Such isomorphism can be obtained by a conformal mapping of the rectangle onto the upper half-plane. Applying analytic continuation through $\{\operatorname{Im} z = 0\}$ by symmetry we may claim that $A_\gamma(T)$ is isomorphic to a subalgebra of the algebra of functions analytic on $C\setminus(\tilde{\gamma}\cup\tilde{\gamma}_c)$ and continuous on C where $\tilde{\gamma}_c$ is the reflection of $\tilde{\gamma}$ with respect to $\operatorname{Im} z = 0$, and C denotes the complex plane. Applying the method used by Hoffman and Singer [5] to prove Theorem 5, we arrive at the conclusion that the maximal ideal space of the algebra $A_\gamma(T)$, and also of the algebra A_γ, is a torus.

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Regular quasimultipliers of some semisimple Banach algebras

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Abstract. If A is a complex nonunital Banach algebra with dense principal ideals we denote by $QM_r(A)$ the pseudo-Banach algebra formed by Esterle's regular quasimultipliers of A. We study the character space $\hat{\hat{A}}$ of $QM_r(A)$ for several concrete algebras A. In particular, for every nondiscrete metrizable compactly generated abelian group G with dual group Γ we prove that $\beta\Gamma$ is homeomorphically embedded into $L^1(G)\hat{}$ (if G is compact $\beta\Gamma$ equals $L^1(G)\hat{}$). We also note that there is a relationship between $QM_r(L^1(G))$ and the space $P(G)$ of pseudomeasures on G. If G is compact, $QM_r(L^1(G)) = P(G)$.

Introduction. Let A be a complex nonunital commutative Banach algebra possessing dense principal ideals and such that $A^\perp = \{0\}$, where $A^\perp = \{a\in A\colon ab = 0 \text{ for all } b\in A\}$. A *quasimultiplier* T of A is an unbounded operator on A whose domain is a dense principal ideal; so T can be written as a quotient $T = a/b$ where $a, b\in A$ and $[bA]^- = A$. We put $QM(A) = \{T\colon T \text{ is a quasimultiplier of } A\}$. A quasimultiplier $T = a/b$ is said to be *regular* if there exist $\lambda > 0$ and $c\in\bigcap_{n=1}^\infty [b^n A]$ satisfying $\sup_n \|\lambda^n T^n c\| < +\infty$; let $QM_r(A) = \{T\in QM(A)\colon T \text{ regular}\}$. These notions and related ideas were introduced by Esterle in [5] to study the problem of existence of topologically simple radical Banach algebras.

The set $QM_r(A)$ is a pseudo-Banach algebra (see [1], [8]), i.e. it can be represented as an inductive limit of Banach algebras. To obtain this representation one needs the following definition. Two commutative Banach algebras A and B are said to be *similar* if there exist a commutative Banach algebra D with dense principal ideals and two continuous homomorphisms $\varphi\colon D\to A$, $\psi\colon D\to B$ such that $\varphi(D)$, $\psi(D)$ are dense ideals in A, B respectively. Then:

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(i) A is similar to B if and only if $QM(A)$ is bornologically isomorphic to $QM(B)$ ([5], p. 120);

(ii) $QM_r(A) = \liminf_{j\in J} \mathrm{Mul}(A_j)$, where $\{A_j\}_{j\in J}$ denotes the set of Banach algebras similar to A, and $\mathrm{Mul}(A_j)$ is the multiplier algebra of A_j, for each $j\in J$ ([5], p. 129).

In particular, $QM_r(A)$ always contains $\mathrm{Mul}(A)$ and, if A is uniform, $QM_r(A) = \mathrm{Mul}(A)$ ([13], p. II-8).

