On vérifie aisément que $EP'[L]$ contient la loi dont la fonction de sauts est $N(x)$:

$$dN^{(+)}(x) = dN^{(-)}(-x) = d[x^{-\beta}] \quad \text{si} \quad 0 < x \leqslant 1,$$
$$= d(\lg e\,x)^{-1} \, \text{si} \quad x > 1,$$

loi qui n'a pas de moments.

Le lecteur pourra prouver (la démonstration est analogue à celle du théorème XI) la proposition:

Pour que $EP[L]$ soit fortement compact, il faut, mais il ne suffit pas, que pour un $\alpha > 0$

$$\int |x|^{\alpha} dF(x) < \infty .$$

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Про сукупність степенів одного закону імовірностей

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(Резюме)

Якщо $S_n = \sum^{n} x_i$, де x_i стохастичні незалежні змінні, то при відповідно дібраних сталих a_n, b_n дистрибуанта $F_n(\xi)$ стохастичної змінної $(S_n - a_n)/b_n$ збігається (для $n \longrightarrow \infty$) — за певними умовами — до границі. Автор досліджує множину тих граничних функцій розподілу, впроваджуючи відповідний функційний простір і поняття класу $K(L)$ степенів закону розподілу L.

On isomorphisms of rings of linear operators

by

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This paper is in connection with the researches on general linear rings which are due to S. Mazur[1]). We consider here especially the rings $\mathfrak{A}(E)$ of linear operators transforming a given Banach space E into another Banach space E'. We shall see, (this was suggested by Mazur), that in any ring $\mathfrak{A}(E)$ the norm is in a certain sense uniquely determined. Furthermore, two rings $\mathfrak{A}(E_1)$, $\mathfrak{A}(E_2)$ are (algebraically) isomorphic if and only if the spaces E_1, E_2 are isomorphic[2]). The isomorphism

$$V = \Phi(U) \quad \text{where} \quad U \in \mathfrak{A}(E_1), \; V \in \mathfrak{A}(E_2),$$

is then of the form

$$V = AUA^{-1},$$

A being a linear operator which yields the isomorphism between E_1 and E_2[3]). In this theorem we may replace the property of $\Phi(U)$ to be additive by the continuity.

§ 1.

1. Consider an arbitrary Banach space E and denote by $\mathfrak{A}(E)$ the set of all linear operators

$$y = U(x)$$

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[1]) S. Mazur, Sur les anneaux linéaires, C. R. Acad. Sc. Paris, 207 (1938) p. 1025—1027.

[2]) As to the definition of isomorphic spaces, see S. Banach, Théorie des opérations linéaires, Monografie Matematyczne, Warszawa 1932, p. 180.

[3]) This theorem was proved for maximal rings of matrices by Weber: Isomorphismus maximaler Matrizenringe, Journ. f. Math. 171 (1934) p. 227—242.

with the domain E and ranges contained in E. In $\mathfrak{A}(E)$ the sum and product of two operators and the product of an operator by a real number are defined in the customary manner; if $U_1, U_2 \in \mathfrak{A}(E)$ and λ is a real number, then

$$U_1 + U_2 = A, \quad U_1 U_2 = B, \quad \lambda U_1 = C,$$

where

$$A(x) = U_1(x) + U_2(x), \quad B(x) = U_1[U_2(x)], \quad C(x) = \lambda U_1(x)$$

for every $x \in E$. Obviously $\mathfrak{A}(E)$ is then a linear ring[4]). The unit element is here the identical operator $U(x) = x$, which will be denoted by I. An operator $U \in \mathfrak{A}(E)$ is said to have an inverse if there exists another operator $V \in \mathfrak{A}(E)$ such that $UV = VU = I$; V is denoted by U^{-1}. If U has an inverse, then U transforms E into E isomorphically. The converse theorem is also true and follows from a general theorem of S. Banach[5]). $\mathfrak{A}(E)$ may be considered as a Banach space with the norm

$$(1) \qquad |U| = \sup_{|x| \leq 1} |U(x)|.$$

This norm satisfies clearly the condition

$$(2) \qquad |U_1 U_2| \leqslant |U_1| \, |U_2|$$

for arbitrary $U_1, U_2 \in \mathfrak{A}(E)$.

