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$\mathscr{L}_\pi$-Spaces and cone summing operators

by

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Abstract. Let E be a real Banach lattice, X a real Banach space, and $T: E \to X$ a linear operator. Suppose $1 \leqslant p < \infty$ and that there is a constant $K > 0$ such that for all $n \in N$ and any $u_1, \ldots, u_n$ in E

$$(\sum_{i=1}^{n} \|Tu\|^p)^{1/p} \leqslant K \sup \{(\sum_{i=1}^{n} \langle \varphi, |u_i| \rangle^p)^{1/p}: \varphi \in \text{ball } E_+^*\}.$$

We show that T has a (sub)factorization through a class of Banach lattices closely related to the $L_p(L_1)$-spaces. We recover as special cases some classical results on p-absolutely summing operators.

1. Introduction.

1.1. Definition. Let E be a Banach lattice, X a Banach space and $1 \leqslant p < \infty$. A linear operator $T: E \to X$ is *cone p-summing* if there is a constant $K > 0$ such that for each positive integer n and any vectors $u_1, \ldots, u_n$ in E,

$$(\sum_{j=1}^{n} \|Tu_j\|^p)^{1/p} \leqslant K \sup \{(\sum_{j=1}^{n} \langle \varphi, |u_j| \rangle^p)^{1/p}: \varphi \in \text{ball } E_+^*\}.$$

We denote by $\hat{\pi}_p(T)$ the least K for which this inequality holds for all n and all choices of n vectors in E; and $\hat{\Pi}_p(E, X)$ is the set of cone p-summing operators $E \to X$.

1.2. Remarks. When $p = 1$ these operators have been studied by Schaefer [7].

Let $1 \leqslant p < \infty$, let E be a Banach lattice and X a Banach space. Let $\Pi_p(E, X)$ denote the p-absolutely summing operators $E \to X$ in the sense of Pietsch [6] and $C_p(E, X)$ the p-concave operators $E \to X$ in the sense of Lindenstrauss and Tzafriri [4]. Then we have the relations:

(i) $\Pi_p(E, X) \subseteq \hat{\Pi}_p(E, X) \subseteq C_p(E, X)$.

(ii) $\Pi_p(E, X) = \hat{\Pi}_p(E, X) = C_p(E, X)$ whenever E is a $C(K)$-space.

(iii) $\hat{\Pi}_1(E, X) = C_1(E, X)$ for all E and all X.

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However, there are significant differences between cone p-summing operators on the one hand and p-absolutely summing operators and p-concave ($p > 1$) on the other. For example:

(a) All p-absolutely summing operators are weakly compact; but, say, the identity operator on any AL-space is cone p-summing for any $p \geqslant 1$, without being weakly compact unless the space is finite-dimensional.

(b) For $1 < p < \infty$, the identity operator on an infinite-dimensional AL_p-space is p-concave but it is not cone p-summing.

1.3. A Pietsch Domination Theorem. *Let E be a Banach lattice, X a Banach space, $1 \leqslant p < \infty$, and $T\colon E \to X$ a cone p-summing operator. Then there is a probability measure ν on $U_+ = \operatorname{ball} E_+^*$ (the positive part of the norm dual of E) and a constant $K > 0$ such that for all $u \in E$*

$$\|Tu\| \leqslant K\Big(\int_{U_+} \langle \varphi, |u|\rangle^p \, \nu(d\varphi)\Big)^{1/p}.$$

Proof. This is an application of the Hahn–Banach separation and Riesz representation theorems, identical to the proof for p-absolutely summing operators [3]. ∎

In this note we obtain a realization of the above domination result as a (sub)factorization through a class of operators that is closely related to the $L_p(L_q)$-spaces.

2. Preliminaries.

2.1. Definition. Let E, F be Banach lattices. A map $u\colon E \to F$ is *order continuous* if for every (upward) directed set (x_α) in E with $\sup x_\alpha = x$, we have $ux = \sup ux_\alpha$ in F.

A map $u\colon E \to F$ is *positive* if $u(E_+) \subseteq F_+$; and a map $w\colon E \to F$ is *regular* if $w = u - v$, where $u, v\colon E \to F$ are positive.