If A, B are Banach algebras with dense principal ideals and $\varphi\colon A\to B$ is a continuous homomorphism such that $\varphi(A)$ or $\varphi(A)B$ is dense in B then one defines $\tilde{\varphi}\colon QM(A)\to QM(B)$ as $\tilde{\varphi}(a/a') = \varphi(a)/\varphi(a')$ for all $a, a'\in A$ with $[a'A]^- = A$. The mapping $\tilde{\varphi}$ is a bornological homomorphism. Moreover, if φ is one-one, so is $\tilde{\varphi}$ ([5]), pp. 83, 84). In particular, if $T = a/b\in QM_r(A)$ we can define the *extended Gelfand transform* $\hat{T}$ of T as $\hat{T}(\varphi) = \hat{a}(\varphi)/\hat{b}(\varphi)$ for every $\varphi\in\hat{A}$, where $\hat{A}$ is the character space of A.

It is interesting to study the character space $\hat{\hat{A}}$ of $QM_r(A)$, called the *extended spectrum* of A by Esterle. If A is radical with bounded approximate identities it is known that $\hat{\hat{A}}$ is very large: it can be mapped continuously onto the character space of H^∞ ([5], p. 135). Nevertheless we do not know much more of the general properties of $\hat{\hat{A}}$ and so it seems a natural starting point to investigate the nature of $\hat{\hat{A}}$ for some concrete Banach algebras. The method consists in exploiting the properties of the extended Gelfand transform $QM_r(A)\to C_\infty(\hat{A}) = \{$continuous bounded functions on $\hat{A}\}$ for semisimple Banach algebras A which are dense in $C_0(\hat{A})$. A part of the results we establish here can be seen as an application to harmonic analysis.

In § 1 several examples of similar Banach algebras are given. These examples are either Banach algebras with an orthogonal Schauder basis or some algebras of integrable functions. The latter are Segal algebras $S(G)$ on a nondiscrete metrizable compact abelian group G, with dual group Γ. In that case we show that $S(G)$ is similar to $c_0(\Gamma)$; therefore $QM_r(S(G))$ is identified with $l^\infty(\Gamma)$ and $\hat{\hat{S(G)}}$ with $\beta\Gamma$, the Stone–Čech compactification of Γ.

In § 2 we consider the Banach algebra $C_0^{(m)}(\boldsymbol{R}^p)$ consisting of all the $C^{(m)}$-functions on $\boldsymbol{R}^p$ which are null at infinity together with their m derivatives. We show that the extended spectrum of $C_0^{(m)}(\boldsymbol{R}^p)$ equals $\beta\boldsymbol{R}^p$.

In § 3 we consider $L^1(G)$ with nondiscrete, metrizable, noncompact and compactly generated abelian group G. In contrast with the above case here it does not seem possible to study the extended spectrum Σ of $L^1(G)$ by means of similarity. However, exploiting the properties of the extended Gelfand transform we prove that $\beta\Gamma$ is homeomorphically embedded into Σ. The problem "$\Sigma = \beta\Gamma$" remains open. However, in § 3 we show that the equality $\Sigma = \beta\Gamma$ is equivalent to one of the following (equivalent) properties: (i) $QM_r(L^1(G))$ is regular—in the sense of Shilov—on its spectrum Σ; (ii) $\hat{T}$ invertible in $C_\infty(\Gamma)$ implies that T is invertible in $QM_r(L^1(G))$.

Recall that for the measure algebra $M(G) = \mathrm{Mul}(L^1(G))$ both properties (i) and (ii) are related and exhibit a classical pathology of $M(G)$, namely $M(G)$ is not regular on its spectrum and there exists μ in $M(G)$ such that $\hat{\mu}$ is bounded below on Γ but μ is not invertible in $M(G)$ ([18], p. 107). In Theorem (3.5) we prove that any μ in $M(G)$ with $\hat{\mu}$ bounded below on Γ is also invertible as regular quasimultiplier.