2. We start with an algebraic characterisation of bounded sets in $\mathfrak{A}(E)$.

Lemma 1. *A set $H \in \mathfrak{A}(E)$ is bounded if and only if the following condition is fulfilled. For every $A \in \mathfrak{A}(E)$ there exists a number $\alpha > 0$ such that the operator $I - \lambda A U$ has an inverse, where λ is any real number, $|\lambda| < \alpha$ and U any operator belonging to H.*

Proof. *Necessity.* Let $|U| \leqslant M$ for $U \in H$ and let $A \in \mathfrak{A}(E)$ be given. Put $\alpha = (M|A|)^{-1}$ (A may be supposed $\neq 0$); then we have

$$|\lambda A U| < 1 \quad \text{for} \quad |\lambda| < \alpha, \quad U \in H,$$

whence it follows that the operator $I - \lambda A U$ has an inverse[6]).

[4]) Cf. loc. cit. [2])

[5]) Loc. cit. [2]) p. 41, th. 5.

[6]) Namely $(I - \lambda A U)^{-1} = I + \lambda A U + (\lambda A U)^2 + \dots$.

Sufficiency. Suppose that the set H is not bounded and let $U_n \in H$ be a sequence for which $|U_n| \to \infty$. It follows then[7]) the existence of an element $x_0 \in E$ such that $\lim_{n\to\infty} |U_n(x_0)| = \infty$. This implies in turn[8]) the existence of a linear functional $f(x)$ such that

$$(3) \qquad \overline{\lim_{n\to\infty}} |f[U_n(x_0)]| = \infty.$$

Put now

$$A(x) = f(x) x_0 \quad \text{for} \quad x \in E.$$

It is easily seen that our condition is not fulfilled for this A. For, let α be an arbitrary positive number. Put $\lambda_n = [f(U_n(x_0))]^{-1}$ and choose n so that $|\lambda_n| < \alpha$. We have then

$$I(x_0) - \lambda_n A[U_n(x_0)] = x_0 - \lambda_n f[U_n(x_0)] x_0 = 0,$$

hence, x_0 being $\neq 0$ by (3), the operator $I - \lambda_n A U_n$ has no inverse.

3. Theorem 1. *Let $|U|^*$ be an arbitrary norm defined on $\mathfrak{A}(E)$ which besides the usual conditions satisfies (2) and with respect to which $\mathfrak{A}(E)$ is a complete space; then this norm is equivalent to the norm (1), i. e. for any sequence $U_n \in \mathfrak{A}(E)$, $|U_n|^* \to 0$ implies $|U_n| \to 0$ and vice-versa.*

Proof. Let $|U_n|^* \to 0$; choose a sequence of real numbers $a_n \to \infty$ so that $|a_n U_n|^* \to 0$. The sequence $\{a_n U_n\}$ being bounded with respect to the norm $|U|^*$, satisfies the condition of the lemma, because the proof of its necessity involves only those properties of the norm (1) which the norm $|U|^*$ also possesses. Hence, by the same lemma, the sequence $\{|a_n U_n|\}$ is bounded, whence we infer that $|U_n| \to 0$.

A general theorem of Banach[9]) gives the converse implication, and so the theorem follows.

§ 2.

1. We need now an algebraic characterisation of the operators of the form

$$U(x) = f(x) y,$$

[7]) Loc. cit. [2]) p. 80, th. 5.

[8]) Ibidem, p. 80, th. 6.

