convergence of

$$\sum_{k=1}^{n} k^{-1}[T^k(f-Tf)-T^{-k}(f-Tf)]$$
$$= f+Tf-\frac{1}{n}(T^{n+1}f+T^{-n}f)-\sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)(T^{k+1}f+T^{-k}f)$$

as $n\uparrow\infty$; by (5) and (6) we see that the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n} k^{-1}[T^k(f-Tf)-T^{-k}(f-Tf)]$$

exists a.e. on X. This completes the proof.

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On the geometry of spaces of C_0K-valued operators

by

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Abstract. Let K be a locally compact Hausdorff space and X a Banach space. We consider operator spaces W in $L(X, C_0K)$ which contain the compact operators and have the property that $T\in W$ implies $M_h\circ T\in W$ for every bounded continuous scalar-valued function h on K (M_h denotes the multiplication operator $f\mapsto hf$ on C_0K).

Our main results center around the M-structure properties of such spaces W. We characterize the centralizer of W if the centralizer of X' is small, and for many classes of Banach spaces X (including e.g. the L^1-preduals) we are able to describe all M-ideals of W, at least in the case of compact K.

These characterizations generalize results of Flinn and Smith who discussed the case $W = L(CK, CK)$ if the scalars are complex.

With our methods we also can treat questions as "Is K determined by W?" or "When can W be a dual space?". We are able to derive answers which generalize recent results of Cambern and Greim.

1. Introduction. Let X be a real or complex Banach space (the scalar field, $\boldsymbol{R}$ or $\boldsymbol{C}$, will be denoted by $\boldsymbol{K}$ in the sequel). The following basic definitions from M-structure theory will be of importance:

1.1. DEFINITION. (i) Let $J\subset X$ be a closed linear subspace. J is called an *M-summand* (resp. *L-summand*) if there is a closed subspace $J^\perp\subset X$ such that $X = J\oplus J^\perp$ algebraically and $\|x+x^\perp\| = \max\{\|x\|, \|x^\perp\|\}$ (resp. $=\|x\|+\|x^\perp\|$) whenever $x\in J$, $x^\perp\in J^\perp$. J is called an *M-ideal* if J^π, the annihilator of J in X', is an L-summand.

(ii) Let $T: X\to X$ be an operator. T is called a *multiplier* if for every extreme functional p (i.e. for every extreme point p of the dual unit ball) there is a scalar $a_T(p)$ such that $p\circ T = a_T(p)p$. $\mathrm{Mult}(X)$ will denote the collection of all multipliers.

A multiplier S is called the *adjoint* of a multiplier T (and we write $S = T^*$ in this case) if $a_S(p)$ is the complex conjugate of $a_T(p)$ for every p.

$Z(X)$, the *centralizer* of X, is the set of all multipliers which admit an adjoint.

These definitions have been introduced by Cunningham ([8, 9]) and Alfsen–Effros ([1]); for a systematic introduction the reader is referred to Behrends ([2]).

Here we only note that $Z(X)$ and $\mathrm{Mult}(X)$ are always commutative

Banach algebras of operators and that M-ideals can be characterized by suitable intersection properties as follows:

1.2. THEOREM. *For a closed subspace $J \subset X$ the following assertions are equivalent*:

(i) *J is an M-ideal.*

(ii) *For every $n \in N$ and every collection $B_i = B(x_i, r_i)$ $(i = 1, \ldots, n)$ of closed balls with $B_i \cap J \neq \emptyset$ and $\bigcap B_i \neq \emptyset$ we have $(\bigcap B_i(x_i, r_i+\varepsilon)) \cap J \neq \emptyset$ for every $\varepsilon > 0$.*

(iii) *For $x_1, \ldots, x_n \in J$, $x \in X$ with $\|x_i\|, \|x\| \leqslant 1$ and $\varepsilon > 0$ there is $y \in J$ with $\|x_i+x-y\| \leqslant 1+\varepsilon$ for $i = 1, \ldots, n$.*

(iv) *The same as* (iii), *with $n = 3$.*

Proof: [2], Theorem 2.20, and [13].

Applications of M-structure theory range from Banach–Stone-type theorems over the study of L^1-preduals to approximation-theoretical questions for operator algebras (cf. Ch. 5 in [2], [4], and [11] and the references cited there). In connection with these operator algebras the M-ideals of such spaces have attracted much attention. For example, all M-ideals have been characterized in $L(X, X)$ (= the space of bounded linear operators from X to X) for several complex Banach spaces X (cf. the references in [4]). However, there seems to be up to now no infinite-dimensional real space X in the literature for which all M-ideals in $L(X, X)$ have been described (in the sequel we will obtain spaces for which such a description is possible).

The aim of the present paper is to investigate the M-structure properties of certain Banach spaces of operators. For the sake of easy reference we introduce the following.

1.3. DEFINITION. Let X be a Banach space and K a locally compact Hausdorff space. A closed subspace $W \subset L(X, C_0 K)$ will be called an *admissible operator space* if it satisfies:

(i) W contains the compact operators.

(ii) For every $T \in W$ we have $M_h \circ T \in W$ for every $h \in C^b K$; here $C^b K$ denotes the space of bounded $\boldsymbol{K}$-valued continuous functions, and M_h: $C_0 K \to C_0 K$ means the multiplication operator $f \mapsto hf$.

Thus, for example, $K(X, C_0 K)$ (= the compact operators) and $L(X, C_0 K)$ are admissible, and further admissible spaces can be obtained by considering the various *operator ideals* treated in the literature (which, of course, have to have the property that they are complete with respect to the operator norm).

Our *main results* are the following (where W is an admissible operator space in $L(X, C_0 K)$):

– If $Z(X')$ (resp. $\mathrm{Mult}(X')$) is one-dimensional, then the operators in $Z(W)$ (resp. in $\mathrm{Mult}(W)$) are precisely the operators $T \mapsto M_h \circ T$ with $h \in C^b K$.

– It is possible to characterize the M-ideals in W provided that K is compact and

X is finite-dimensional and contains no nontrivial L-summands,
or X' is uniformly convex,
or X' is an abstract L^1-space (in fact it suffices to assume that X' contains "sufficiently many" one-dimensional L-summands);

more precise information is obtained if $X = c_0(I)$ or $X = l^p(I)$, $1 < p < \infty$, for some set I.

The methods are *elementary*, and they apply to the real and complex case as well. The results imply:

– *Banach–Stone-type theorems* for operator spaces: if $W_i \subset L(X_i, C_0 K_i)$ are admissible, if the $Z(X_i)$ are one-dimensional $(i = 1, 2)$, then the Stone–Čech compactifications of K_1 and K_2 are homeomorphic provided that W_1 and W_2 are isometrically isomorphic; these theorems contain the results of Cambern ([6]) and parts of the results of Cambern and Greim ([7]) as special cases;

– the *characterization of the M-ideals* in $L(CK, CK)$, K compact, $\boldsymbol{K} = \boldsymbol{C}$ by Flinn and Smith ([11]) (which is proved by using techniques from the theory of complex Banach algebras and which works by relating the M-ideals of $L(CK, CK)$ with certain elements of the bidual of CK; a characterization which is much better accessible is due to Werner and Werner [14]);

– the existence of infinite-dimensional *real Banach spaces X for which one knows all M-ideals in $L(X, X)$*; take $X = CK$, K infinite and compact.