Finally, in § 4 we sketch the relationship between the quasimultipliers on G and the set $P(G)$ of all pseudomeasures on G. If G is compact, $QM_r(L^1(G)) = P(G)$ and so it is possible to give a definition of $P(G)$ with no reference to the dual group Γ of G. The same thing happens for the convolution product of pseudomeasures (if $T = a/b$, $T' = a'/b'$, TT' is defined as aa'/bb' in $QM(A)$). We conclude the paper by proving that, for G noncompact compactly generated, the regular quasimultipliers of $L^1(G)$ with compact support are precisely the pseudomeasures on G with compact support.

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§ 1. Examples of similar Banach algebras.

a. *Banach algebras with orthogonal Schauder basis.* These algebras have been studied in [9], [10]. These are Banach algebras A whose underlying Banach space has a Schauder basis $(e_n)_{n=1}^\infty$ such that $e_n e_m = e_n$ if $n = m$, and $e_n e_m = 0$ if $n\neq m$. If $x\in A$ then there is a unique sequence of complex scalars $(x_n)_{n=1}^\infty$ such that $x = \sum_{n=1}^\infty x_n e_n$, and if e_n^* denotes the linear functional $e_n^*(x) = x_n$, $x\in A$, $n\geqslant 1$, then the sequence $(e_n^*)_{n=1}^\infty$ is equicontinuous. Moreover, each e_n^* is a character on A since $xy = \sum_{n=1}^\infty e_n^*(x)e_n^*(y)e_n$ for all $x, y\in A$. Actually, $\hat{A} = (e_n^*)_{n=1}^\infty$. The *Gelfand transform* of A is the mapping $x\in A\mapsto (e_n^*(x))_{n=1}^\infty\in c_0$, where c_0 is the Banach algebra of sequences which are null at infinity.

We denote by l^∞ the Banach algebra of bounded sequences. If E is a Banach space, an *unconditional basis* for E is, by definition, a Schauder basis $(e_n)_{n=1}^\infty$ such that

$$\Big\|\sum_{n=1}^\infty \lambda_n e_n^*(x)e_n\Big\| \leqslant (\sup_n|\lambda_n|)\Big\|\sum_{n=1}^\infty e_n^*(x)e_n\Big\|$$

for all $x\in E$ and $(\lambda_n)_{n=1}^\infty\in l^\infty$.

PROPOSITION (1.1). (i) *If A is a Banach algebra with an orthogonal Schauder basis then A is similar to c_0 and so $QM_r(A) = l^\infty$ and $\hat{\hat{A}} = \beta N$.*

(ii) *If E is a Banach space with an unconditional basis $(e_n)_{n=1}^{\infty}$ then E, endowed with the product $xy = \sum_{n=1}^{\infty} e_n^*(x) e_n^*(y) e_n$, $x, y \in A$, is a Banach algebra (in an equivalent norm) and $(e_n)_{n=1}^{\infty}$ is an orthogonal Schauder basis for E. Moreover,* $\mathrm{Mul}(E) = QM_r(E) = l^{\infty}$.

Proof. (i) Let D be the vector subspace of A formed by the elements x of A such that $\sum_{n=1}^{\infty} |e_n^*(x)| < +\infty$. D is a Banach algebra under the product induced by A and the norm $\|x\|_D = \sum_{n=1}^{\infty} |e_n^*(x)|$, $x \in D$. As such an algebra, D is isometric to the usual l^1, with coordinatewise product, and so D is a dense ideal of c_0. Further, D contains the set $\{x \in A: e_n^*(x) = 0 \text{ eventually}\}$,

$$\|x\| = \Big\|\sum_{n=1}^{\infty} e_n^*(x) e_n\Big\| \leqslant \sum_{n=1}^{\infty} |e_n^*(x)| = \|x\|_D$$

if $x \in D$, and

$$\sum_{n=1}^{\infty} |e_n^*(xy)| = \sum_{n=1}^{\infty} |e_n^*(x)|\,|e_n^*(y)| \leqslant \big(\sup_n |e_n^*(y)|\big) \sum_{n=1}^{\infty} |e_n^*(x)| < +\infty$$

for all $x \in D$, $y \in A$. It follows that D is a dense ideal of A and the inclusion $D \to A$ is continuous, hence A and c_0 are similar (note that D possesses dense principal ideals: it suffices to take $x \in D$ such that $e_n^*(x) \neq 0$ for every $n \geqslant 1$ to have $[xD]^- = D$). Finally, note that $QM_r(A) = l^{\infty}$ since c_0 is a uniform algebra.

