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Gaussian measures in Urbanik's sense and a characterization theorem for abelian groups

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Abstract. Gaussian measures on the real line can be characterized as follows. Let $\xi_1, \ldots, \xi_s, \ldots$ be independent identically distributed random variables with distribution γ. If for any system of integers $\{a_0, \ldots, a_s\}$, $s \geqslant 2$, with $a_0^2 = a_1^2 + \ldots + a_s^2$ the linear forms $a_0 \xi_1$ and $a_1 \xi_1 + \ldots + a_s \xi_s$ are identically distributed, then γ is a Gaussian distribution. In this paper a complete description is given of the class of those locally compact abelian groups for which an analogous theorem is valid.

Gaussian measures on the real line can be characterized as follows.

THEOREM A. *Let $\xi_1, \ldots, \xi_s$ be independent identically distributed random variables with distribution γ. If for any system of integers $\{a_0, \ldots, a_s\}$, $s \geqslant 2$, satisfying*

$$a_0^2 = a_1^2 + \ldots + a_s^2 \tag{1}$$

the linear forms $a_0 \xi_1$ and $a_1 \xi_1 + \ldots + a_s \xi_s$ are identically distributed, then γ is a symmetric Gaussian distribution (1).

Theorem A can be equivalently formulated as follows: If the characteristic function $\hat{\gamma}(y)$ of a distribution γ satisfies the system of equations

$$\hat{\gamma}(a_0 y) = \hat{\gamma}(a_1 y) \ldots \hat{\gamma}(a_s y), \tag{2}$$

where $\{a_0, \ldots, a_s\}$ runs through all systems of integers satisfying (1), then γ is a symmetric Gaussian distribution.

In this paper we give a complete description of the class of those locally compact abelian groups for which an analogous theorem is valid. It turns out that this characterization of Gaussian distributions is closely connected with measures on groups, Gaussian in Urbanik's sense (see [8], [2]). We will

(1) As was shown by Linnik (see e.g. [6]), γ is Gaussian if it is only assumed that the linear forms $a_0 \xi_1$ and $a_1 \xi_1 + \ldots + a_s \xi_s$ are identically distributed for one fixed system of real numbers $\{a_0, \ldots, a_s\}$ satisfying (1). The case $s = 2$ was earlier studied by Pólya. This result of Linnik is a much more subtle characterization of Gaussian distributions than Theorem A.

use some results from the structure theory of locally compact abelian groups and from Pontryagin's duality theory (see e.g. [4]).

Let X be a locally compact abelian separable metric group, $Y = X^*$ its character group, (x, y) the value of the character $y \in Y$ on $x \in X$, and C_X the zero component of X. If G is a closed subgroup of X, then $G^\perp$ denotes its annihilator, $G^\perp = \{y \in Y\colon (x, y) = 1 \text{ for all } x \in G\}$. We have the isomorphism $G^* \approx Y/G^\perp$. We denote by $\boldsymbol{R}$, $\boldsymbol{Z}$ and $\boldsymbol{T}$ the groups of reals, of integers and the circle group respectively. The degenerate distribution concentrated at $x \in X$ will be denoted by E_x. If μ is a distribution on X, then the distribution $\bar{\mu}$ is defined by $\bar{\mu}(E) = \mu(-E)$ for every Borel set E. The support of μ is denoted by $\sigma(\mu)$.

A distribution μ is called *idempotent if* $\mu^{*2} = \mu * E_x$ for some $x \in X$. The set of all idempotent distributions on X will be denoted by $I(X)$. It coincides with the set of all translations of the Haar distributions m_K of compact subgroups K of X (see e.g. [7]).

DEFINITION 1 ([7]). A distribution γ on X is called *Gaussian* if its characteristic function can be written in the form

$$\hat{\gamma}(y) = (x, y)\exp\{-\varphi(y)\} \tag{3}$$

for some $x \in X$, where $\varphi(y)$ is a continuous nonnegative function on Y satisfying

$$\varphi(y_1 + y_2) + \varphi(y_1 - y_2) = 2[\varphi(y_1) + \varphi(y_2)] \tag{4}$$

for all $y_1, y_2 \in Y$.