As the preceding short survey indicates, M-structure properties of X' have to be known to obtain information on admissible operator spaces $W \subset L(X, C_0 K)$. We therefore collect together some facts which concern spaces for which $Z(X')$ resp. $\mathrm{Mult}(X')$ consist only of the constant multiples of the identity operator.

1.4. PROPOSITION. *Let X be a Banach space*; *to avoid notational complications, we will assume in* (ii) *and* (iii) *below that X is not isometrically isomorphic to the two-dimensional real space l_2^∞.*

(i) *$Z(X')$ is one-dimensional iff X contains no L-summands other than* $\{0\}$ *and X.*

(ii) *Each of the following conditions implies that* $\mathrm{Mult}(X')$ (*and thus* $Z(X')$) *is one-dimensional*:

(a) *X' is smooth.*

(b) *X' is strictly convex.*

(c) *X' is an abstract L^1-space* (*e.g.* $X = C_0 K$).

(iii) *Each of the following conditions implies that $Z(X')$ is one-dimensional:*

(a) *X has a nontrivial (i.e. different from $\{0\}$ and X) M-ideal.*

(b) *X' has no nontrivial M-ideals.*

(c) *X has a nontrivial L^p-summand for some $p \in]1, \infty[$ (the definition is similar to that of M-summands or L-summands, but the relevant norm condition is now $\|x+x^{\perp}\|^p = \|x\|^p + \|x^{\perp}\|^p$).*

(d) *X' has a nontrivial L^p-summand for some $p \in [1, \infty[$.*

(e) *X' is an abstract L^p-space for some $1 \leqslant p < \infty$.*

Proof. (i) Suppose that J is a nontrivial L-summand in X. Then the transpose E' of the natural projection E onto J is not in $\mathbb{K}\,\mathrm{Id}$ so that $Z(X')$ is not one-dimensional (E' is in $Z(X')$ by 1.5 and 3.15 in [2]). Conversely, suppose that $Z(X')$ is at least two-dimensional. Since $Z(X')$ is isometrically isomorphic to a space $C(K_X)$, where K_X is compact and extremally disconnected (3.10 and 5.10 in [2]), $Z(X')$ contains a nontrivial idempotent which gives rise to an M-summand in X' and thus to an L-summand in X (3.15, 5.6, and 1.5 in [2]).

(ii) (a) and (c) are proved in Prop. 5.2 of [3], and (b) has been shown in [12] (see also [15]).

(iii) (a) This follows from (i) and [2], Prop. 2.4.

(b) This is proved in 3.16 of [2].

(c) and (d) follow from (a) and (b) by using [2], Theorem 6.2.

(e) is an immediate consequence of (d).

The author wants to express his gratitude to D. Werner and W. Werner for making available their manuscript [14]. It was their results which stimulated the research leading to the present paper.

2. The multiplier and the centralizer for admissible spaces. It is well known that, for every locally compact space K, $Z(C_0K)$ and $\mathrm{Mult}(C_0K)$ consist precisely of the operators M_h, where $h \in C^b K$. These operators give rise to elements in the centralizer of every admissible operator space $W \subset L(X, C_0K)$ which has been observed for the case $W = L(X, C_0K)$ already in [14]:

2.1. Proposition. *Let $W \subset L(X, C_0K)$ be an admissible operator space and $h \in C^b K$. By ϕ_h: $W \to W$ we denote the operator $T \mapsto M_h \circ T$. Then ϕ_h lies in the centralizer of W, and $h \mapsto \phi_h$ is an isometric algebra homomorphism from $C^b K$ into $Z(W)$.*

Proof. It is obvious that $h \mapsto \phi_h$ is an isometric algebra homomorphism from $C^b K$ into $L(W, W)$ so that we only have to show that $\phi_h \in Z(W)$ for every h.

It is straightforward to prove that ϕ_h is M-bounded in the sense of Definition 3.2 in [2] so that, by ([2], Theorem 3.3) ϕ_h lies in $\mathrm{Mult}(W)$. This proves the assertion in the real case.

In the complex case, we consider $\phi_{\bar h}$ ($\bar h$ = the complex conjugate of h). The operators $(1/2)(\phi_h + \phi_{\bar h})$ and $(1/(2i))(\phi_h - \phi_{\bar h})$ are obviously hermitian operators on W so that the eigenvalues of the transposes are necessarily real. This proves that $a_{\phi_{\bar h}} = \overline{a_{\phi_h}}$ so that $\phi_{\bar h} = \phi_h^*$. Therefore ϕ_h admits an adjoint, i.e. $\phi_h \in Z(W)$.

In order to treat the problem whether all $\phi \in Z(W)$ have the form ϕ_h we need two preliminary results. The first seems to be well-known; nevertheless, a proof is included since we are not aware of a reference.

2.2. Lemma. *Let X be a Banach space and $T \in \mathrm{Mult}(X)$. Then, for x in the kernel of T and $y = T\tilde y$ in the range of T we have $\|ax+by\| = \max\{\|ax\|, \|by\|\}$ for arbitrary scalars a, b* (which just means that $\ker T$ and range T satisfy the norm condition for M-summands in Definition 1.1; in general, however, these spaces are not M-summands of X since they do not span the whole space).

Proof. Without loss of generality we may assume that $a = b = 1$, $x \neq 0 \neq y$. We first note that $a_T(p) = 0$ for every extreme functional p such that $p(x) \neq 0$ (since $0 = p(Tx) = a_T(p)\,p(x)$).

Choose a p with $p(x) = \|x\|$. Then $p(y) = a_T(p)\,p(\tilde y) = 0$ so that $\|x\| = p(x) = p(x+y) \leqslant \|x+y\|$. Similarly it follows that $\|y\| \leqslant \|x+y\|$, since $p(y) = \|y\|$ yields $a_T(p) \neq 0$ and thus $p(x) = 0$.

Conversely, let p be given with $p(x+y) = \|x+y\|$. If $a_T(p) \neq 0$ then necessarily $p(x) = 0$ so that $\|x+y\| = p(y) \leqslant \|y\|$. If $a_T(p) = 0$, then $p(y) = 0$, and in this case we get $\|x+y\| = p(x) \leqslant \|x\|$.

2.3. Lemma. *Let $W \subset L(X, C_0K)$ be an admissible operator space.*

(i) *Let T be an operator from X to C_0K* with $\|T\| \leqslant 1$ *such that $\|x' \otimes h \pm T\| \leqslant 1$ for every $x' \in X'$, $h \in C_0K$ with $\|x'\| = \|h\| = 1$; here $x' \otimes h$ denotes the operator $x \mapsto x'(x)h$. Then $T = 0$.*

(ii) *Let $\phi \in \mathrm{Mult}(W)$ be given such that $\{x' \otimes h \mid x' \in X',\ h \in C_0K\}$ lies in the kernel of ϕ. Then $\phi = 0$.*

Proof. (i) For $x \in X$ with $\|x\| = 1$ and $k_0 \in K$ choose $x' \in X'$, $h \in C_0K$ with $\|h\| = h(k_0) = \|x'\| = x'(x) = 1$. Then, by assumption,

$$|x'(x)\,h(k_0) \pm (Tx)(k_0)| = |1 \pm (Tx)(k_0)| \leqslant 1$$

which yields $(Tx)(k_0) = 0$. Since x and k_0 are arbitrary, it follows that $T = 0$.