(ii) The first part is well known ([10], p. 346). Now, if $(\lambda_n)_{n=1}^{\infty}$ belongs to l^{∞} it is clear that the mapping

$$\sum_{n=1}^{\infty} x_n e_n \in E \mapsto \sum_{n=1}^{\infty} \lambda_n x_n e_n \in E$$

is a multiplier of E and we have $l^{\infty} \subset \mathrm{Mul}(E) \subset QM_r(E) = l^{\infty}$. ■

Now we give examples of Banach algebras which satisfy the conditions of the proposition.

1) In the usual Banach sequence spaces c_0, l^p $(1 \leqslant p < +\infty)$ the sequence $(\delta_n)_{n=1}^{\infty}$, where $\delta_n = (0, \ldots, 0, \overset{n}{1}, 0, \ldots)$, $n \geqslant 1$, is an unconditional basis.

2) Let BV_0 be the Banach subalgebra of c_0 formed by the sequences $(x_n)_{n=1}^{\infty} \in c_0$ such that $\sum_{n=2}^{\infty} |x_n - x_{n-1}| < +\infty$ endowed with the norm

$$\|(x_n)_{n=1}^{\infty}\| = \max\Big(\sup_n |x_n|,\ \sum_{n=2}^{\infty} |x_n - x_{n-1}|\Big).$$

$(\delta_n)_{n=1}^{\infty}$ is an orthogonal Schauder basis for BV_0.

3) James' Banach space J is a Banach subalgebra of c_0, under a certain norm ([2], p. 1083). The sequence $(\delta_n)_{n=1}^{\infty}$ is also an orthogonal Schauder basis for J.

BV_0 and J are semisimple Banach algebras with "small" multiplier algebras: $\mathrm{Mul}(BV_0) = BV_0 \oplus C$, $\mathrm{Mul}(J) = J \oplus C \cong J^{**}$ (see [2], [13]). By Proposition (1.1), $QM_r(BV_0) = QM_r(J) = l^{\infty}$.

4) The Hardy spaces $H^p(\boldsymbol{D})$, $1 \leqslant p < +\infty$, where $\boldsymbol{D}$ is the open unit disk in $\boldsymbol{C}$ are Banach algebras with the Hadamard product

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w} g(zw^{-1})\, dw,$$

where $f, g \in H^p(\boldsymbol{D})$ and $|z| < r < 1$ (see [16]). The sequence $e_n(z) = z^n$, $z \in \boldsymbol{D}$, $n \geqslant 1$, is an orthogonal Schauder basis for $H^p(\boldsymbol{D})$ if $1 < p < +\infty$, whence $QM_r(H^p(\boldsymbol{D})) = l^{\infty}$ $(1 < p < \infty)$.

Now, let $A(\boldsymbol{D})$ be the Banach space of functions which are continuous on $\bar{\boldsymbol{D}}$ and analytic on $\boldsymbol{D}$, endowed with the norm $\|f\|_{\infty} = \sup_{|z| \leqslant 1} |f(z)|$. Under the Hadamard product, $A(\boldsymbol{D})$ is also a Banach algebra, say $A(\boldsymbol{D})_*$ ([16], p. 94). Actually, $A(\boldsymbol{D})_*$ is an ideal of each $H^p(\boldsymbol{D})$; $\|f\|_p \leqslant \|f\|_{\infty}$ if $f \in A(\boldsymbol{D})$; $A(\boldsymbol{D})$ is dense in $H^p(\boldsymbol{D})$ if $1 \leqslant p < +\infty$ ([16], p. 84), and $A(\boldsymbol{D})_*$ possesses dense principal ideals (it suffices to consider $f \in A(\boldsymbol{D})$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \boldsymbol{D})$, with $\sum_{n=0}^{\infty} |a_n| < +\infty$ and $a_n \neq 0$ for every $n \geqslant 0$, to have $[f_* A(\boldsymbol{D})_*]^- = A(\boldsymbol{D})_*$ since the polynomials are dense in $A(\boldsymbol{D})$). In short, $A(\boldsymbol{D})_*$ and $H^p(\boldsymbol{D})$ are similar and $QM_r(A(\boldsymbol{D})_*) = QM_r(H^p(\boldsymbol{D})) = l^{\infty}$ for $1 \leqslant p < +\infty$.