2.2. Definition. Let S, T be compact topological spaces, $\pi\colon T \to S$ be a continuous surjection, ν a measure on S and $1 \leqslant p \leqslant \infty$. A linear operator $u\colon C(T) \to L_p(\nu, S)$ is *π-modular* if $u(f \cdot g\circ\pi) = g \cdot u(f)$ for all f in $C(T)$, and all g in $C(S)$. We set

$$\mathscr{L}_\pi^\times(C(T), L_p(\nu, S)) = \{u \mid u \ \pi\text{-modular and } u \text{ order continuous}\}.$$

When S is Stonean we define the space $\mathscr{L}_\pi^\times(C(T), C(S))$ in a similar fashion. This class of operator spaces has been studied by Haydon [2], and our definitions are extensions of Haydon's idea.

2.3. Notes. The structure of $\mathscr{L}_\pi^\times$-spaces seems to be of considerable intrinsic interest. In particular, if we define

$$\mathscr{L}_\pi(L_p(\mu), L_q(\nu)) = \{u \mid u\colon L_p(\mu, T) \to L_q(\nu, S),\ u(f \cdot g\circ\pi) = g \cdot u(f) \text{ for all } f \in L_p(\mu), \text{ and all } g \in C(S)\}$$

then it can be shown that:

(i) Every operator in $F = \mathscr{L}_\pi(L_p, L_q)$ is regular; so F is a Banach lattice.

(ii) If $1 \leqslant p < q \leqslant \infty$ and ν has no atoms then $\mathscr{L}_\pi(L_p, L_q) = \{0\}$.

(iii) If $1 \leqslant q < p \leqslant \infty$, $\mathscr{L}_\pi^\times(L_p, L_q)$-spaces are ultrastable.

(iv) If $\pi\colon [0, 1] \times [0, 1] \to [0, 1]$ then for $1 \leqslant p, q \leqslant \infty$, $q' + q = qq'$, and $1 + 1/p = 1/q + 1/r$, we have

$$\mathscr{L}_\pi^\times(L_{q'}([0, 1] \times [0, 1]), L_r[0, 1]) = L_p(L_q),$$

the $L_q[0, 1]$-valued, Bochner p-integrable functions on $[0, 1]$.

2.4. Proposition. *Let E be any Banach lattice. Then there are compact Hausdorff spaces S, T and a continuous surjection $\pi\colon T \to S$ such that E embeds as a sublattice of the Banach lattice $\mathscr{L}_\pi^\times(C(T), C(S))$.*

Proof. Let $U_+ = \operatorname{ball} E_+^*$, $U_{++} = \max \operatorname{ball} E_+^*$, the maximum being taken in the canonical order on E^*. Given $\varphi \in U_{++}$ define

$$N_\varphi = \{u \in E\colon \langle \varphi, |u|\rangle = 0\},$$

the absolute kernel of φ. Set E_φ = completion of E/N_φ with norm $\|\hat{u}\| = \langle \varphi, |u|\rangle$, $u \in \hat{u} \in E/N_\varphi$. Then E_φ is an AL-space [7]; so E_φ^* is an AM-space with unit. Let $C = (\sum_{\varphi\in U_{++}}^{\oplus} E_\varphi^*)_\infty$. Then C is a commutative C^*-algebra so that $C = C(T)$, T a compact Hausdorff space; more explicitly, T is the Stone–Čech compactification of $\bigsqcup_{\varphi\in U_{++}} T_\varphi$, where $E_\varphi^* = C(T_\varphi)$ for $\varphi \in U_{++}$.

Now let U_{++}^e be the set of extreme points of U_{++} (so that the weak* closure of U_{++}^e is the Shilov boundary of E_+ considered as the cone of continuous positive real functions on the weak* compact set U_+), and set $C(S) = l_\infty(U_{++}^e)$. Then S is the Stone–Čech compactification of U_{++}^e; and $C(S)$ embeds as a subalgebra of $C(T)$ via $f \mapsto (f(\varphi)1_\varphi) \in C$, where 1_φ is the unit in E_φ^*, $\varphi \in U_{++}$. This embedding induces a continuous surjection $\pi\colon T \to S$. More explicitly, $\pi(t_\varphi) = \varphi$ for all $t_\varphi \in T_\varphi$ and $\varphi \in U_{++}$.