(ii) This is a consequence of Lemma 2.2 and part (i).

2.4. THEOREM. *Suppose that $W \subset L(X, C_0 K)$ is an admissible operator space.*

(i) *If* Mult (X') *is one-dimensional, then* $\mathrm{Mult}(W) = \{\phi_h \mid h \in C^b K\}$.

(ii) *If* $Z(X')$ *is one-dimensional, then* $Z(W) = \{\phi_h \mid h \in C^b K\}$.

(*Therefore* $\mathrm{Mult}(W) \cong \mathrm{Mult}(C_0 K) \cong C^b K$ *in the first and* $Z(W) \cong Z(C_0 K) \cong C^b K$ *in the second case.*)

Proof. (i) Let $\phi \in \mathrm{Mult}(W)$ be given. We first note that, since all ϕ_h are multipliers, we have $\phi \circ \phi_h = \phi_h \circ \phi$ ([2], p. 54) which means that

$$(*) \qquad M_h \circ \phi(T) = \phi(M_h \circ T) \quad (\text{all } h \in C^b K,\ T \in W).$$

Now let $k_0 \in K$ be given. We choose a function $h \in C_0 K$ with $\|h\| = 1$ and $h|_U = 1$ for a suitable neighbourhood U of k_0. $\omega_{k_0}\colon X' \to X'$ will be defined by $x' \mapsto \delta_{k_0} \circ \phi(x' \otimes h)$; here δ_{k_0} denotes the usual evaluation functional.

1. $\omega_{k_0} \in \mathrm{Mult}(X')$.

This follows from the fact that ϕ is M-bounded from which we conclude that ω_{k_0} is M-bounded as well. So one only has to apply [2], Theorem 3.3.

2. *The definition of* ω_{k_0} *does not depend on the choice of* h.

Suppose that h and h' are functions in $C_0 K$ with $\|h\| = \|h'\| = 1$, $h|_U = 1$, $h'|_{U'} = 1$ for suitable neighbourhoods U, U' of k_0. We choose a function $g \in C_0 K$ with $g(k_0) = \|g\| = 1$ which has its support in $U \cap U'$. It follows that $(h-h')g = 0$ so that, for every x',

$$\begin{aligned} 0 = \delta_{k_0} \circ \phi(x' \otimes (h-h')g) &= \delta_{k_0} \circ \phi\big(M_g \circ (x' \otimes (h-h'))\big) \\ &= \delta_{k_0} \circ \big(M_g \circ \phi(x' \otimes (h-h'))\big) \quad (\text{by } (*)) \\ &= \delta_{k_0} \circ \phi(x' \otimes (h-h')) \end{aligned}$$

which proves that $\delta_{k_0} \circ \phi(x' \otimes h) = \delta_{k_0} \circ \phi(x' \otimes h')$.

By assumption there is an $h_0(k_0) \in K$ such that $\omega_{k_0} = h_0(k_0)\,\mathrm{Id}$. We claim that

3. h_0 *is bounded and continuous.*

h_0 is clearly bounded by $\|\phi\|$. Let $k_0 \in K$ be given and h as in the definition of ω_{k_0}. Let U be an open neighbourhood of k_0 such that $h|_U = 1$. By "2." we can define ω_k for all $k \in U$ by taking the same function h, i.e.

$$\delta_k \circ \phi(x' \otimes h) = \omega_k(x') = h_0(k)\,x' \quad \text{for } k \in U.$$

This means that $k \mapsto h_0(k)x'$ coincides on U with $\phi(x' \otimes h)$. Therefore $k \mapsto h_0(k)\,x'(x)$ is continuous on U for arbitrary x, x' which proves that h_0 is continuous at k_0.

Now suppose that $g_1, g_2 \in C_0 K$, $k_0 \in K$, and that g_1 and g_2 coincide in a neighbourhood of k_0. We claim that

4. $\phi(x' \otimes g_1) = \phi(x' \otimes g_2)$ *in a neighbourhood of* k_0 (*all* $x' \in X'$).

The proof is similar to the proof of "2.", the details are therefore omitted.

5. $\phi(x' \otimes g) = M_{h_0} \circ (x' \otimes g)$ *for* $x' \in X'$ *and* $g \in C_0 K$.

Let $k_0 \in K$ and let $h \in C_0 K$ be a function with $\|h\| = 1$ and $h = 1$ in a neighbourhood U of k_0. Then, by definition, $\phi(x' \otimes h) = M_{h_0} \circ (x' \otimes h)$ on the interior of U. Since hg and g coincide on U, "4." implies that

$$\begin{aligned} \delta_{k_0} \circ \phi(x' \otimes g) = \delta_{k_0} \circ \phi(x' \otimes hg) &= \delta_{k_0} \circ \phi\big(M_g \circ (x' \otimes h)\big) \\ &= \big(g(k_0)\big)\,\delta_{k_0} \circ \phi(x' \otimes h) \quad \text{by } (*) \\ &= g(k_0)\,h_0(k_0)\,x' = \delta_{k_0} \circ M_{h_0} \circ (x' \otimes g). \end{aligned}$$

This proves our claim since k_0 was arbitrary.

Finally, we show that $\phi = \phi_{h_0}$. But this is easy, since $\phi - \phi_{h_0}$ lies in $\mathrm{Mult}(W)$, and all $x' \otimes g$ are in the kernel by "5.". Thus we only have to apply the preceding lemma.

(ii) Let $\phi \in Z(W)$ be given (the adjoint of ϕ will be denoted by ϕ^*). The preceding proof will lead us to the desired result provided that we are able to show that the operators $\omega_{k_0}\colon X' \to X'$ not only are multipliers but lie in the centralizer of X'.

If $K = R$ nothing has to be proved so that we may assume that the scalars are complex.

We fix k_0 and define $\hat{\omega}_{k_0}\colon X' \to X'$ by $x' \mapsto \delta_{k_0} \circ \phi^*(x' \otimes h)$, where h is as in the definition of ω_{k_0}. Clearly $\hat{\omega}_{k_0} \in \mathrm{Mult}(X')$, and we claim that $\hat{\omega}_{k_0} = \omega_{k_0}^*$. It suffices to show that $\omega_1 := (1/2)(\omega_{k_0} + \hat{\omega}_{k_0})$ and $\omega_2 := \big(1/(2i)\big)(\omega_{k_0} - \hat{\omega}_{k_0})$ are hermitian. To this end, let $\varphi \in \big(L(X')\big)'$ be given with $\varphi(\mathrm{Id}_{X'}) = \|\varphi\| = 1$. We define $\tilde{\varphi} \in \big(L(W)\big)'$ by $\tilde{\varphi}(\tilde{\phi}) := \varphi(\tilde{\omega}_{k_0})$, where $\tilde{\omega}_{k_0}(x') := \delta_{k_0} \circ \tilde{\phi}(x' \otimes h)$. Then $\tilde{\varphi}(\mathrm{Id}_W) = \|\tilde{\varphi}\| = 1$ so that, since $(1/2)(\phi + \phi^*)$ and $\big(1/(2i)\big)(\phi - \phi^*)$ are hermitian, $\varphi(\omega_1) = \tilde{\varphi}\big((1/2)(\phi + \phi^*)\big)$ and $\varphi(\omega_2) = \tilde{\varphi}\big((1/(2i))(\phi - \phi^*)\big)$ are real which completes the proof.