[9]) Ibidem, p. 41, th. 6.

where $f(x)$ is a linear functional and y an element in E. Such operators will be called, in the sequel, *one-dimensional.*

Lemma 2. *An operator $U_0 \in \mathfrak{A}(E)$ is one-dimensional if and only if the following condition is fulfilled. For every $U \in \mathfrak{A}(E)$ there exists a number λ such that*

(4) $$(UU_0)^2 = \lambda U U_0 .$$

Proof. *Necessity.* It is evident that for an one-dimensional operator V there exists a number λ such that $V^2 = \lambda V$. But U_0 being an one-dimensional operator, UU_0 is always also one-dimensional and so (4) follows.

Sufficiency. It is sufficient to show that the range of $U_0(x)$ is one-dimensional. For we have then $U_0(x) = g(x)\, x_0$ where $g(x)$ is a functional and $x_0 \in E$. But $U_0(x)$ being linear, it follows easily that $g(x)$ is linear too. Thus suppose that the range of $U_0(x)$ contains two linearly independent elements,

(5) $$U_0(x_1) = y_1, \quad U_0(x_2) = y_2, \qquad x_1,\ x_2,\ y_1,\ y_2 \in E_1 .$$

Then there exist two linear functionals $f_1(x)$, $f_2(x)$ such that

$$f_1(y_1) = 0, \quad f_1(y_2) = 1, \quad f_2(y_1) = 1, \quad f_2(y_2) = 0 .$$

Putting

$$U(x) = f_1(x)\, x_1 + f_2(x)\, x_2$$

we have $UU_0(x_1) = x_2$, $UU_0(x_2) = x_1$, hence, on setting $V = (UU_0)^2$, $V(x_2) = x_2$. But (4) implies $V(x_2) = \lambda U[U_0(x_2)]$ i. e. $x_2 = \lambda x_1$, consequently $y_2 = \lambda y_1$, contrary to our assumption.

2. Let E_1, E_2 be two Banach spaces. The corresponding rings $\mathfrak{A}(E_1)$, $\mathfrak{A}(E_2)$ are said to be *algebraically isomorphic* if there exists a one-to-one transformation $V = \Phi(U)$ of $\mathfrak{A}(E_1)$ into $\mathfrak{A}(E_2)$ (the whole ring $\mathfrak{A}(E_2)$) additive and multiplicative i. e. such that

$$\Phi(U_1 + U_2) = \Phi(U_1) + \Phi(U_2), \quad \Phi(U_1 U_2) = \Phi(U_1)\, \Phi(U_2)$$
$$\text{for any } U_1,\ U_2 \in \mathfrak{A}(E_1).$$

Theorem 2. *Two rings $\mathfrak{A}(E_1)$, $\mathfrak{A}(E_2)$ are algebraically isomorphic if and only if the corresponding spaces E_1, E_2 are isomorphic; moreover $V = \Phi(U)$ being the isomorphism between $\mathfrak{A}(E_1)$ and $\mathfrak{A}(E_2)$ there exists an isomorphism $A(x) = y$ between E_1 and E_2 such that*

(6) $$\Phi(U) = AUA^{-1} \quad \text{for} \quad U \in \mathfrak{A}(E_1).$$

Proof. The sufficiency is evident.

Necessity. We first show that $\Phi(U)$ is homogenous. Let λ be an arbitrary number and $U \in \mathfrak{A}(E_1)$ an arbitrary operator. Since $\lambda I_1 . U = U . \lambda I_1$ (I_1 is the unit element of $\mathfrak{A}(E_1)$) we have $\Phi(\lambda I_1)\, \Phi(U) = \Phi(U)\, \Phi(\lambda I_1)$ i. e. the operator $\Phi(\lambda I_1)$ is permutable with every operator of $\mathfrak{A}(E_2)$. Hence [10])

$$\Phi(\lambda I_1) = u\, I_2$$

where I_2 denotes the unit element of $\mathfrak{A}(E_2)$ and u is a suitable number. But $\Phi(U)$ being an isomorphism, the real function $u = u(\lambda)$ is obviously additive and multiplicative. Moreover $u(1) = 1$ [11]). By a well known theorem of Darboux this implies $u(\lambda) \equiv \lambda$, and consequently $\Phi(\lambda I_1) = \lambda I_2$. Now we have for an arbitrary $U \in \mathfrak{A}(E_1)$

$$\Phi(\lambda U) = \Phi(\lambda I_1 U) = \Phi(\lambda I_1)\, \Phi(U) = \lambda\, \Phi(U)$$

i. e. $\Phi(U)$ is homogenous.