A Gaussian distribution is called *symmetric* if $x = 0$ in (3).

The sets of all Gaussian distributions and of all symmetric Gaussian distributions on X will be denoted by $\Gamma(X)$ and $\Gamma^s(X)$ respectively.

Let $\mathscr{A}$ denote the set of all systems of integers $A = \{a_0, \ldots, a_s\}$, $s \geqslant 2$, satisfying (1). Let $\Gamma_A(X)$, $A \in \mathscr{A}$, be the set of all distributions γ on X with the following property: if $\xi_1, \ldots, \xi_s$ are independent identically distributed random variables with values in X and with distribution γ, then the linear forms $a_0\xi_1$ and $a_1\xi_1 + \ldots + a_s\xi_s$ are identically distributed. Just as in the case of the real line, $\gamma \in \Gamma_A(X)$ iff $\hat{\gamma}(y)$ satisfies (2).

Let $\Gamma_\infty(X) = \bigcap_{A \in \mathscr{A}} \Gamma_A(X)$ and consider still another class of distributions on X, closely connected with $\Gamma_\infty(X)$. If $y \in Y$, then $y(\gamma)$ denotes the image of the distribution γ under the homomorphism $y\colon X \to \boldsymbol{T}$.

DEFINITION 2 (cf. [8], [2]). A distribution γ on X is called *Gaussian in Urbanik's sense* if $y(\gamma) \in \Gamma(\boldsymbol{T})$ for each $y \in Y$.

The set of distributions on X which are Gaussian in Urbanik's sense will be denoted by $\Gamma_U(X)$.

PROPOSITION 1. *Let γ be a distribution on X. Then the following conditions are equivalent*:

(i) $\gamma \in \Gamma_\infty(X)$.

(ii) $\gamma = \bar{\gamma}$ *and for each character* $y \in Y$ *either* $y(\gamma) \in \Gamma(\boldsymbol{T})$ (*i.e.* $\gamma \in \Gamma_U(X)$), *or* $y(\gamma) = m_{\boldsymbol{T}}$.

(iii) *The characteristic function* $\hat{\gamma}(y)$ *is real-valued and satisfies the system of equations*

$$\hat{\gamma}(ny) = (\hat{\gamma}(y))^{n^2}, \quad n \in \boldsymbol{Z},\ n \geqslant 2. \tag{5}$$

For the proof we will need the following

LEMMA 1. *Let $X \approx \boldsymbol{T}$. Then $\Gamma_\infty(X) = \{m_X\} \cup \Gamma^s(X) \cup E_\zeta * \Gamma^s(X)$, where ζ is the element of order two in X.*

Proof. Clearly, the characteristic function of a distribution $\gamma \in \Gamma_\infty(X)$ on any group X satisfies the system (5). Without loss of generality we can assume that $X = \boldsymbol{T}$, $Y = \boldsymbol{Z}$. Let $\gamma \in \Gamma_\infty(\boldsymbol{T})$. If $\hat{\gamma}(1) = 0$, then $\hat{\gamma}(-1) = 0$ and (5) shows that $\hat{\gamma}(n) = 0$ for all $n \in \boldsymbol{Z}$, $n \neq 0$, i.e. $\gamma = m_{\boldsymbol{T}}$. If $\hat{\gamma}(1) \neq 0$, then we put $\hat{\gamma}(1) = \exp\{a + ib\}$, where $a \leqslant 0$, $-\infty < b < \infty$. Then $\hat{\gamma}(-1) = \exp\{a - ib\}$ and (5) implies that

$$\hat{\gamma}(n) = \begin{cases} \exp\{an^2 + ibn^2\} & \text{if } n \geqslant 0, \\ \exp\{an^2 - ibn^2\} & \text{if } n < 0. \end{cases}$$

Since the function $\hat{\gamma}(n)$ also satisfies the equation $\hat{\gamma}(2n) = (\hat{\gamma}(n))^3\hat{\gamma}(-n)$, we have $\exp\{2ibn^2\} = 1$ for all $n \in \boldsymbol{Z}$. This yields $b = \pi k$ for some $k \in \boldsymbol{Z}$, and so $\gamma \in \Gamma^s(\boldsymbol{T}) \cup E_\zeta * \Gamma^s(\boldsymbol{T})$. The proof of the lemma is complete.