We now turn to some *applications* of the preceding theorem.

2.5. THEOREM. *Suppose that* $Z(X')$ *is one-dimensional and that* $W \subset L(X, C_0 K)$ *is an admissible operator space. Then there are as many M-summands in W as there are clopen subsets of K. More precisely, the M-summands of W are just the subspaces* $\{T \mid T \in W,\ \delta_k \circ T = 0$ *for every* $k \in D\}$, *where* $D \subset K$ *is clopen.*

Proof. This follows at once from the fact that the M-summands are in one-to-one correspondence with the idempotents of the centralizer.

2.6. THEOREM. *Suppose that X_1 and X_2 are Banach spaces such that the dual spaces have one-dimensional centralizer. Then the following Banach–Stone-type theorem is valid: If $W_i \subset L(X_i, C_0 K_i)$, $i = 1, 2$, are admissible operator spaces, then the existence of an isometric isomorphism between W_1 and W_2 implies that βK_1 and βK_2 are homeomorphic ($\beta K :=$ the Stone–Čech compactification of K).*

Proof. $W_1 \cong W_2$ yields $C(\beta K_1) \cong C^b K_1 \cong Z(W_1) \cong Z(W_2) \cong C^b K_2 \cong C(\beta K_2)$ so that $\beta K_1 \cong \beta K_2$ is a consequence of the classical Banach–Stone theorem.

Notes. 1) The result is only a first step towards an understanding of such Banach–Stone-type theorems for operator spaces. For example, it would be interesting to know whether not only βK_1 and βK_2 are homeomorphic but also K_1 and K_2, and whether or not X'_1 and X'_2 are necessarily isometrically isomorphic (as the simple case $K = \{k_0\}$ shows it is not to be expected that $W_1 = W_2$ yields $X_1 = X_2$).

2) For $W_i = L(E, CK_i)$, $i = 1, 2$, where E is a reflexive space with strictly convex dual and where the K_i are extremally disconnected and compact, the preceding theorem has been proved by Cambern [6] (note that by Prop. 1.4, $Z(E')$ is one-dimensional). The case $W_i = L(E_i, CK_i)$, $i = 1, 2$, where $Z(E_i)$ are one-dimensional and K_i are compact and hyperstonean, is treated by Cambern and Greim in [7]. There also some information on the structure of isometric isomorphisms between $L(E_1, CK_1)$ and $L(E_2, CK_2)$ is obtained.

2.7. THEOREM. *Suppose that $W \subset L(X, C_0 K)$ is an admissible operator space and that $Z(X')$ is one-dimensional. If W is a dual Banach space, then βK is necessarily hyperstonean.*

Proof. The centralizer of a dual Banach space is always of the form $C\hat{K}$, where $\hat{K}$ is compact and hyperstonean (see [5], 3.2 and [2], 5.10). Therefore our assertion follows from Theorem 2.5 and the Banach–Stone theorem.

3. M-ideals in admissible operator spaces: general results. In this section W will be a fixed admissible operator space in $L(X, C_0 K)$. The following definition is due to D. Werner and W. Werner [14]; they used it for the case $W = L(CK, CK)$ to obtain an intrinsic description of the M-ideals of this space.

3.1. DEFINITION. For every closed subset $D \subset \beta K$ let $J_{(D)}$ be the subspace $\{T \mid T \in W, \|\delta_k \circ T\| \to 0 \text{ for } k \in K, k \to D\}$.

3.2. PROPOSITION. *$J_{(D)}$ is an M-ideal in W.*

Proof. By 1.2 we have to show that, for $T \in W$, $T_1, T_2, T_3 \in J_{(D)}$ with $\|T\|, \|T_i\| \leqslant 1$, $\varepsilon > 0$, there is an $S \in J_{(D)}$ such that $\|T_i + T - S\| \leqslant 1 + \varepsilon$ for $i = 1, 2, 3$.

Let such operators T, T_1, T_2, T_3 be given. Since the T_i are contained in $J_{(D)}$ there is an open neighbourhood U of D such that $\|\delta_k \circ T_i\| \leqslant \varepsilon$ for $k \in K \cap U$ and $i = 1, 2, 3$.

Choose a function $h \in C^b K$ such that its canonical extension h_0 to βK satisfies $0 \leqslant h_0 \leqslant 1$, $h_0|_D = 1$, $\operatorname{supp} h_0 \subset U$. We define $S := M_{1-h} T$; S is in W since W is admissible. It is easy to check that S lies in $J_{(D)}$ and that $\|\delta_k(T_i + T - S)\| \leqslant 1 + \varepsilon$ for every $k \in K$ so that $\|T_i + T - S\| \leqslant 1 + \varepsilon$.

Since, as is well known, the M-ideals in $C_0 K$ are precisely the subspaces of the form $\{f \mid f|_{D \cap K} = 0\}$, where $D \subset \beta K$ is closed, the M-ideals $J_{(D)}$ are in a sense just those M-ideals which are determined by $C_0 K$. We will say that *the M-ideals of W are determined by the M-ideals of $C_0 K$* if there are no other M-ideals, i.e. if for every M-ideal $J \subset W$ there is a closed set $D \subset \beta K$ such that $J = J_{(D)}$.

3.3. THEOREM. *Let $W \subset L(X, C_0 K)$ be an admissible operator space. Suppose that:*

(i) *For every $k_0 \in \beta K$ and every M-ideal J which is strictly larger than $J_{(\{k_0\})}$ we have $J = W$ (i.e. the $J_{(\{k_0\})}$ are maximal M-ideals).*

(ii) *For every $k_0 \in \beta K$ for which there exists a $T \in W$ with*

$$\limsup_{k \to k_0} \|\delta_k \circ T\| > 0$$

there also exists an $S \in W$ with

$$\liminf_{k \to k_0} \|\delta_k \circ S\| > 0$$

(which, for example, is automatically satisfied if K is compact).

Then the M-ideals of W are determined by the M-ideals of $C_0 K$.

Proof. Let $J \subset W$ be an M-ideal. We define

$$D := \{k_0 \mid k_0 \in \beta K, \|\delta_k \circ T\| \to 0 \text{ as } k \to k_0 \text{ for every } T \in J\}.$$

Suppose for the moment that it has been verified that D is closed.