Having proved this, the continuity of $\Phi(U)$ follows easily by the lemma 1. For let H be any bounded set in $\mathfrak{A}(E_1)$; the condition of the lemma 1 is then satisfied and the properties of $\Phi(U)$ imply that the range $\Phi(H)$ of the set H satisfies the same condition. Hence, using again the lemma 1, we see that the set $\Phi(H)$ is also bounded, which involves the continuity of $\Phi(U)$ [12]).

We determine now the operator $A(x_0)$ which yields the isomorphism between E_1 and E_2. Choose a fixed linear functional $f_0(x)$ in E_1 and an element $x_0 \in E_1$ such that $f_0(x_0) = 1$ and put

$$U_0(x) = f_0(x)\, x_0 .$$

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[10]) It follows simply from the fact that $\Phi(\lambda I_1)$ is permutable with every one-dimensional operator.

[11]) For we have for every $U \in \mathfrak{A}(E_1)$, $\Phi(U) = \Phi(I_1 U) = (\Phi I_1)\, \Phi(U)$ and therefore $\Phi(I_1) = I_2$.

[12]) Observe that that conclusion is valid also without the property of $\Phi(U)$ to be a one-to-one dimensional transformation.

U_0 satisfies the condition of lemma 2 and the properties of $\Phi(U)$ imply that this condition is fulfilled by the operator $V_0=\Phi(U_0)$ too, consequently

$$V_0(y)=g_0(y)y_0 \quad \text{for } y\in E_2,$$

where $y_0\in E_2$ and $g_0(y)$ is a linear functional in E_2. U_0 being $\neq 0$ we have also $V_0\neq 0$, hence $y_0\neq 0$ and $g_0(y)\not\equiv 0$.

We define now the required operator $A(x)$ as follows. Let $z\in E_1$ be given; we choose then an arbitrary operator $U\in\mathfrak{A}(E_1)$ such that [13])

$$U(x_0)=z$$

and set

$$A(z)=V(y_0)$$

where $V=\Phi(U)$. We prove now:

a) $A(z)$ is uniquelyd etermined. For, if $U_1(x_0)=U_2(x_0)=z$, $U_1, U_2\in A(E_1)$, then $U_1[U_0(x)]=U_2[U_0(x)]=f_0(x)z$, consequently $V_1V_0=V_2V_0$, where $V_1=\Phi(U_1)$, $V_2=\Phi(U_2)$ whence $V_1(y_0)=V_2(y_0)$.

b) $z_1\neq z_2$ implies $A(z_1)\neq A(z_2)$. Let $U_1(x_0)=z_1$, $U_2(x_0)=z_2$, $U_1, U_2\in\mathfrak{A}(E_1)$, $V_1=\Phi(U_1)$, $V_2=\Phi(U_2)$. We have then $U_1U_0 \neq U_2U_0$, consequently $V_1V_0\neq V_2V_0$ i. e. $V_1(y_0)\neq V_2(y_0)$.

c) The range of the operator $A(x)$ is identical with the whole space E_2. For, if an arbitrary element $y\in E_2$ is given, there exists an operator $V\in\mathfrak{A}(E_2)$ such that $V(y_0)=y$. Putting $U=\Phi^{-1}(V)$, $U(x_0)=z$, we have $y=A(z)$.

