\operatorname{span}(Y_1 \cup Y_2) = X$, which together with the inclusions (5) implies (a). Similarly, to prove (b) write the operator I in the form

$$I = T_0 V + \sum_{i=1}^{n} T_i R_i$$

with $R_i \in \mathscr{A}_0$ and $T_i \in B(X)$. This implies $(0) = \ker I \supset \ker V \cap \ker \mathscr{A}_0 = X_2 \cap X_1$, and so (b) follows.

Now, having proved (i) we can write the projection P_2 of X onto X_2 in the form

$$P_2 = VT_0 + \sum_{i=1}^{n} R_i T_i \quad (T_i \in B(X),\ R_i \in \mathscr{A}_0).$$

Since $\operatorname{im} P_2 = X_2$ and $\operatorname{im} \sum_{i=1}^{n} R_i T_i \subset Y_1 \subset X_1$, we have $P_2 = VT_0$. This implies $\operatorname{im} V \supset \operatorname{im} P_2 = X_2$. By (5) we obtain $\operatorname{im} V = X_2$. Since $\ker V = X_2$, this implies that the restriction $V|X_1$ is a one-to-one map of X_1 onto X_2 and so (ii) follows.

In order to prove (iii) write the projection P_1 of X onto X_1 in the form

$$P_1 = VT_0 + \sum_{i=1}^{n} R_i T_i \quad (T_i \in B(X),\ R_i \in \mathscr{A}_0).$$

As before this implies $P_1 = \sum_{i=1}^{n} R_i T_i$, and so $X_1 = \operatorname{im} P_1 = \operatorname{im} \sum_{i=1}^{n} R_i T_i \subset \operatorname{im} \mathscr{A}_0 = Y_1 \subset X_1$, and (iii) follows.

Propositions 1 and 3 immediately imply the following.

THEOREM. 4. *A real or complex Banach space X is a "square" (i.e. it has a direct sum decomposition $X_1 \oplus X_2$ with X_1 and X_2 isomorphic) if and only if the algebra $B(X)$ is algebraically generated by two subalgebras of square zero, one of them being of dimension one.*

Remark 5. Relation (iii) of Proposition 3 implies that under the assumptions of that proposition we have $\mathscr{A}_0 \subset \mathscr{A}_0(X_1)$. However, as we see in the following example, we cannot claim that the assumption of Proposition 3 implies $\mathscr{A}_0 = \mathscr{A}_0(X_1)$.

EXAMPLE 6. Put $X = \mathbf{R}^4$ (or $\mathbf{C}^4$) and decompose $X = X_1 \oplus X_2$, where $X_1 = \operatorname{span}(e_1, e_2)$, $X_2 = \operatorname{span}(e_3, e_4)$, with $e_1, \ldots, e_4$ the standard basis in X. Setting

$$A = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad B = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad C = \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad V = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}$$

we see that $\mathscr{A}_0(X_1) = \operatorname{span}(A, B, C, D)$, and setting $\mathscr{A}_0 = \operatorname{span}(A, B)$ we obtain a proper subalgebra of $\mathscr{A}_0(X_1)$. The relations $C = BVA$ and $D = AVB$ together with Proposition 1 show that $\mathscr{A}_0$ and V algebraically generate $B(X)$, while $\mathscr{A}_0 \neq \mathscr{A}_0(X_1)$.

In the above example we cannot claim that (1) holds. In fact, it must fail as follows from the following

PROPOSITION 7. *Under the assumptions and notation of Proposition 3 the following are equivalent:*

(i) $\mathscr{A}_0 = \mathscr{A}_0(X_1)$.

(ii) *For each T in $B(X)$ there are elements $R_1, \ldots, R_4 \in \mathscr{A}_0$ such that* (1) *holds.*

Moreover, if (i) *or* (ii) *is satisfied, then the operators R_i in* (1) *are uniquely determined by T and depend upon it in a continuous way.*

Proof. The implication (i) $\Rightarrow$ (ii) follows readily from Propositions 3 and 1, or rather from the proof of the latter.

Assume now (ii). First we show that the operators $R_1, \ldots, R_4$ are uniquely determined by T. If not, then taking for some T two different representations in the form (1) and substracting we obtain

$$R_1 + R_2 V + VR_3 + VR_4 V = 0. \tag{6}$$

Thus it is sufficient to show that (6) implies $R_i = 0$, $i = 1, \ldots, 4$. By (6), $VR_3 + VR_4 V = -R_1 - R_2 V$. Since the image of the left-hand operator is contained in X_2 and that of the right-hand one is contained in X_1 (cf. notation of Proposition 3) the relation $X_1 \cap X_2 = (0)$ implies $VR_3 + VR_4 V = 0$ and $R_1 + R_2 V = 0$. Now $VR_3 = -VR_4 V$ implies $VR_3 = 0$ and $VR_4 V = 0$ since the kernels of the left- and right-hand sides span together the whole of X. One can easily see that the latter equalities imply $R_3 = R_4 = 0$. Similarly $R_1 = -R_2 V$ implies $R_1 = R_2 = 0$.