We will show that $J = J_{(D)}$. $J \subset J_{(D)}$ is trivially true so that it remains to prove the reverse inclusion. Let $T \in J_{(D)}$ and $\varepsilon > 0$ be given. We claim that

(*) For every $k_0 \in \beta K$ there are an open neighbourhood U_{k_0} of k_0 and an operator $S_{k_0} \in J$ such that

$$\|\delta_k \circ (T - S_{k_0})\| \leqslant \varepsilon \quad \text{for } k \in U_{k_0} \cap K.$$

If $k_0 \in D$, we simply take $S_{k_0} = 0$ and any neighbourhood U_{k_0} of k_0 such that $\|\delta_k \circ T\| \leqslant \varepsilon$ for $k \in U_{k_0} \cap K$. For $k_0 \notin D$ we have $J \not\subset J_{(\{k_0\})}$ by the definition of D and $J_{(\{k_0\})}$ so that $J + J_{(\{k_0\})}$ is an M-ideal which is strictly larger than $J_{(\{k_0\})}$ (note that the sum of two M-ideals is also an M-ideal; [2], Prop. 2.7). By (i) it follows that $J + J_{(\{k_0\})} = W$ so that in particular $T - S_{k_0} \in J_{(\{k_0\})}$ for a suitable $S_{k_0} \in J$. Clearly then $\|\delta_k \circ (T - S_{k_0})\| \leqslant \varepsilon$ in a neighbourhood U_{k_0} of k_0.

Thus (*) is true for every k_0, and we may cover βK by finitely many $U_{k_1}, \ldots, U_{k_n}$. If $\tilde{h}_1, \ldots, \tilde{h}_n \in C(\beta K)$ is a partition of unity subordinate to this cover we have, with $h_i := \tilde{h}_i|_K$,

$$\|\delta_k \circ (\sum M_{h_i} \circ S_{k_i} - T)\| \leqslant \varepsilon \quad \text{for every } k \in K$$

so that $\|\sum M_{h_i} \circ S_{k_i} - T\| \leqslant \varepsilon$. But M-ideals are invariant with respect to every operator in the centralizer ([2], Prop. 3.14) so that, by Proposition 2.1, the operator $S := \sum M_{h_i} \circ S_{k_i}$ belongs to J. Since $\varepsilon > 0$ was arbitrary, this shows that $T \in J$.

To complete the proof we have to verify that D is closed. Suppose that $k_0 \notin D$. Then, by definition, $J + J_{(\{k_0\})}$ is strictly larger than $J_{(\{k_0\})}$ so that this M-ideal is all of W. By (ii) there is an $S \in W$ such that

$$\liminf_{k \to k_0} \|\delta_k \circ S\| > 0.$$

We write $S = T + R$ with $T \in J$, $R \in J_{(\{k_0\})}$. Then clearly also

$$\liminf_{k \to k_0} \|\delta_k \circ T\| > 0,$$

and this implies that $U \cap D = \emptyset$ for a suitable neighbourhood U of k_0.

Note. Since the M-ideals which contain $J_{(\{k_0\})}$ are in one-to-one correspondence with the M-ideals in $W/J_{(\{k_0\})}$ ([2], Prop. 2.9(ii)) the condition (i) may be rephrased by saying that this quotient contains no nontrivial M-ideals.

We close this section with *some simple applications* of the preceding theorem.

The first corollary is included only to illustrate our methods and to prepare a proof which will be given later. It can be derived independently by well-known methods from M-structure theory.

3.4. Corollary. *Let X be a finite-dimensional Banach space. Then the following assertions are equivalent*:

(i) *For every admissible operator space $W \subset L(X, C_0 K)$ and every K, the M-ideals of $C_0 K$ determine the M-ideals of W.*

(ii) *X' contains no nontrivial M-ideals*

(or, by well-known results, X contains no nontrivial L-summands, or $Z(X')$ is one-dimensional).

Proof. (i)⇒(ii). An application of (i) with $K = \{k_0\}$ guarantees that $W = L(X, C\{k_0\}) = X'$ has precisely the M-ideals X' $(= J_{(\emptyset)})$ and $\{0\}$ $(= J_{(\{k_0\})})$.

(ii)⇒(i). Let K and W be given. From the continuity of the functions $k \mapsto \|\delta_k \circ T\|$ for all $T \in W$ it follows that

$$W/J_{(\{k_0\})} \cong \begin{cases} X' & \text{if } k_0 \in K, \\ 0 & \text{if } k_0 \in \beta K \setminus K, \end{cases}$$

so that 3.3(i) is satisfied. It is obvious that 3.3(ii) is also valid.

3.5. Corollary. *Let $1 < p < \infty$, I a set and X the space $l^p(I)$. Then, if $W \subset L(X, C_0 K)$ is an admissible operator space, we have*:

(i) *For every $k_0 \in K$, the space $W/J_{(\{k_0\})}$ can be written as $Y_{k_0} \oplus Y_{k_0}^{\perp}$, where $\|y + y^{\perp}\|^q = \|y\|^q + \|y^{\perp}\|^q$ for $y \in Y_{k_0}$, $y^{\perp} \in Y_{k_0}^{\perp}$, i.e. Y_{k_0} is an L^q-summand in the sense of 1.4(iii)(c); in addition, Y_{k_0} is isometrically isomorphic to $X' \cong l^q(I)$ (here, of course, q is defined by $1/p + 1/q = 1$).*

(ii) *The M-ideals of W are determined by the M-ideals of $C_0 K$ provided that K is compact and* $\operatorname{card} I \geqslant 2$.

Proof. The case of finite I follows from 3.4 since $l^q(I)$ never has nontrivial M-ideals. We therefore may assume that I is infinite.

(i) Let $k_0 \in K$ be given. We define two subspaces $W_{k_0}^c$, $W_{k_0}^0$ of W by

$$W_{k_0}^c := \{T \mid T \in W,\ \|\delta_k \circ T - \delta_{k_0} \circ T\| \to 0 \text{ as } k \to k_0\},$$

$$W_{k_0}^0 := \{T \mid T \in W,\ \delta_{k_0} \circ T = 0\}.$$

Clearly these subspaces are closed, and $W_{k_0}^c \cap W_{k_0}^0 = J_{(\{k_0\})}$. Let Y_{k_0} and $Y_{k_0}^{\perp}$ be the canonical images of $W_{k_0}^c$ and $W_{k_0}^0$ in $W/J_{(\{k_0\})}$, respectively. It is routine to show that $Y_{k_0} \cong X'$ (not only in the present case). Since the quotient norm of an operator $T \in W$ in $W/J_{(\{k_0\})}$ is given by $\liminf_{k \to k_0} \|\delta_k \circ T\|$, the claimed norm condition is a consequence of the following observation: if $x'_\alpha, x' \in l^q(I) = l^p(I)'$ are such that the x'_α tend to 0 with respect to the weak* topology, then

$$\limsup \|x' + x'_\alpha\|^q = \|x'\|^q + \limsup \|x'_\alpha\|^q;$$

this is an immediate consequence of the fact that the x'_α have to tend pointwise on I to zero.