5) The space $L^p(\boldsymbol{T})$, $1 \leqslant p < +\infty$, where $\boldsymbol{T}$ is the circle group is a Banach algebra with convolution. If $1 < p < +\infty$, the sequence of trigonometric polynomials $e_n(t) = t^n$, $t \in \boldsymbol{T}$, $n \in \boldsymbol{Z}$, is an orthogonal Schauder basis for $L^p(\boldsymbol{T})$ (if $p = 2$, $(e_n)_{n=-\infty}^{\infty}$ is an unconditional basis), hence $L^p(\boldsymbol{T})$ is similar to c_0. Moreover, $L^1(\boldsymbol{T})$ is also similar to $L^p(\boldsymbol{T})$, $p > 1$, and to c_0 (the details are given in Proposition (1.3) in part b of this section where we study more specifically algebras of integrable functions), and we see that $QM_r(L^p(\boldsymbol{T})) = l^{\infty}$, $1 \leqslant p < +\infty$.

Remarks. (i) By Proposition (1.1) (i), any Banach algebra with an orthogonal Schauder basis is similar to c_0. The converse is not true: $L^1(\boldsymbol{T})$ is similar to c_0 as we have observed in 5), but it has no Schauder basis orthogonal for convolution.

To see this, assume that $(b_n)_{n=-\infty}^{\infty}$ is an orthogonal Schauder basis in $L^1(\boldsymbol{T})$. Then

$$b_n^*(e_k)\, b_n^*(e_j) = \begin{cases} b_n^*(e_k) & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \tag{1}$$

since each b_n^* is a character. Further, the Fourier series of each b_n is an idempotent in $c_0(\boldsymbol{Z})$ and so b_n is a finite linear combination with coefficients 0 or 1 of the elements e_m, $m \in \boldsymbol{Z}$. It follows that

$$e_k = \sum_{n=-\infty}^{\infty} b_n^*(e_k)\, b_n, \quad b_n = \sum_{k=-\infty}^{\infty} e_k^*(b_n)\, e_k, \quad k, n \in \boldsymbol{Z};$$

therefore

(2) $$b_m^*(e_j)\,b_m = \sum_{n=-\infty}^{\infty} b_n^*(e_j)\,b_n b_m = e_j b_m = e_j^*(b_m)\,e_j, \qquad j, m \in \mathbf{Z}.$$

Let n_0 be fixed. If $b_{n_0}^*(e_k) = 0$ for all $k \in Z$, then $e_k^*(b_{n_0}) = 0$ for all $k \in Z$ by (2) and $b_{n_0} = 0$, a contradiction. From this and (1), there is a unique $k_0 \in \mathbf{Z}$ such that $b_{n_0} = e_{k_0}$. This shows that the Schauder basis $(b_n)_{n=-\infty}^{\infty}$ in $L^1(T)$ is a rearrangement of $(e_n)_{n=-\infty}^{\infty}$ (note that the same argument shows that any orthogonal Schauder basis in any Banach algebra is a rearrangement of a given one). But this is impossible (see [15], Th. 9, p. 24).