Proof of Proposition 1. (i)$\Rightarrow$(ii). Let γ be any distribution on X and let $y \in Y$. Then the characteristic function of the distribution $y(\gamma)$ is

$$(y(\gamma))\hat{}\,(n) = \hat{\gamma}(ny), \quad n \in \boldsymbol{Z}. \tag{6}$$

Hence $y(\gamma) \in \Gamma_\infty(\boldsymbol{T})$ whenever $\gamma \in \Gamma_\infty(X)$, and (ii) follows from Lemma 1.

(ii)$\Rightarrow$(iii). Let $y \in Y$. Then it is easily seen that the characteristic function of the distribution $y(\gamma)$ satisfies

$$(y(\gamma))\hat{}\,(n) = ((y(\gamma))\hat{}\,(1))^{n^2}, \quad n \in \boldsymbol{Z},\ n \geqslant 2.$$

Now, (6) gives (5).

(iii)$\Rightarrow$(i) is obvious.

COROLLARY 1. *We have the equality* $\{\gamma \in \Gamma_U(X)\colon \hat{\gamma}(y) > 0 \text{ for all } y \in Y\} = \{\gamma \in \Gamma_\infty(X)\colon \hat{\gamma}(y) > 0 \text{ for all } y \in Y\}$.

Note that (4) shows that $\varphi(ny) = n^2\varphi(y)$ for any $y \in Y$, $n \in \mathbf{Z}$. Therefore $\Gamma^s(X) \subset \Gamma_\infty(X)$. However, in contrast to the classical situation, in general there may exist on X non-Gaussian distributions belonging to $\Gamma_\infty(X)$. Set $I_\infty(X) = I(X) \cap \Gamma_\infty(X)$. The set $I_\infty(X)$ can be easily described.

For any $n \in \mathbf{Z}$, $n \neq 0$, consider the homomorphism $X \to X$ defined by $x \to nx$. The image of X under this map will be denoted by nX.

Proposition 2. *Let K be a compact subgroup of X. Then the following conditions are equivalent*:

1° $m_K \in \Gamma_\infty(X)$.
2° *K is connected.*

Proof. Let $n \in \mathbf{Z}$, $n \neq 0$. By a standard argument one can show the equivalence of the following conditions:

(a) $nK = K$.
(b) If $ny \in K^\perp$, then $y \in K^\perp$.

Note also that $\hat{m}_K(y) = 1$ for $y \in K^\perp$, and $\hat{m}_K(y) = 0$ otherwise.

$1° \Rightarrow 2°$. The characteristic function $\hat{m}_K(y)$ satisfies the system (5). Fix $n \in \mathbf{Z}$, $n \geqslant 2$. Then if $ny \in K^\perp$ we have $\hat{m}_K(ny) = 1$ and (5) shows that $\hat{m}_K(y) = 1$, i.e. $y \in K^\perp$. It follows that (b), and hence (a), is fulfilled, i.e. $nK = K$ for any $n \in \mathbf{Z}$, $n \geqslant 2$. This implies that K is connected (see [4]).

$2° \Rightarrow 1°$. If K is connected, then $nK = K$ for every $n \in \mathbf{Z}$, $n \geqslant 2$. Hence (b) holds. We show that the function $\hat{m}_K(y)$ satisfies (5). If $y \in K^\perp$, then (5) is obviously fulfilled, since $\hat{m}_K(y) = \hat{m}_K(ny) = 1$. If $y \notin K^\perp$, then (b) yields $ny \notin K^\perp$. We have $\hat{m}_K(y) = \hat{m}_K(ny) = 0$. It follows that the characteristic function $\hat{m}_K(y)$ satisfies condition (iii) of Proposition 1, and so $m_K \in \Gamma_\infty(X)$.