(ii) Let K be compact. By (i), the spaces $W/J_{(\{k_0\})}$ contain nontrivial L^q-summands for every $k_0 \in \beta K = K$ so that all M-ideals of these spaces are trivial (if a space would contain a nontrivial M-ideal and a nontrivial L^q-summand, then its dual would contain both a nontrivial L-summand and a nontrivial L^p-summand in contradiction to [2], Theorem 6.2).

3.6. Corollary. *Let I be an index set and $X := c_0(I)$. Then, if $W \subset L(X, C_0 K)$ is an admissible operator space, we have:*

(i) *For every $k_0 \in K$, $W/J_{(\{k_0\})}$ is the direct sum of two L-summands Y_{k_0}, $Y_{k_0}^\perp$; Y_{k_0} is isometrically isomorphic to $X' \cong l^1(I)$.*

(ii) *The M-ideals of W are determined by the M-ideals of $C_0 K$ if K is compact, $\operatorname{card} K > 2$, and $\operatorname{card} I \geqslant 2$.*

Proof. This can be shown similarly to the proof of Cor. 3.5.

3.7. Corollary *Let $1 < p_1 < p_2 < \infty$, let I_0, I_1, I_2 be index sets, K_0, K_1, K_2 compact spaces and W_0, W_1, W_2 admissible operator spaces in $L(c_0(I_0), CK_0)$, $L(l^{p_1}(I_1), CK_1)$, $L(l^{p_2}(I_2), CK_2)$, respectively. Then W_0, W_1, W_2 are pairwise not isometrically isomorphic provided that $\operatorname{card} I_j \geqslant 2$ $(j = 0, 1, 2)$.*

Proof. We prove that, for example, $W_0 \not\cong W_1$. Suppose that there were an isometric isomorphism φ: $W_0 \to W_1$. Since for any fixed $k_0 \in K_0$ the subspace $J_{(\{k_0\})} \subset W_0$ is a maximal M-ideal, $\varphi(J_{(\{k_0\})}) \subset W_1$ necessarily also has this property and thus is of the form $J_{(\{k_1\})}$ for a suitable $k_1 \in K_1$. The quotients $W_0/J_{(\{k_0\})}$ and $W_1/J_{(\{k_1\})}$ are then also isometrically isomorphic. But this is impossible: the first space has a nontrivial L-summand whereas the second admits a nontrivial L^{q_1}-summand which never occurs at the same time ([2], Th. 6.2).

4. M-ideals in admissible operator spaces $W \subset L(X, C_0 K)$ with K compact, X' uniformly convex. Here we will show that Corollary 3.5(ii) generalizes to the case of arbitrary X with uniformly convex dual.

4.1. Theorem. *Let X be a Banach space such that X' is uniformly convex, and let K be a compact Hausdorff space. Then CK determines the M-ideals of every admissible operator space $W \subset L(X, CK)$, i.e. the M-ideals of W are precisely the subspaces $J_{(D)}$, $D \subset K$ closed* (see Def. 3.1).

Proof. Since condition 3.3(ii) is obviously valid it remains to show that 3.3(i) is satisfied, too.

To this end let J be an M-ideal of W with $J \supsetneq J_{(\{k_0\})}$, where $k_0 \in K$. This means that there is a $T \in J$ with $\limsup_{k \to k_0} \|\delta_k \circ T\| > 0$, and by considering $M_h \circ T$ instead of T for a suitable $h \in CK$ we may assume that $\|T\| = \limsup \|\delta_k \circ T\| = 1$.

Now let $x'_0 \in X'$ with $\|x'_0\| = 1$ be arbitrary and $\varepsilon_0 > 0$. Since X' is uniformly convex we find an $\varepsilon > 0$ with

$$(*) \qquad \|x'\| \geqslant 1 - \varepsilon,\ \|x' \pm z'\| \leqslant 1 + \varepsilon \Rightarrow \|z'\| \leqslant \varepsilon_0.$$

By using Theorem 1.2 we obtain an operator $S \in J$ with $\|S\| \leqslant 1 + \varepsilon$ and $\|\pm T + (x'_0 \otimes 1 - S)\| \leqslant 1 + \varepsilon$ so that, by $(*)$, $\|x'_0 - \delta_k \circ S\| \leqslant \varepsilon_0$ whenever $\|\delta_k \circ T\| \geqslant 1 - \varepsilon$. It follows that $\limsup_{k \to k_0} \|\delta_k \circ S\| \geqslant 1 - \varepsilon_0$.

We are now able to prove that $J = W$. Let $R \in W$ be given, $\|R\| = 1$, and $\varepsilon_0 > 0$. Choose ε for ε_0 as in $(*)$ and an operator $T_\varepsilon \in J$ with

$$\|T_\varepsilon\| = \limsup \|\delta_k \circ T_\varepsilon\| = 1, \qquad \liminf \|\delta_k \circ T_\varepsilon\| > 1 - \varepsilon;$$

such a T_ε exists by the first part of the proof. Another application of Theorem 1.2 provides us with an $S \in J$ with $\|S\| \leqslant 1 + \varepsilon$, $\|\pm T_\varepsilon + (R - S)\| \leqslant 1 + \varepsilon$, and $(*)$ implies that $\|\delta_k \circ (R - S)\| \leqslant \varepsilon_0$ in a suitable neighbourhood of k_0. Choose a continuous function h: $K \to [0, 1]$ with support in this neighbourhood and $h(k_0) = 1$. Then $\|M_h \circ (R - S)\| \leqslant \varepsilon_0$, and consequently $\|R - (M_{1-h} \circ R + M_h \circ S)\| \leqslant \varepsilon_0$. But $M_{1-h} \circ R + M_h \circ S \in J$ since $J \supset J_{(\{k_0\})}$ and M-ideals are invariant with respect to operators in the centralizer. It follows that $R \in J$, and this completes the proof of the theorem.

5. M-ideals in admissible operator spaces $W \subset L(X, C_0 K)$ with K compact and sufficiently many one-dimensional L-summands in X'. In the preceding section we used uniform convexity to conclude that $\|z'\|$ is small whenever $\|x' \pm z'\| \leqslant 1 + \varepsilon$, where $\|x'\| \geqslant 1 - \varepsilon$. Here we will develop and apply a subtle technique which allows similar conclusions provided that there are "sufficiently many" one-dimensional L-summands.

5.1. Lemma. *Let Y be a Banach space, $\eta \in Y$ such that $\|\eta\| = 1$ and $\mathbf{K}\eta$ is an L-summand. Further, let $x, y \in Y$ and $\varepsilon_i \geqslant 0$ $(i = 1, \ldots, 5)$ be given such that*

– $1 - \varepsilon_1 \leqslant \|x\| \leqslant 1 + \varepsilon_2$,

– $\|x + \theta\eta\| \geqslant 2 - \varepsilon_3$ *(for all $\theta \in \mathbf{K}$ with $|\theta| = 1$),*

– $\|y\| \leqslant 1 + \varepsilon_4$.

Then $\|\theta x + (\eta - y)\| \leqslant 1 + \varepsilon_5$ (for all $|\theta| = 1$) implies that $\|\eta - y\| \leqslant \varepsilon_4 + 2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_5)$.