(ii) Proposition (1.1) (ii) gives examples of not necessarily uniform Banach algebras A with the property $\mathrm{Mul}(A) = QM_r(A)$ (see the introduction and [13], p. II-8).

b. *Algebras of integrable functions.* Throughout this paper, we denote by G a nondiscrete locally compact abelian group and by Γ its dual group. As usual, $L^1(G)$ denotes the convolution algebra of integrable functions on G with respect to the normalized Haar measure; if $f \in L^1(G)$ we set $\|f\|_1 = \int_G |f|$ and we denote by $\hat{f}$ the Fourier transform of f, which is a continuous function vanishing at infinity on Γ, i.e. $\hat{f} \in C_0(\Gamma)$.

Now we consider Segal algebras $S(G)$ on G (we refer to [17], pp. 127, 128 for the definition, examples and a list of general properties of Segal algebras). It is clear from the definition and properties of Segal algebras that $S(G)$ is a convolution Banach algebra, under a certain norm $\|\ \|_S$, on G; and $S(G)$ is also a dense ideal of $L^1(G)$. Next, we will characterize the nondiscrete locally compact abelian groups G for which the theory of regular quasimultipliers can be applied to any Segal algebra $S(G)$, in particular to $L^1(G)$.

LEMMA (1.2). *Let G be a nondiscrete locally compact abelian group and let $S(G)$ be a Segal algebra on G. Then:*

(i) *For $f \in S(G)$, $f * S(G)$ is $\|\ \|_S$-dense in $S(G)$ iff $\hat{f}(\gamma) \neq 0$ for every $\gamma \in \Gamma$.*

(ii) *$S(G)$ possesses dense principal ideals iff G is metrizable.*

Proof. (i) This part is clear in the case where $S(G) = L^1(G)$ by the Wiener tauberian theorem. In general, if $f \in S(G)$ and $\hat{f}(\gamma) \neq 0$ for all $\gamma \in \Gamma$ we write I_S for the $\|\ \|_S$-closure in $S(G)$ of $f * S(G)$. By [17], p. 129, there is a unique closed ideal I of $L^1(G)$ such that $I_S = I \cap S(G)$. Moreover, $I = [I_S]^-$ in $L^1(G)$. Because $S(G)$ is $\|\ \|_1$-dense in $L^1(G)$ ([17], p. 127), so is $f * S(G)$ (again by the Wiener tauberian theorem). Therefore $I = L^1(G)$ and $I_S = I \cap S(G) = S(G)$ as desired. The converse follows from the density of $S(G)$ in $L^1(G)$.

(ii) First recall that G is metrizable if and only if Γ is countable at infinity ([17], p. 94).

Suppose that G is not metrizable, and take $f \in L^1(G)$. Set $K_n = \{\gamma\colon |\hat{f}(\gamma)| \geqslant 1/n\}$, a compact set in Γ, and set $K = \bigcup K_n$. Then $K \neq \Gamma$. For $\gamma \in \Gamma \setminus K$, $\hat{f}(\gamma) = 0$, and so $[f * S(G)]^- \neq S(G)$.

Conversely, suppose that G is metrizable. Then there is a sequence (K_n) of compact sets such that $\Gamma = \bigcup K_n$. Since $S(G)$ is dense in $L^1(G)$, for each n there exists $f_n \in S(G)$ with $\hat{f}_n(\gamma) \neq 0$, $\gamma \in K_n$. Set

$$g = \sum_{n=1}^{\infty} \frac{f_n * \tilde{f}_n}{2^n(1+\|f_n\|_S^2)},$$

where $\tilde{f}_n(x) = f_n(-x)$, $x \in G$. Then $\hat{g}(\gamma) \neq 0$ and so, by (i), $[g * S(G)]^- = S(G)$. ∎

For G nondiscrete and metrizable it is evident that $L^1(G)$ is similar to any Segal algebra $S(G)$, and a natural way to identify $QM_r(S(G))$—and $QM_r(L^1(G))$ in particular—consists in finding a Segal algebra $S(G)$ for which the description of $QM_r(S(G))$ is sufficiently simple. For instance, this method is available if G is compact as the following proposition shows.