In order to prove (i) assume that $\mathscr{A}_0 \neq \mathscr{A}_0(X_1)$ and try to get a contradiction. By Remark 5 we have $\mathscr{A}_0 \subset \mathscr{A}_0(X_1)$ and so there is some $R \in \mathscr{A}_0(X_1) \backslash \mathscr{A}_0$. By (ii) we can write $R = R_1 + R_2 V + VR_3 + VR_4 V$ with $R_1, \ldots, R_4 \in \mathscr{A}_0$. Since $R_i \in \mathscr{A}_0(X_1)$, $i = 1, \ldots, 4$, the same representation works as well if we replace $\mathscr{A}_0$ by $\mathscr{A}_0(X_1)$. But then there is another representation $R = R$. This contradicts the already proved uniqueness of R_i in (1) and so (i) follows. If we have (i) or (ii), then the construction of R_i in Proposition 1 shows that these operators depend continuously upon T. The conclusion follows.

We now prove that if the space X is an "nth power" for some $n > 1$, then $B(X)$ is also algebraically generated by two subalgebras with square zero. We propose here one of many possible constructions.

Proposition 8. *Let X be a real or complex Banach space and suppose that X can be decomposed into a direct sum of closed linear subspaces*

$$X = X_0 \oplus \ldots \oplus X_n, \quad n \geqslant 1,$$

with the X_i all isomorphic to one another. Then the algebra $B(X)$ is algebraically generated by two subalgebras with square zero.

Proof. Denote by L_i a linear homeomorphism of X_0 onto X_i, $i = 1, \ldots, n$, and put

$$V_i x = \begin{cases} L_i x & \text{for } x \in X_0, \\ 0 & \text{for } x \in X_1 \oplus \ldots \oplus X_n, \end{cases}$$

$$U_i x = \begin{cases} L_i^{-1} & \text{for } x \in X_i, \\ 0 & \text{for } x \in Z_i, \end{cases}$$

where $Z_i = \bigoplus_{0 \leqslant j \leqslant n, j \neq i} X_j$, $i = 1, \ldots, n$.

We check easily that the operators $P_i = V_i U_i$, $i = 1, \ldots, n$, are the projections of X onto X_i, corresponding to the direct sum decomposition $X = X_0 \oplus \ldots \oplus X_n$, and $P_0 = U_j V_j$ is the projection of X onto X_0 for any j, $1 \leqslant j \leqslant n$. We have $V_i V_j = 0$ for $1 \leqslant i, j \leqslant n$ and $\mathscr{A}_1 = \operatorname{span}(V_1, \ldots, V_n)$ is an n-dimensional subalgebra of $B(X)$ with square zero. We shall show that $\mathscr{A}_1$ and the subalgebra given by $\mathscr{A}_0 = \mathscr{A}_0(X_0)$ (see (2)) algebraically generate the whole of $B(X)$. In fact, just as in Proposition 1, we can write any operator T in $B(X)$ in the form

$$T = \Big(\sum_{i=0}^{n} P_i\Big) T \Big(\sum_{i=0}^{n} P_i\Big) = \Big(U_1 V_1 + \sum_{i=1}^{n} V_i U_i\Big) T \Big(U_1 V_1 + \sum_{j=1}^{n} V_j U_j\Big)$$

$$= U_1 V_1 T \sum_{i=1}^{n} V_i U_i + \sum_{i=1}^{n} V_i U_i T U_1 V_1 + \sum_{i,j=1}^{n} V_i U_i T V_j U_j + U_1 V_1 T U_1 V_1$$

$$= R_0 + \sum_{i=1}^{n} V_i R_i V_1 + \sum_{i=1}^{n} V_i R_{i+n} + R_{2n+1} V_1,$$

where $R_0 = U_1 V_1 T \sum_i V_i U_i$, $R_i = U_i T U_1$, $R_{n+i} = \sum_j U_i T V_j U_j$ and $R_{2n+1} = U_1 V_1 T U_1$, $i = 1, \ldots, n$. Just as in Proposition 1 we show that $R_i \in \mathscr{A}_0$, $0 \leqslant i \leqslant 2n+1$, and so the conclusion follows.

We do not know whether a converse result is true, in particular, whether the fact that $B(X)$ is algebraically generated by two subalgebras of square zero, one of them being n-dimensional, implies that $X = X_0 \oplus \ldots \oplus X_n$, with the X_i isomorphic to one another. Similarly to (i) of Proposition 3 we can prove

Proposition 9. *Suppose that for a real or complex Banach space X the algebra $B(X)$ is algebraically generated by two subalgebras $\mathscr{A}_0$ and $\mathscr{A}_1$ with square zero. Then there is a direct sum decomposition $X = X_0 \oplus X_1$ into closed linear subspaces, where $X_0 = \ker \mathscr{A}_0$ and $X_1 = \ker \mathscr{A}_1$.*