Since S is Stonean, $\mathscr{L}_\pi^\times(C(T), C(S))$ is a Banach lattice. Indeed, it is a 1-injective Banach lattice [2] being a projection band in the Dedekind complete 1-injective Banach lattice $\mathscr{L}(C(T), C(S)) = (\sum_{\varphi\in U_{++}^e}^{\oplus} C(T)^*)_\infty$. We now define

$$J\colon E \to \mathscr{L}_\pi^\times(C(T), C(S))$$

as follows: Given $u \in E$, $f \in C(T)$, $\varphi \in U_{++}^e$ define

$$(Ju)(f)(\varphi) = \langle J_\varphi u, f_\varphi\rangle$$

where $f = (f_\varphi) \in C(T) = (\sum_{\varphi\in U_{++}^e}^{\oplus} E_\varphi^*)_\infty$, $J_\varphi\colon E \to E_\varphi$ is the lattice homomorphism in the construction of E_φ, $\varphi \in U_{++}$ [7, p. 243], and $\langle\ ,\ \rangle$ is the duality (E_φ, E_φ^*).

Clearly $(Ju)(f) \in C(S)$. Moreover, for $u \in E_+$

$$\|Ju\| = \sup\{|(Ju)(f)(\varphi)|: f \in \text{ball } C(T),\ \varphi \in U^{e}_{++}\}$$
$$= \sup\{|\langle J_\varphi u, f_\varphi\rangle|: f_\varphi \in E^*_\varphi,\ \varphi \in U^*_{++}\}$$
$$= \sup\{|\langle \varphi, u\rangle|:\ \varphi \in U^{e}_{++}\} = \|u\|$$

so that $\|J\| = 1$. It remains to show that Ju is π-modular and order continuous and that J is a lattice homomorphism.

Let $g \in C(S)$, $f \in C(T)$, $u \in E$, $\varphi \in U^{e}_{++}$. Then

$$(Ju)(f \cdot g \circ \pi)(\varphi) = \langle J_\varphi u, (f \cdot g \circ \pi)_\varphi\rangle = \langle J_\varphi u, f_\varphi \cdot g(\varphi) 1_\varphi\rangle$$
$$= g(\varphi)\langle J_\varphi u, f_\varphi\rangle = g(\varphi)(Ju)(f)(\varphi).$$

Thus Ju is π-modular. It is easy to see that it is also order continuous since $C(S)$ is order complete. Finally, let $f \in C(T)$, $u \in E$, $\varphi \in U^{e}_{++}$. Then since J_φ is a lattice homomorphism, we have

$$(J|u|)(\varphi) = \langle J_\varphi |u|, f_\varphi\rangle = \langle |J_\varphi u|, f_\varphi\rangle = \sup\{|(J_\varphi u)(g_\varphi)|:\ 0 \leqslant |g| \leqslant f\}$$
$$= \sup\{|(Ju)(g)(\varphi):\ 0 \leqslant |g| \leqslant f\} = |Ju|(f)(\varphi),$$

so that J is lattice homomorphism. ■

2.5. Proposition. *Let E be a Banach lattice realized as a sublattice of $\mathscr{L}^\times_\pi(C(T), C(S))$. Suppose $\varphi \in \text{extmax ball } E^*_+$. Then there exists a positive linear functional $\lambda(\varphi)$ on $C(S)$ such that:*

(i) *$\lambda(\varphi) \in \text{ball } C(S)^*_+$;*

(ii) *$\langle \varphi, |u|\rangle \leqslant \langle \lambda(\varphi), (J|u|)(1_T)\rangle$, for all $u \in E$, where J is the embedding in Proposition 2.4 above.*

Proof. Given $\varphi \in \text{extmax ball } E^*_+$, define a linear functional on $F = \mathscr{L}^\times_\pi(C(T), C(S))$ by $\tilde{\varphi}(V) = (V1_T)(\varphi)$. Then

$$\|\tilde{\varphi}\| = \sup\{|(V1_T)(\varphi)|:\ V \in \text{ball } F\} \leqslant 1.$$

Thus $\tilde{\varphi} \in \text{ball } F^*_+$. Now given $f \in C(S)$ define

$$\lambda(\varphi)(f) = \sup\{\tilde{\varphi}(f \cdot V):\ V \in \text{ball } F_+\},$$

where $(f \cdot V)(h) = V(h \cdot f \circ \pi)$ for all h in $C(T)$.