Remark 1. Clearly, $E_x \in \Gamma_\infty(X) \Leftrightarrow 2x = 0$. Suppose $\lambda = m_K * E_{x_0} \in I_\infty(X)$. Then the group K is connected, and the restriction of the characteristic function $\hat{\lambda}(y)$ to $K^\perp$ can be written in the form $\hat{\lambda}(y) = ([x_0], y)$, where $[x_0] \in X/K$, $2[x_0] = 0$, i.e. $2x_0 = x' \in K$. Since K is connected, we have in particular $2K = K$. Hence $x' = 2x''$, $x'' \in K$. Put $x = x_0 - x''$. Then $2x = 0$ and $[x_0] = [x]$, and therefore $\lambda = m_K * E_x$.

Proposition 3. *Let $\gamma \in \Gamma_\infty(X)$. Then there is $x \in X$ with $2x = 0$ such that $\sigma(\gamma * E_x) \subset C_X$.*

Proof. Denote by Y_0 the subgroup of all compact elements in Y. Let $y_0 \in Y_0$. Then for some sequence of positive integers $n_j \to \infty$ we have $n_j y_0 \to 0$. If $|\hat{\gamma}(y_0)| < 1$, then (5) shows that

$$1 = \hat{\gamma}(0) = \lim_{j\to\infty} \hat{\gamma}(n_j y_0) = \lim_{j\to\infty} (\hat{\gamma}(y_0))^{n_j^2} = 0.$$

Hence $|\hat{\gamma}(y)| \equiv 1$ on Y_0. Since $Y_0 \approx (X/C_X)^*$, we have $\hat{\gamma}(y) = ([x_0], y)$, $y \in Y_0$, for some $[x_0] \in X/C_X$ with $2[x_0] = 0$. Arguing now as in Remark 1 we obtain the existence of an element $x \in X$ with $2x = 0$ such that $\hat{\gamma}(y) = (x, y)$ for $y \in Y_0$. Set $\gamma' = \gamma * E_x$. Then $\hat{\gamma}'(y) \equiv 1$ on Y_0, and so $\sigma(\gamma') \subset Y_0^\perp$. But $Y_0^\perp = C_X$ (see [4]). The proof of Proposition 3 is complete.

Obviously, the set $\Gamma_\infty(X)$ is a semigroup with respect to convolution. Therefore

$$I_\infty(X) * \Gamma^s(X) \subset \Gamma_\infty(X).$$

Our problem is the description of those groups X for which

$$I_\infty(X) * \Gamma^s(X) = \Gamma_\infty(X). \tag{7}$$

Note that if X satisfies (7), then every distribution $\gamma \in \Gamma_\infty(X)$ is invariant with respect to some connected subgroup K of X, and γ induces a Gaussian distribution on the factor group X/K.

Elements $x_1, \ldots, x_n \in X$ are called *independent* if $k_1 x_1 + \ldots + k_n x_n = 0$, $k_j \in \mathbf{Z}$, implies $k_1 = \ldots = k_n = 0$.

Theorem 1. *A group X satisfies the equality (7) if and only if the following condition (α) is fulfilled: in the group C_X^*, any two elements which are not infinitely divisible are dependent.*

For the proof we will need the following lemmas.

Lemma 2 ([1]). *Suppose X has no subgroup isomorphic to $\mathbf{T}$. Assume that $\gamma \in \Gamma(X)$ and $\gamma = \gamma_1 * \gamma_2$, where the γ_j are distributions on X. Then $\gamma_j \in \Gamma(X)$.*

Lemma 3 ([2]). *Suppose X satisfies the following condition (β): either $C_X \approx \mathbf{T}$, or any factor group of the group Y contains an infinitely divisible element. Then*

$$\Gamma(X) = \Gamma_U(X).$$

On the other hand, if condition (β) is not fulfilled, then there is a distribution $\gamma \in \Gamma_U(X)$ such that $\hat{\gamma}(y) > 0$ for all $y \in Y$ and $\gamma \notin \Gamma(X)$.

Note also that by the structure theorem any connected group X is isomorphic to $\mathbf{R}^m + G$, where $m \geqslant 0$ and G is a compact connected group. Then $Y \approx \mathbf{R}^m + H$, where $H = G^*$ is a discrete group consisting of elements of infinite order (see [4]).