Since many Banach spaces are "squares" or "nth powers", Propositions 1 and 8 show that for these spaces the algebra $B(X)$ is algebraically generated by two subalgebras of square zero. It is so for $X = L_p(\Omega, \Sigma, \mu)$, $C(K)$ for a metrizable compact K, the disc algebra, $C^{(k)}[0, 1]$ and several other spaces. However, it is not known in general when a Banach space is a "square", or an "nth power". Certainly the James space J ([3]), or the space $C(\Gamma_{\omega_1})$, where Γ_{ω_1} is the compact space of all ordinal numbers not greater than ω_1, the first uncountable ordinal ([6]), cannot be "nth powers". It is known that for $X = J$ and for $X = C(\Gamma_{\omega_1})$, the algebra $B(X)$ has a nonzero multiplicative-linear functional ([4], [7]), and thus it cannot be generated by any family of subalgebras with square zero. We do not know whether for these spaces $B(X)$ can be generated by two abelian subalgebras. Neither do we know whether for any X, $B(X)$ can be generated by two subalgebras of square zero without being algebraically generated by such subalgebras. As mentioned in the introduction we do not know whether for any Banach space X the algebra $B(X)$ is generated by two commutative subalgebras, or even algebraically generated by such subalgebras. Nor do we know whether for $X = H$ a Hilbert space, the algebra $B(H)$ is generated, or even algebraically generated, by two commutative C^*-subalgebras.

If we consider the τ_s-generation of $B(X)$, where τ_s is the strong operator topology, then it was shown in [2] that, if H is separable, $B(H)$ is τ_s-generated by two operators, and so by two commutative subalgebras. It was shown in [5] that $B(H)$ is τ_s-generated by two selfadjoint operators, and so by two commutative C^*-subalgebras. The general problem of τ_s-generation is settled by the following.

Proposition 10. *Let X be a real or complex Banach space*, $\dim X > 1$. *Then $B(X)$ is τ_s-generated by two subalgebras of square zero.*

The proof will appear in [10].

Finally, let us remark that the algebras $\mathscr{A}_0(X_0)$ given by (2) are maximal subalgebras of $B(X)$ with respect to the property of being of square zero, and all such maximal subalgebras are of the form (2) (see [8]).

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Śniadeckich 8, 00-950 Warszawa, Poland

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L^2-Angles between one-dimensional tubes

by

MACIEJ SKWARCZYŃSKI (Warszawa)

Abstract. Let $D_i \subset C^N$, $i = 1, 2$, be two domains with nonempty intersection, such that $D = D_1 \cup D_2$ is contained neither in $\bar{D}_1$ nor in $\bar{D}_2$. Denote by F_i, $i = 1, 2$, the closed subspace in $L^2(D)$ consisting of functions which are holomorphic in D_i and otherwise arbitrary. As indicated in [7], [8] the Bergman projection in D can be described in terms of the orthogonal projections P_i: $L^2(D) \to F_i$. In some cases, the relevent alternating projections procedure can be carried out by explicit analytic calculations [7], [9]. In this context it is natural to study the angle $\gamma \in [0, \pi/2]$ between the subspaces F_1 and F_2. We call it (for brevity) the L^2-angle between D_1 and D_2. It was shown in [5] that the L^2-angle between two halfplanes (bounded by parallel lines) is 0. In the present paper we are concerned with the more general situation when D_i, $i = 1, 2$, are arbitrary tubes in the complex plane. We determine the L^2-angle $\alpha(r)$ between the strip $\{0 < \operatorname{Re} z < 1\}$ and a halfplane $\{\operatorname{Re} z > r\}$ $(0 < r < 1)$, as well as the L^2-angle $\beta(r, s)$ between two strips $\{-\pi < \operatorname{Re} z < s\pi\}$, $\{r\pi < \operatorname{Re} z < \pi\}$ $(-1 < r < s < 1)$. It turns out that

$$\cos^2\alpha(r) = r \quad \text{and} \quad \cos^2\beta(r, s) = \frac{1-s}{1+s}\cdot\frac{1+r}{1-r}.$$

We include a brief presentation of the Genchev transform [2], [4] since it plays an important role in our considerations.

1. The abstract definition of an angle. Let F_1, F_2 be closed subspaces in an abstract Hilbert space $\mathscr{H}$. Assume that $F = F_1 \cap F_2$ is a proper subset in each of F_i, $i = 1, 2$. (This implies that each F_i contains a nonzero element orthogonal to F.)

DEFINITION 1. The *angle* $\gamma \in [0, \pi/2]$ between F_1 and F_2 is defined by

$$\cos\gamma = \sup_{\substack{f_i \in F_i\setminus\{0\} \\ f_i \perp F}} \frac{|\langle f_1, f_2\rangle|}{\|f_1\|\cdot\|f_2\|}. \tag{1}$$

We shall restate this definition in a way which is less symmetric, but more convenient for our purpose.

LEMMA 1. *Denote by P_2 the orthogonal projection of $\mathscr{H}$ onto F_2. Then*

$$\cos\gamma = \sup_{\substack{f_1 \in F_1\setminus\{0\} \\ f_1 \perp F}} \frac{\|P_2 f_1\|}{\|f_1\|}. \tag{2}$$