It was shown in [5] that $\lambda(\varphi)$ extends to a positive linear functional on $C(S)$ (which we shall also denote by $\lambda(\varphi)$) satisfying:

(i) $\|\lambda(\varphi)\| = \|\tilde{\varphi}\|$.

(ii) $\tilde{\varphi}(V) \leqslant \langle \lambda(\varphi), V1_T\rangle$ for all V in F_+.

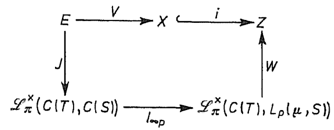
Now for $V = J|u|$, u in E, we have

$$\tilde{\varphi}(J|u|) = (J|u|)(1_T)(\varphi) = \langle J_\varphi |u|, 1_\varphi\rangle = \langle \varphi, |u|\rangle.$$

Hence $\langle \varphi, |u|\rangle \leqslant \langle \lambda(\varphi), (J|u|)(1_T)\rangle$. ■

3. The main result. We conclude with the factorization and some of its specializations to the case of classical Banach spaces. This illustrates the general connections between cone p-summing and p-absolutely summing operators.

3.1. Theorem. *Let E be a Banach lattice, X a Banach space, V: $E \to X$ a linear operator and $1 \leqslant p < \infty$. Then V is cone p-summing if and only if V factors as follows:*



where J is a lattice homomorphism, μ is a probability measure on ball E^*, *$Z = l_\infty(\text{ball } X^*)$, $I_{\infty p}$ is the canonical lattice injection, W is a linear operator such that $\|W\| = \hat{\pi}_p(V)$, i is an isometric embedding, and S, T are compact Hausdorff spaces.*

Proof. Suppose $i \circ V = W \circ I_{\infty p} \circ J$. Since $J \geqslant 0$ and i is an isometry, it suffices to prove that $I_{\infty p}$ is cone p-summing. Let $u_1, \ldots, u_n$ be in $F = \mathscr{L}^\times_\pi(C(T), C(S))$. Then

$$\sum_{i=1}^{n} \|I_{\infty p} u_i\|^p = \sum_{i=1}^{n} [\sup\{\|u_i(f)\|_p:\ f \in \text{ball } C(T)\}]^p$$
$$= \sum_{i=1}^{n} [\sup(\int_S |u_i(f)(s)|^p \mu(ds))^{1/p}:\ f \in \text{ball } C(T)]^p$$
$$\leqslant \sum_{i=1}^{n} (\int_S (|u_i|\, 1_T)(s)^p \mu(ds))^{1/p})^p$$
$$= \int_S \sum_{i=1}^{n} (|u_i|\, 1_T)(s)^p \mu(ds)$$
$$\leqslant \mu(S) \sup\{\sum_{i=1}^{n} (|u_i|\, 1_T)(s)^p:\ s \in S\}$$
$$\leqslant \mu(S) \sup\{\sum_{i=1}^{n} \langle \varphi, |u_i|\rangle^p:\ \varphi \in \text{ball } F^*_+\},$$

where we have used the fact that $u \mapsto u1_T(s)$, $s \in S$, is a norm one linear functional on F. Hence $\hat{\pi}_p(I_{\infty p}) \leqslant \mu(S) = 1$ and we now have

$$\hat{\pi}_p(V) \leqslant \|W\|\hat{\pi}_p(I_{\infty p}) \leqslant \|W\|\mu(S) = \|W\|.$$