Proof of Theorem 1. Necessity. Suppose condition (α) is not satisfied for X. Then $C_X \neq \{0\}$. Clearly, it suffices to construct a distribution $\gamma \in \Gamma_\infty(C_X)$ with $\gamma \notin I_\infty(C_X) * \Gamma^s(C_X)$. Therefore without loss of generality we may assume that X itself is connected. Two cases are possible:

1) X is not compact, and in Y there is an element which is not infinitely divisible. Let $X = \mathbf{R}^m + G$, where $m \geqslant 1$ and G is a compact connected group. The elements of the group $Y = \mathbf{R}^m + H$, where $H = G^*$, will be denoted by $y = (s, h)$, where $s \in \mathbf{R}^m$, $h \in H$. If $y_0 = (s_0, h_0)$ is not infinitely divisible, then neither is $\eta = (0, h_0)$. Denote by L_η the subgroup of Y consisting of all elements depending on η, i.e. of elements $y \in Y$ such that $py = q\eta$ for some $p, q \in \mathbf{Z}$. Since η is not infinitely divisible, we have $L_\eta \approx \mathbf{Z}$. Consider the subgroup $H_\eta = \mathbf{R}^m + L_\eta$ of Y. Put $K_\eta = H_\eta^*$. Then $K_\eta \approx \mathbf{R}^m + \mathbf{T}$. By Lemma 3 there is on K_η a distribution $\gamma_0 \in \Gamma_U(K_\eta)$ such that $\hat{\gamma}_0(y) > 0$ for $y \in H_\eta$ and $\gamma_0 \notin \Gamma(K_\eta)$. Define

$$f(y) = \begin{cases} \hat{\gamma}_0(y) & \text{if } y \in H_\eta, \\ 0 & \text{if } y \in Y \setminus H_\eta. \end{cases}$$

Obviously, $f(y)$ is a positive-definite function on Y. Since the subgroup H is open, $f(y)$ is continuous. By the Bochner–Khinchin theorem, $f(y) = \hat{\gamma}(y)$, where γ is a distribution on X. We will check that $\gamma \in \Gamma_\infty(X)$. By Proposition 1 it suffices to verify that $f(y)$ satisfies the system of equations (5). By Corollary 1, $\gamma_0 \in \Gamma_\infty(K_\eta)$. Hence each of the equations (5) is satisfied for $y \in H_\eta$. If $y \notin H_\eta$, then by construction $ny \notin H_\eta$ and so each of the equations (5) is satisfied too. Therefore $\gamma \in \Gamma_\infty(X)$. Since $\gamma_0 \notin \Gamma(K_\eta)$, we have $\gamma \notin I(X) * \Gamma(X)$.

2) X is compact and there are in Y two independent elements η, ζ which are not infinitely divisible. Since X is compact and connected, Y is discrete and consists of elements of infinite order. Consider the subgroups L_η and L_ζ. We have $L_\eta \approx L_\zeta \approx \mathbf{Z}$ and $L_\eta \cap L_\zeta = \{0\}$. Without loss of generality we can assume η and ζ to be generators of the groups L_η and L_ζ respectively. Define on Y the function

$$f(y) = \begin{cases} \exp\{-\alpha n^2\} & \text{if either } y = n\eta, \text{ or } y = n\zeta,\ n \in \mathbf{Z}, \\ 0 & \text{if } y \in Y \setminus L_\eta \cup L_\zeta, \end{cases}$$

where α is chosen so that $\sum_{n \neq 0} \exp\{-\alpha n^2\} < \frac{1}{2}$. Then

$$\varrho(x) = \sum_{y \in Y} f(y)\overline{(x, y)} \geqslant 0$$

and so $f(y)$ is the characteristic function of a distribution γ on X. By construction, $\gamma \in \Gamma_\infty(X)$ and $\gamma \notin I(X) * \Gamma(X)$. The proof of the necessity is complete.