d) $A(x)$ is additive. Let $z_1, z_2\in E_1$ and $U_1, U_2\in\mathfrak{A}(E_1)$, $U_1(x_0)=z_1$, $U_2(x_0)=z_2$; we have then $U_1(x_0)+U_2(x_0)=z_1+z_2$ and $\Phi(U_1+U_2)=V_1+V_2$ where $V_1=\Phi(U_1)$, $V_2=\Phi(U_2)$, consequently $A(z_1+z_2)=V_1(y_0)+V_2(y_0)=A(z_1)+A(z_2)$.

e) $A(x)$ is continuous. Let $z_n\in E_1$, $z_n\to 0$; put $U_n(x)=f_0(x)z_n$. We have $|U_n|=|f_0|\,|z_n|\to 0$, hence $|V_n|=|\Phi(U_n)|\to 0$; further, since $U_n(x_0)=z_n$, $A(z_n)=V_n(y_0)$, consequently $A(z_n)\to 0$.

By a), b), c), d), e) $A(x)$ is an isomorphism between E_1 and E_2.

We are now able to finish the proof of our theorem. Let $z\in E_1$ and $U\in\mathfrak{A}(E_1)$ be arbitrary chosen and put $V=\Phi(U)$.

13) E. g. $U(x)=f_0(x)z$.

We determine an operator $U_1\in\mathfrak{A}(E_1)$ such that $U_1(x_0)=z$ and put $V_1=\Phi(U_1)$. We have then $UU_1(x_0)=U(z)$ and $\Phi(UU_1)=VV_1$, hence

$$A[U(z)]=V[V_1(y_0)]=V[A(z)];$$

z being arbitrary, it follows $V=AUA^{-1}$ which proves the theorem.

Corollary. If, in particular, $E_1=E_2=E$ we obtain the following theorem: *Every automorphism of the ring $\mathfrak{A}(E)$ is an inner one.*

3. We shall now prove a slightly different theorem replacing the additivity of $\Phi(U)$ by its continuity.

Theorem 3. *Let $U=\Phi(U)$ be an one-to-one continuous and multiplicative transformation of the ring $\mathfrak{A}(E_1)$ into the ring $\mathfrak{A}(E_2)$, E_1 being a space at least two-dimensional; $\Phi(U)$ is then also additive, consequently, by the theorem 2, of the form* (6)[14]).

Proof. As before we obtain (6) knowing now merely that $A(x)$ is a one-to-one continuous transformation of E_2 into E_2 (the whole space). Moreover the equality $\Phi(\lambda U)=u(\lambda)\Phi(U)$ for arbitrary λ, U yields

$$A(\lambda x)=u(\lambda)A(x) \tag{7}$$

for any real λ and $x\in E_1$. In order to prove our theorem, it is sufficient to show that $A(x)$ is additive, for, by (6), it follows then the additivity of $\Phi(U)$[15]).

Suppose first that $x_1, x_2\in E_1$ are linearly independent. We put $y_1=A(x_1)$, $y_2=A(x_2)$ and distinguish two cases:

1) $A^{-1}(y_1+y_2)=x_3$ is linearly independent of x_1, x_2,

2) the contrary case occurs.

In the first case we can determine an operator $U\in\mathfrak{A}(E_1)$ such that

$$U(x_1)=x_1,\quad U(x_2)=x_2,\quad U(x_3)=x_1+x_2 \tag{8}$$

14) In the one-dimensional case the theorem is false. E. g. we may put for $U(x)=ux$, $\Phi(U)=u^3x$, x and u being real numbers.

15) That (6) together with the continuity of $A(x)$ implies $A(x)$ to be linear was proved otherwise in the case of the ring of finite matrices by M. Nagumo: Über eine kennzeichnende Eigenschaft der Linearkombinationen von Vektoren und ihre Anwendung, Gött. Nachr. (1933) p. 36.

(e. g. $U(x) = f_1(x)\,x_1 + f_2(x)\,x_2 + f_3(x)\,x_3$, where $f_i(x_k) = 0$ for $i \neq k$, $=1$ for $i = k$).