Proof. We write $x = \alpha\eta + x_1$, $y = \beta\eta + y_1$, where x_1, y_1 are in $(\mathbf{K}\eta)^\perp$. We have

$$2 - \varepsilon_3 \leqslant \|x + \theta\eta\| = |\theta + \alpha| + \|x_1\|$$

for every $|\theta| = 1$ so that

$$2 - \varepsilon_3 \leqslant 1 - |\alpha| + \|x_1\| = 1 - |\alpha| + \|x\| - |\alpha| \leqslant 2 - 2|\alpha| + \varepsilon_2.$$

It follows that $|\alpha| \leqslant (1/2)(\varepsilon_2 + \varepsilon_3) =: \varepsilon_6$, and consequently $\|x_1\| = \|x\| - |\alpha| \geqslant 1 - \varepsilon_1 - \varepsilon_6$.

On the other hand, we have

$$\begin{aligned} 1 + \varepsilon_5 \geqslant \|\theta x + (\eta - y)\| &= |\theta\alpha + (1 - \beta)| + \|\theta x_1 - y_1\| \\ &\geqslant |1 - \beta| - \varepsilon_6 + \|\theta x_1 - y_1\|. \end{aligned}$$

Thus $1+\varepsilon_5+\varepsilon_6-|1-\beta| \geqslant \|\theta x_1-y_1\|$ for all $|\theta|=1$ which yields, since $\|x_1\| \geqslant 1-\varepsilon_1-\varepsilon_6$, that $1+\varepsilon_5+\varepsilon_6-|1-\beta| \geqslant 1-\varepsilon_1-\varepsilon_6$, i.e. $|1-\beta| \leqslant \varepsilon_1+\varepsilon_5+2\varepsilon_6$. This implies that $|\beta| \geqslant 1-\varepsilon_1-\varepsilon_5-2\varepsilon_6$, and we get

$$\|\eta-y\| = |1-\beta|+\|y_1\| \leqslant \varepsilon_1+\varepsilon_5+2\varepsilon_6+\|y\|-|\beta|$$
$$\leqslant 2\varepsilon_1+2\varepsilon_5+4\varepsilon_6+\varepsilon_4 = 2(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)+\varepsilon_4.$$

In order to apply this lemma it will be essential to know that, given x with norm close to one, there is an L-summand $K\eta$ such that $\|x+\theta\eta\|$ is close to two. We show that such an η can always be found if there are sufficiently many one-dimensional L-summands.

5.2. Lemma. *Let X be a Banach space such that X' contains n_0 one-dimensional L-summands $K\eta_1, \ldots, K\eta_{n_0}$, where $\|\eta_n\|=1$ for all n. By $P_n\colon X' \to X'$ we denote the natural projection onto $K\eta_n$, $n=1, \ldots, n_0$. Further, let K be a compact Hausdorff space, $\varepsilon_1, \varepsilon_2 \geqslant 0$, and $\omega\colon K \to X'$ a weak*-continuous mapping with $\|\omega(k)\| \leqslant 1+\varepsilon_1$ for every k.*

(i) For every $k_0 \in K$ and every set $A \subset K$ such that $k_0 \in A^-$ there is an $n_1 \in \{1, \ldots, n_0\}$ such that k_0 is in the closure of $\{k \mid k\in A,\ \|P_{n_1}(\omega(k))\| \leqslant (1+\varepsilon_1)/n_0\}$.

(ii) Suppose that $\|\omega(k_0)-\eta_{n_1}\| \leqslant \varepsilon_2$ for some $k_0 \in K$, $n_1 \in \{1, \ldots, n_0\}$. Then, for every $n_2 \in \{1, \ldots, n_0\}$ different from n_1 and every $\varepsilon_3 > 0$, there is a neighbourhood U of k_0 such that

$$\|P_{n_2}(\omega(k))\| \leqslant \varepsilon_1+\varepsilon_2+2\varepsilon_3 \quad \text{for all } k\in U.$$

Proof. (i) Define A_n by

$$A_n := \{k \mid k\in A,\ \|P_n(\omega(k))\| \leqslant (1+\varepsilon_1)/n_0\}.$$

If $k_0 \notin A_n^-$ for $n=1, \ldots, n_0$, then $\bigcup A_n \subsetneqq A$. But this means that there exists a $k\in A$ with $\|P_n(\omega(k))\| > (1+\varepsilon_1)/n_0$ for $n=1, \ldots, n_0$ in contradiction to

$$1+\varepsilon_1 \geqslant \|\omega(k)\| \geqslant \sum \|P_n(\omega(k))\|.$$

(ii) Let $\varepsilon_3>0$ be given. $J := \ker \eta_{n_2}$ is an M-ideal, and η_{n_1} lies in $(J^\pi)^\perp$. Since $\|\eta_{n_1}\| = \|\eta_{n_1}|_J\|$ by [2], p. 35, there is an $x_0 \in J$ with $\|x_0\|=1$, $|\eta_{n_1}(x_0)| \geqslant 1-\varepsilon_3$. Choose a neighbourhood U of k_0 such that $|(\omega(k)-\omega(k_0))(x_0)| \leqslant \varepsilon_3$ on U. For these k we have $|(\omega(k))(x_0)| \geqslant 1-\varepsilon_2-2\varepsilon_3$ so that

$$|((\mathrm{Id}_{X'}-P_{n_2})(\omega(k)))(x_0)| \geqslant 1-\varepsilon_2-2\varepsilon_3$$

(note that $P_{n_2}(\omega(k)) \in K\eta_{n_2}$). Consequently $\|(\mathrm{Id}-P_{n_2})(\omega(k))\| \geqslant 1-\varepsilon_2-2\varepsilon_3$ so that

$$\|P_{n_2}(\omega(k))\| = \|\omega(k)\| - \|(\mathrm{Id}-P_{n_2})(\omega(k))\| \leqslant \varepsilon_1+\varepsilon_2+2\varepsilon_3.$$

5.3. Lemma. *Let Y be a Banach space, and let $\eta \in Y$ with $\|\eta\|=1$ be such that $K\eta$ is an L-summand. Then:*

(i) $\|\theta\eta+y\| \leqslant 1+\varepsilon$ (for all θ with $|\theta|=1$) implies that $\|y\| \leqslant \varepsilon$.

(ii) $\|\theta\eta+y\| \geqslant 2-2\varepsilon_2-\varepsilon_1$ for all θ with $|\theta|=1$ provided that $1-\varepsilon_1 \leqslant \|y\|$ and also $\|Py\| \leqslant \varepsilon_2$ (where $P=$ the natural projection onto $K\eta$).

Proof: elementary.

Our final lemma is a straightforward consequence of Theorem 1.2.

5.4. Lemma. *Let J be an M-ideal in a Banach space Y, and let $\varepsilon>0$. Then, for $y_i \in J$, $y\in Y$ with $\|y_i\|, \|y\| < 1+\varepsilon$ $(i=1, 2, 3)$, there is a $z\in J$ with $\|z\| < 1+\varepsilon$ and $\|\theta y_i+(y-z)\| < 1+\varepsilon$ for all $|\theta|=1$.*

5.5. Theorem. *Let X be a Banach space such that X' contains at least 33 one-dimensional L-summands: $K\eta_1, \ldots, K\eta_{33}$, where $\|\eta_n\|=1$ for all n.*

Then, for every compact Hausdorff space K and every admissible operator space $W \subset L(X, CK)$, the M-ideals of W are determined by the M-ideals of CK, i.e. they are precisely the subspaces $\{T \mid T\in W,\ \|\delta_k \circ T\| \to 0 \text{ for } k\to D\}$, where $D=D^- \subset K$.