Sufficiency. If $C_X = \{0\}$, then the equality (7) follows from Proposition 3. Let $C_X \neq \{0\}$. In view of Proposition 3 we can assume X to be connected. Suppose $\gamma \in \Gamma_\infty(X)$. Put $\nu = \gamma * \bar{\gamma} \in \Gamma_\infty(X)$, $E = \{y \in Y\colon \hat{\gamma}(y) \neq 0\}$. There are two possibilities:

(i) Y consists of infinitely divisible elements. We first check that E is a subgroup in Y. Let $y_1, y_2 \in E$, $y_1 \neq 0$, $y_2 \neq 0$. Two cases are possible:

1. $L_{y_1} = L_{y_2}$. Then $py_1 = gy_2$ for some $p, q \in \mathbf{Z}$. Hence in view of (5) we obtain

$$\hat{\gamma}(y_1 + y_2) \neq 0 \Leftrightarrow \hat{\gamma}(p(y_1 + y_2)) \neq 0 \Leftrightarrow \hat{\gamma}((p+q)y_2) \neq 0 \Leftrightarrow \hat{\gamma}(y_2) \neq 0$$

provided that $p + q \neq 0$. If $p + q = 0$, then obviously $\hat{\gamma}(y_1 + y_2) \neq 0$.

2. $L_{y_1} \neq L_{y_2}$. Then $L_{y_1} \cap L_{y_2} = \{0\}$. Put $L = L_{y_1} + L_{y_2}$ and define a monomorphism $\psi\colon L \to \mathbf{R}^2$ in the following way. For any $y \in L$, $y \neq 0$, there are $n, n_1, n_2 \in \mathbf{Z}$, $n \neq 0$, such that $ny = n_1 y_1 + n_2 y_2$. Let $\psi(y) = (r_1, r_2)$, $r_j = n_j/n$, for $y \neq 0$, $\psi(0) = 0$. On the group $\psi(L) \subset \mathbf{R}^2$ we consider the positive-definite function $l(r_1, r_2) = l(\psi(y)) = \hat{\nu}(y)$. From the equality

$$1 - \operatorname{Re} g(t_1 + t_2) \leqslant 2[(1 - \operatorname{Re} g(t_1)) + (1 - \operatorname{Re} g(t_2))],$$

valid for any positive-definite function $g(y)$ on Y and for all $t_1, t_2 \in Y$, we obtain

(8) $$1 - l(r_1, r_2) \leqslant 2[(1 - l(r_1, 0)) + (1 - l(0, r_2))], \quad (r_1, r_2) \in \psi(L).$$

Consider the group $\psi(L_{y_j})$ with discrete topology and write $K_j = (\psi(L_{y_j}))^*$. The functions $l(r_1, 0)$, $(r_1, 0) \in \psi(L_{y_1})$, and $l(0, r_2)$, $(0, r_2) \in \psi(L_{y_2})$, are the characteristic functions of some distributions $\mu_j \in \Gamma_\infty(K_j)$. Since the elements y_1 and y_2 are infinitely divisible, the groups $\psi(L_{y_j})$ consist of infinitely divisible elements. Clearly, all their factor groups also have this property. Applying Corollary 1 and Lemma 3 to the groups K_j and the distributions μ_j, we see that $\mu_j \in \Gamma^s(K_j)$. Consequently, $l(r_1, 0) = \exp\{-\alpha_1 r_1^2\}$, $l(0, r_2) = \exp\{-\alpha_2 r_2^2\}$. Since the subgroups

$$H_1 = \{r_1\colon \psi(y) = (r_1, 0),\ y \in L_{y_1}\}, \quad H_2 = \{r_2\colon \psi(y) = (0, r_2),\ y \in L_{y_2}\}$$

are dense in $\mathbf{R}$, it follows from (8) that the function $l(r_1, r_2)$ is continuous at zero on $\psi(L)$ in the topology induced from $\mathbf{R}^2$.

We now use the inequality

(9) $$|g(t_1) - g(t_2)|^2 \leqslant 2(1 - \operatorname{Re} g(t_1 - t_2)),$$

valid for any positive-definite function $g(y)$ on Y and for all $t_1, t_2 \in Y$. (9) implies that $l(r_1, r_2)$ is uniformly continuous on $\psi(L)$. Since $\psi(L)$ is dense in $\mathbf{R}^2$, the function $l(r_1, r_2)$ can be extended to a continuous positive-definite function $l(s_1, s_2)$ on $\mathbf{R}^2$, also satisfying the system of equations (2). It is easily seen that $l(s_1, s_2)$ is then the characteristic function of a Gaussian distribution. Therefore $l(r_1, r_2) \neq 0$ for $(r_1, r_2) \in \psi(L)$. In particular, $l(1, 1) = \hat{\nu}(y_1 + y_2) = |\hat{\gamma}(y_1 + y_2)|^2 \neq 0$. We have thus proved that E is a subgroup of Y (open, of course).