We have then by (6) and (8), putting $V = \Phi(U)$,

$$V(y_1 + y_2) = A[U(x_3)] = A(x_1 + x_2),$$

further

$$V(y_1) + V(y_2) = A[U(x_1)] + A[U(x_2)] = A(x_1) + A(x_2),$$

hence

$$A(x_1 + x_2) = A(x_1) + A(x_2).$$

In the second case let

$$x_3 = \lambda_1 x_1 + \lambda_2 x_2 .$$

We determine an operator $U \in \mathfrak{A}(E_1)$ such that $U(x_1) = x_1$, $U(x_2) = 0$. By (6) we have then, putting $V = \Phi(U)$,

$$V(y_1 + y_2) = A[U(\lambda_1 x_1 + \lambda_2 x_2)],$$

and on the other hand

$$V(y_1) + V(y_2) = A[U(x_1)] + A[U(x_2)],$$

hence

$$A[U(\lambda_1 x_1 + \lambda_2 x_2)] = A[U(x_1] + A[U(x_2)],$$

or by (7) $u(\lambda_1)\,A(x_1) = A(x_1)$. This implies $u(\lambda_1) = 1$, whence $\lambda_1 = 1$, since $u(\lambda') \neq u(\lambda'')$ for $\lambda' \neq \lambda''$ and $u(1) = 1$.

In the same way we obtain $\lambda_2 = 1$, i. e.

$$A^{-1}(y_1 + y_2) = A^{-1}(y_1) + A^{-1}(y_2).$$

It remains to prove that $A(x)$ is homogenous. Let $x \in E_1$ be an arbitrary element and n an integer. We choose a sequence $\{x_k\}$ linearly independent of x such that $\lim\limits_{k\to\infty} x_k = nx$. We have then

$$A((n+1)\,x) = \lim_{k\to\infty} A(x + x_k) = A(x) + A(xn).$$

By induction we see at once that $A(nx) = nA(x)$ for an arbitrary integer. Thus the homogeneity of $A(x)$ follows in a familiar way from its continuity.

Corollary. In the above proof the continuity of $A(x)$ was used only to prove the homogeneity of $A(x)$. *Hence we may replace in theorem 2 the additivity of $\Phi(U)$ by its homogeneity.*

Remark. In the theorems 2, 3 it is essential that the considered spaces and rings are real, i. e. we use only products of elements and operators by real numbers. In the case of complex spaces and rings, we have to suppose explicite that the transformation $\Phi(U)$ takes every operator λI_1 into the operator λI_2; the hypothesis of $\Phi(U)$ to be additive resp. continuous is then superfluous.

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Про ізоморфізми кільців лінійних операторів

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(Резюме)

Нехай E будьякий простір типу (B), $\mathfrak{A}(E)$ кільце лінійних операторів, що відображують E в себе (при звичайних дефініціях алгебричних дій).

Теорема I. Якщо в $\mathfrak{A}(E)$ є дана норма $|A|$ $(A \in \mathfrak{A}(E))$ при якій $\mathfrak{A}(E)$ є простір типу (B) і яка задовольняє умову

$$|AB| \leqslant |A|\,|B| \quad \text{для} \quad A, B \in \mathfrak{A}(E),$$

то ця норма еквівалентна звичайній нормі:

$$\|A\| = \sup_{|x| \leqslant 1} |A(x)|.$$

Теорема II. Два кільця $\mathfrak{A}(E_1)$, $\mathfrak{A}(E_2)$ є алгебрично ізоморфні тоді і тільки тоді, коли простори E_1 і E_2 є ізоморфні. Ізоморфізм

$$V = \Phi(U), \quad U \in \mathfrak{A}(E_1), \quad V \in \mathfrak{A}(E_2)$$

має тоді форму

$$V = AUA^{-1},$$

де A є лінійний оператор, який дає ізоморфізм між E_1 і E_2.

В цій теоремі можна заступити адитивність операції $\Phi(U)$ її неперервністю або однорідністю.