PROPOSITION (1.3). *Let G be a nondiscrete metrizable compact abelian group and let $S(G)$ be a Segal algebra on G. Then $S(G)$ is similar to $c_0(\Gamma)$, $QM_r(S(G)) = l^\infty(\Gamma)$ and $S(G)^{\widehat{}} = \beta\Gamma$.*

Proof. For G compact, Γ is discrete and countable. The Segal algebra $L^2(G)$ is isometric to $l^2(\Gamma) = \{(a_\gamma)_{\gamma\in\Gamma}\colon \sum_{\gamma\in\Gamma} |a_\gamma|^2 < +\infty\}$ which is a dense ideal of $c_0(\Gamma)$ as an easy computation shows. It follows that $L^2(G)$ is similar to $c_0(\Gamma)$. Since $c_0(\Gamma)$ is uniform, $QM_r(c_0(\Gamma)) = \mathrm{Mul}(c_0(\Gamma)) = l^\infty(\Gamma)$. ∎

Remark. For each locally compact abelian group G we denote by $M(G)$ the convolution algebra of Borel finite measures on G normed by the total variation. As is well known, $M(G)$ equals $\mathrm{Mul}(L^1(G))$, the multiplier algebra of $L^1(G)$. If G is metrizable, $M(G)$ is contained in $QM_r(L^1(G))$ ([5]). Moreover, if G is also nondiscrete and compact we have identified $QM_r(L^1(G))$ to be $l^\infty(\Gamma)$. It may be of some interest to compare certain properties of $M(G)$ and $QM_r(L^1(G))$ in that case:

1) The character space of $M(G)$ remains somewhat mysterious so far ([20]).

1′) $QM_r(L^1(G)) = l^\infty(\Gamma)$ so its character space is $\beta\Gamma$.

2) $M(G)$ is not regular on its character space ([18]).

2′) $QM_r(L^1(G)) = l^\infty(\Gamma)$ is regular on $\beta\Gamma$.

3) (related to 2) There exists $\mu \in M(G)$ such that its Fourier transform $\hat{\mu}$ is invertible in $l^\infty(\Gamma)$ but μ is not invertible in $M(G)$ ([18], [19], [23]).

3′) Clearly, T is invertible in $QM_r(L^1(G))$ if and only if $\hat{T}$ is invertible in $l^\infty(\Gamma)$. In particular, if $\mu \in M(G)$ and $\hat{\mu}$ is invertible in $l^\infty(\Gamma)$ then μ is invertible in $QM_r(L^1(G))$. Even more, every quasimultiplier T on G whose extended Gelfand transform $\hat{T}$ is bounded on Γ is regular.

The study of $QM_r(L^1(G))$, trivial as it has been observed to be if G is compact, is much more difficult if G is noncompact locally compact. In § 3 we shall deal with this case.

Note that the theory of quasimultipliers is trivial if G is discrete since in that case $l^1(G)$ has an identity.

§ 2. **A general method of investigating the extended spectrum.** In § 1 we have obtained some results about similarity of Banach algebras by considering their Gelfand representation. Here we exploit a bit more this canonical mapping to establish a general procedure for investigating the extended spectrum when the easy similarities are not available. We have the following result, whose proof is routine:

LEMMA (2.1). *Let A be a unital pseudo-Banach algebra and B a unital Banach algebra. Suppose that $\varphi\colon A \to B$ is a bounded injective unital homomorphism such that $\overline{\varphi(A)} = B$. Then $\hat{B}$ is homeomorphically embedded into $\hat{A}$ by means of the mapping $\varphi^*\colon \hat{B} \to \hat{A}$ given by $\varphi^*(x) = x \circ \varphi$ for every $x \in \hat{B}$.*

Furthermore, $\hat{A} \equiv \hat{B}$ if and only if each $a \in A$ such that $|\varphi(a)\hat{\ }(x)| \geqslant c$ $(x \in \hat{B})$ for some $c > 0$ is invertible in A.