Denote by δ the distribution on the factor group $X/E^{\perp}$ whose characteristic function is the restriction of $\hat{\gamma}(y)$ to E. Put $\mu = \delta * \bar{\delta}$. Then $\mu \in \Gamma_{\mathrm{U}}(X/E^{\perp})$ by Corollary 1, since $\mu \in \Gamma_{\infty}(X/E^{\perp})$. Since Y consists of infinitely divisible elements, it is easily verified that so does E, and therefore any factor group of E has an infinitely divisible element. Hence $\mu \in \Gamma(X/E^{\perp})$ by Lemma 3. Since the group $X/E^{\perp}$ has no subgroup isomorphic to T, Lemma 2 shows that $\delta \in \Gamma(X/E^{\perp})$. Observe now that the subgroup $E^{\perp}$ is connected. We have thus obtained the following representation for $\hat{\gamma}(y)$:

$$\hat{\gamma}(y) = \begin{cases} ([x_0], y) \exp\{-\varphi_0(y)\} & \text{if } y \in E, \\ 0 & \text{if } y \in Y \setminus E, \end{cases}$$

where $[x_0] \in X/E^{\perp}$ and the function $\varphi_0(y)$ on E is as in (3). Obviously, $2([x_0] = 0$. Arguing as in Remark 1 we obtain the existence of an element $x \in X$ with $2x = 0$ such that $([x_0], y) = (x, y)$ for $y \in E$. We extend $\varphi_0(y)$ to a function $\varphi(y)$ on Y with the same properties (see e.g. [5]). Let $\gamma_0 \in \Gamma^s(X)$ and $\hat{\gamma}_0(y) = \exp\{-\varphi(y)\}$. Then $\gamma = m_{E^{\perp}} * E_x * \gamma_0$, where $\lambda = m_{E^{\perp}} * E_x \in I_{\infty}(X)$, i.e. $\gamma \in I_{\infty}(X) * \Gamma^s(X)$. We have thus proved the sufficiency in case (i).

(ii) Y contains an element which is not infinitely divisible. Then it easily follows from the assumptions of the theorem that X is compact, and so Y is discrete and consists of elements of infinite order. As above, we first check that E is a subgroup of Y. Let $y_1, y_2 \in E$, $y_1 \neq 0$, $y_2 \neq 0$. If y_1 and y_2 are either dependent or infinitely divisible, then, as shown in case (i), $y_1 + y_2 \in E$. It remains to consider the case where y_1 is infinitely divisible and y_2 is not. Then $L_{y_1} \cap L_{y_2} = \{0\}$. Put $L = L_{y_1} + L_{y_2}$. Let $y_3 \in L \setminus L_{y_1} \cup L_{y_2}$. We have $ny_3 = n_1 y_1 + n_2 y_2$ for some $n, n_1, n_2 \in \boldsymbol{Z}$ different from zero. Hence $(n_2 - n_1) y_1 + ny_3 = n_2(y_1 + y_2) \in L_{y_1} + L_{y_3}$. Now ($\alpha$) shows that y_3 is infinitely divisible.

If $y_3 \in E$, then, as shown in case (i), $L_{y_1} + L_{y_3} \subset E$. This implies that $n_2(y_1 + y_2) \in E$, and (5) shows that $y_1 + y_2 \in E$.