Conversely, let V be cone p-summing. Then by the Domination

Theorem 1.3 above, there is a probability measure ν on S such that $\|Vu\| \leqslant \int_S \langle \varphi, |u| \rangle^p \nu(d\varphi)$ for all u in E. Let S, T be the topological spaces constructed in Proposition 2.4 above; let J be the lattice homomorphism in that proposition; and define a measure μ on S by

$$\mu(A) := \int_A \lambda(\varphi)\, \nu(d\varphi), \quad A \subseteq S,$$

where $\lambda(\varphi)$, for φ in S, is the linear functional constructed in Proposition 2.5. Then μ is well defined since $\varphi \mapsto \lambda(\varphi)$ is measurable with respect to ν. Hence

$$\begin{aligned} \|Vu\|^p &\leqslant K \int_S \langle \varphi, |u| \rangle^p \nu(d\varphi), \quad K = \hat{\pi}_p(V) \\ &\leqslant K \int_S \langle \lambda(\varphi), |u| 1_T \rangle^p \nu(d\varphi), \quad \text{by Proposition 2.5 above} \\ &= K \int_S |u| 1_T(\varphi)^p \mu(d\varphi) \\ &\leqslant K \| \, |u| \, \|_{F_p}^p, \quad F_p = \mathscr{L}_\pi^\times (C(T), L_p(\mu, S)). \end{aligned}$$

Define $\tilde{W}$: $I_{\infty p} JE \to X$ by $\tilde{W}(I_{\infty p} Ju) = Vu$ for all u in E. Then

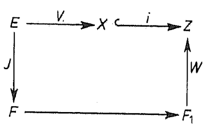
$$\|\tilde{W}(I_{\infty p} Ju)\|^p = \|Vu\|^p \leqslant K \| \; |u| \; \|_{F_p}$$

so that $\|\tilde{W}\| \leqslant K$. Now Z is an injective Banach space [3] so we can extend $\tilde{W}$ to a linear map W: $F_p \to Z$ with $\|W\| \leqslant K$. ∎

We recover the following result due to Schaefer [7]:

3.2. Corollary. *Let E be a Banach lattice, X a Banach space, and V: $E \to X$ a cone 1-summing operator. Then there is an AL-space L, a lattice homomorphism V_1: $E \to L$, and a bounded linear operator V_2: $L \to X$ such that $V = V_2 \circ V_1$.*

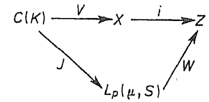
Proof. By the Main Result above we have a factorization



Now $F_1 = \mathscr{L}_\pi^\times (C(T), L_1(\mu, S))$ is a band in the Banach lattice of regular operators $\mathscr{L}^r(C(T), L_1(\mu, S))$ which in turn is an AL-space [7]. Thus F_1 is an AL-space. Let $V_1 = I_{\infty 1} J$. Then $V_1 E$ is a sublattice of F_1 and so the closure of $V_1 E$ in F_1 is an AL-space [4] which we shall denote by L. The linear map V_2: $V_1 E \to X$ defined by $V_2(V_1 u) = Vu$ is continuous of norm $\leqslant 1$ and its (continuous) extension $L \to X$ will also be denoted by V_2. ∎

Finally, we recover the factorization of p-absolutely summing operators defined on $C(K)$-spaces. On these spaces we have already noted in 1.2(ii) above that there is a coincidence of concepts.

3.3. Corollary. *Let V: $C(K) \to X$ be a p-absolutely summing operator $(1 \leqslant p < \infty)$. Then V factors as follows:*



Proof. In the construction in Section 2 above it suffices to consider J_φ: $C(K) \to E_\varphi$ with $\varphi \in K$ (K = the set of evaluation functionals on $C(K)$). Then $E_\varphi = \mathbf{R}$ (where $\mathbf{R}$ is the space of real numbers) and $C(T) = (\sum_{\varphi \in K}^{\oplus} \mathbf{R})_\infty = l_\infty(K, \mathbf{R})$. Thus T is the Stone–Čech compactification of K. Moreover, $C(S) = l_\infty(K, \mathbf{R})$ so that $S = T$ and π: $T \to S$ is the identity map. The π-modular operators u: $C(T) \to C(S)$ reduce to multiplication by a fixed element of $C(S)$. Thus $F = \mathscr{L}_\pi^\times (C(T), C(S)) = C(T) = C(S)$. Similarly the π-modular operators u: $C(T) \to L_p(\mu, S)$ reduce to multiplication by a fixed element of $L_p(\mu, S)$ and we have $F_p = \mathscr{L}_p^\times (C(T), L_p(\mu, S)) = L_p(\mu, S)$. The rest of the proof proceeds as in the usual p-summing case [3].

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