Proof. We will show that Theorem 3.3 can be applied. Condition 3.3(ii) is obviously valid, and it remains to prove that 3.3(i) holds, too.

To this end, let $k_0 \in K$ be given and let $J \supsetneqq J_{(\{k_0\})}$ be an M-ideal. We will show that $d(T_0, J) \leqslant 65/66$ for every $T_0 \in W$ with $\|T_0\|=1$ so that $J=W$.

Since $J \supsetneqq J_{(\{k_0\})}$, there is an operator $T\in J$ with $\limsup_{k\to k_0} \|\delta_k \circ T\| > 0$. As in the proofs of the preceding sections we may assume that $\|T\| = \limsup \|\delta_k \circ T\| = 1$.

Now let $\varepsilon>0$ be a fixed number such that $32/33+74\varepsilon \leqslant 65/66$. With $A := \{k \mid \|\delta_k \circ T\| > 1-\varepsilon\}$ we have $k_0 \in A^-$ so that, by Lemma 5.2(i) which we apply with $\omega(k) := \delta_k \circ T$, there are a subset $A_1 \subset A$ and an $n_1 \in \{1, \ldots, 33\}$ with $\|P_{n_1}(\delta_k \circ T)\| \leqslant 1/33$ for $k\in A_1$.

Lemma 5.4 provides us with an $S\in J$ such that $\|S\| < 1+\varepsilon$, $\|\theta T+(\eta_{n_1} \otimes 1-S)\| < 1+\varepsilon$ for all $|\theta|=1$.

We apply Lemma 5.1 with $x := \delta_k \circ T$, $y := \delta_k \circ S$ (where $k\in A_1$ is arbitrary) and $\varepsilon_1=\varepsilon_4=\varepsilon_5=\varepsilon$, $\varepsilon_2=0$, $\varepsilon_3=2/33+\varepsilon$ (that this number is admissible follows from Lemma 5.3(ii)). Therefore we have

$$\|\eta_{n_1}-\delta_k \circ S\| \leqslant \varepsilon+2(\varepsilon+0+2/33+\varepsilon+\varepsilon) = 7\varepsilon+4/33$$

for $k\in A_1$ so that, since $k_0 \in A_1^-$ and $k \mapsto \delta_k \circ S$ is weak*-continuous, $\|\eta_{n_1}-\delta_{k_0} \circ S\| \leqslant 7\varepsilon+4/33$.

Now choose any n_2 different from n_1. Lemma 5.2(ii) yields (with $\omega(k) := \delta_k \circ S$, $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 7\varepsilon + 4/33$, $\varepsilon_3 = \varepsilon/2$) the existence of a neighbourhood U_1 of k_0 such that $\|P_{n_2}(\delta_k \circ S)\| \leqslant 9\varepsilon + 4/33$ for $k \in U_1$.

we apply once more Lemma 5.4: we have $\|S\| < 1+\varepsilon$, $\|\eta_{n_2} \otimes \mathbb{1}\| = 1$, so that there is an $R \in J$ with $\|R\| < 1+\varepsilon$ and $\|\theta S + (\eta_{n_2} \otimes \mathbb{1} - R)\| < 1+\varepsilon$ for all $|\theta| = 1$.

Choose a neighbourhood $U \subset U_1$ of k_0 with $\|\delta_k \circ S\| > 1 - 8\varepsilon - 4/33$ on U (this is possible since $k \mapsto \delta_k \circ S$ is weak*-continuous at k_0 and $\|\delta_{k_0} \circ S\| \geqslant 1 - 7\varepsilon - 4/33$).

Since we also have

– $\|\delta_k \circ S\| \leqslant 1+\varepsilon$,

– $\|\delta_k \circ R\| \leqslant 1+\varepsilon$,

– $\|\theta\eta_{n_2} - \delta_k \circ S\| \geqslant 2 - 26\varepsilon - 12/33$ (Lemma 5.3(ii))

for these k, it follows from Lemma 5.1 that

$$\|\eta_{n_2} - \delta_k \circ R\| \leqslant \varepsilon + 2\big((8\varepsilon + 4/33) + \varepsilon + (26\varepsilon + 12/33) + \varepsilon\big)$$
$$= 32/33 + 73\varepsilon.$$

Now let $T_0 \in W$ with $\|T_0\| = 1$ be arbitrarily given. A third application of Lemma 5.4 provides us with an operator $\tilde{R} \in J$ such that $\|\theta R + (T_0 - \tilde{R})\| \leqslant 1+\varepsilon$ (for all $|\theta| = 1$).

But then $\|\theta\eta_{n_2} + \delta_k \circ (T_0 - \tilde{R})\| \leqslant 1 + 32/33 + 74\varepsilon \leqslant 1 + 65/66$ so that, by Lemma 5.3(i), $\|\delta_k \circ (T_0 - \tilde{R})\| \leqslant 65/66$ for $k \in U$. We choose any continuous $h\colon K \to [0, 1]$ with $\operatorname{supp} h \subset U$ and $h(k_0) = 1$. Then $T := M_h \circ \tilde{R} + M_{1-h} \circ T_0$ belongs to J since $M_h \circ \tilde{R} \in J$ and $M_{1-h} \circ T_0 \in J_{(\{k_0\})}$. Moreover, $\|\delta_k \circ (T - T_0)\| \leqslant 65/66$ for every k so that $\|T - T_0\| \leqslant 65/66$.

As a consequence we have

5.6. Theorem. *Let X be a Banach space such that X' is an abstract L^1-space; in the case of real scalars we assume that X is not isometrically isomorphic to l_2^∞.*

Then, for every compact Hausdorff space and every admissible operator space $W \subset L(X, CK)$, the M-ideals of W are determined by the M-ideals of CK.

Proof. The case of finite-dimensional X follows from Cor. 3.4 and the fact that L^1-spaces do not have nontrivial M-ideals. If X is infinite-dimensional, then X' admits in fact infinitely many one-dimensional L-summands: Kp is an L-summand for every extreme functional $p \in X'$ ([10], Prop. 2.1).

5.7. Corollary ([11], [14]). *The M-ideals of $L(CK, CK)$ are determined by the M-ideals of CK for every compact Hausdorff space (in the case of real scalars we have to assume that* card $K > 2$).

Note. Flinn and Smith also can describe the M-ideals for $L(C_0K, C_0K)$ provided that K is a locally compact Hausdorff space which behaves not too pathologically. Our methods also apply to situations which are more general than those treated in this section. The results, however, are far from being satisfactory up to now, the difficulties arise mainly from the fact that one has not much information about the behaviour of $k \mapsto \delta_k \circ T$ $(T \in L(X, C_0K))$ near the points $k_0 \in \beta K \setminus K$.

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