Consider now the case where $y \notin E$ for any $y \in L \setminus L_{y_1} \cup L_{y_2}$; we show that this is impossible. Let the monomorphism $\psi\colon L \to \boldsymbol{R}^2$, the function $l(r_1, r_2)$ on $\psi(L)$ and the subgroups H_1 and H_2 of $\boldsymbol{R}$ be defined as in case (i). In our present situation, H_1 is dense in $\boldsymbol{R}$ and $H_2 \approx \boldsymbol{Z}$. Without loss of generality we can assume that $H_2 = \boldsymbol{Z}$. Just as in case (i) we verify that $l(r_1, 0) = \exp\{-\alpha_1 r_1^2\}$, $(r_1, 0) \in \psi(L_{y_1})$. It is also clear that $l(0, n) = \exp\{-\alpha_2 n^2\}$, $(0, n) \in \psi(L_{y_2})$. The function $l(r_1, n)$ is continuous at zero on the group $\psi(L)$ in the topology induced from $\boldsymbol{R} + \boldsymbol{Z}$, and so in view of the inequality (9) it is uniformly continuous on $\psi(L)$. Since $\psi(L)$ is dense in $\boldsymbol{R} + \boldsymbol{Z}$, we can extend $l(r_1, n)$ to a positive-definite function $l(s, n)$ on $\boldsymbol{R} + \boldsymbol{Z}$. Since by assumption $(L \setminus L_{y_1} \cup L_{y_2}) \cap E = \emptyset$, it follows that $l(r_1, n) = 0$ for all $(r_1, n) \in \psi(L)$ with $r_1 n \neq 0$. But this contradicts the continuity of $l(s, n)$ at $(0, 1)$. The proof that E is a subgroup of Y is complete.

Just as in case (i), let δ be the distribution on $X/E^{\perp}$ whose characteristic function is the restriction of $\hat{\gamma}(y)$ to E. We show that $\delta \in \Gamma(X/E^{\perp})$.

Suppose $X/E^{\perp} \not\approx T$. It is easily seen that any factor group of E then has an infinitely divisible element and the factor group $X/E^{\perp}$ has no subgroup isomorphic to T. The argument in this case is the same as in case (i). On the other hand, if $X/E^{\perp} \approx T$, then $\delta \in \Gamma(X/E^{\perp})$ by Lemma 1. The proof is now finished as in case (i). This completes the proof of Theorem 1.

Remark 2. Condition (β) of Lemma 3 is necessary and sufficient for X to have the following property: if $\gamma \in \Gamma_{\infty}(X)$ and $\hat{\gamma}(y) \neq 0$ for all $y \in Y$, then $\gamma \in \Gamma(X)$.

Proof. If X does not satisfy (β), then by Lemma 3 there is a distribution $\gamma \in \Gamma_{\mathrm{U}}(X)$ such that $\hat{\gamma}(y) > 0$ for all $y \in Y$ and $\gamma \notin \Gamma(X)$. The necessity now follows from Corollary 1.

Sufficiency. Suppose $\gamma \in \Gamma_{\infty}(X)$ and $\hat{\gamma}(y) \neq 0$ for all $y \in Y$. Then $\gamma \in \Gamma_{\mathrm{U}}(X)$ by Proposition 1, and so $\gamma \in \Gamma(X)$ by Lemma 3.

We complement Theorem 1 with the following assertion:

Proposition 4. *Suppose $\gamma \in \Gamma_{\infty}(X)$ and γ is an infinitely divisible distribution. Then $\gamma \in I_{\infty}(X) * \Gamma^s(X)$.*

For the proof we need

Lemma 4 ([3]). *Suppose that an infinitely divisible distribution γ on a group X has no idempotent factors. Let $\nu = \gamma * \bar{\gamma}$. Then*

$$\hat{\nu}(2y) \geqslant (\hat{\nu}(y))^4 \tag{10}$$

for every $y \in Y$. Equality holds in (10) *for all $y \in Y$ if and only if $\gamma \in \Gamma(X)$.*

Proposition 4 can now be proved as follows. Consider $E = \{y \in Y\colon \hat{\nu}(y) \neq 0\}$. Since ν is an infinitely divisible distribution, E is a subgroup of Y ([7]). Since $\nu \in \Gamma_{\infty}(X)$, the restriction of $\hat{\nu}(y)$ to E satisfies the equation (5) for $n = 2$. By Lemma 4 this restriction is the characteristic function of a Gaussian distribution. Clearly, in the class of infinitely divisible distributions a Gaussian distribution has only Gaussian factors; it follows that the restriction of $\hat{\gamma}(y)$ to E is also the characteristic function of a Gaussian distribution. The argument is now finished just as in the proof of the sufficiency in Theorem 1.

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