In the remainder of this section we apply the foregoing lemma to a Banach algebra of differentiable functions which has a bounded approximate identity. For m and p nonnegative integers we write $|k| = k_1 + \ldots + k_p$, $k! = k_1! \ldots k_p!$ for $k = (k_1, \ldots, k_p) \in N^p$; $\partial^k f$ denotes the partial derivative $\partial^k f/\partial x^k$ for $0 \leqslant |k| \leqslant m$ and $f \in C^{(m)}(R^p)$. We let $C_0^{(m)}(R^p)$ be the Banach algebra of all $C^{(m)}$-functions f on R^p such that $\partial^k f(\infty) = 0$, $0 \leqslant |k| \leqslant m$, endowed with the norm

$$\|f\| = \sum_{0 \leqslant |k| \leqslant m} \sup_x |\partial^k f(x)|/k!.$$

The sequence $(e_n)_{n=1}^{\infty}$ where

$$e_n(x) = \exp(-\|x\|^2 n^{-1}), \qquad \|x\|^2 = x_1^2 + \ldots + x_p^2,$$

$x = (x_1, \ldots, x_p) \in R^p$, $n \in N$, is a bounded approximate identity for $C_0^{(m)}(R^p)$ ([21]). Any principal ideal $f C_0^{(m)}(R^p)$ where f is nowhere zero on R^p is dense in $C_0^{(m)}(R^p)$ since each function in $C_0^{(m)}(R^p)$ with compact support belongs to $f C_0^{(m)}(R^p)$.

The following result is well known:

LEMMA (2.2). *Let ω be a positive continuous function on R^p. Then for every continuous function f on R^p there exists a $C^{(\infty)}$-function g on R^p such that*

$$|g(x) - f(x)| < \omega(x) \quad \textit{for all } x \in R^p.$$

In particular, for every zero-free continuous function f on R^p there is always a zero-free $C^{(\infty)}$-function g on R^p satisfying $|g(x)| < |f(x)|$ for all $x \in R^p$.

The following lemma will also be used in § 3.

LEMMA (2.3). (i) *Let F be a continuous function on R^p with values in a unital Banach algebra A, and α a positive continuous function on R^p such that $\lim_{\|x\| \to \infty} \alpha(x) = 0$. Then there exists a function $v \in C_0^{(m)}(R^p) \cap C^{(\infty)}(R^p)$ such that $v(x) \neq 0$ for all $x \in R^p$ and*

$$\sup_{x \in R^p} \|F(x)\,\partial^k v(x)\|_A\, \alpha(x)^{-1} \leqslant 1, \qquad 0 \leqslant |k| \leqslant m.$$

(ii) *If F is also of class $C^{(m)}$, then v can be chosen so that*

$$\sup_{x \in R^p} \|\partial^k (Fv)(x)\|_A\, \alpha(x)^{-1} \leqslant 1, \qquad 0 \leqslant |k| \leqslant m.$$

(iii) *Moreover, if F is also bounded by 1 on R^p, then v can be chosen so that*

$$\sup_{x \in R^p} \|\partial^k (F^n v)(x)\|_A\, \alpha(x)^{-1} \leqslant n^m, \qquad n \geqslant 1.$$

Proof. If we set $g(x) = [1 + \|F(x)\|_A]^{-1} \alpha(x) \beta(x)$ for $x \in R^p$, where $\beta(x) = O(\|x\|^{-n})$ at infinity for every $n \geqslant 1$, β positive and continuous on R^p, then the function

$$\varphi(t) = \min_{\|x\|^2 = t} g(x)$$

is continuous on $[0, \infty)$. We can take a positive decreasing infinitely differentiable function h on $[0, +\infty)$ such that $h(t) < \varphi(t)$ for every $t \in [0, \infty)$. We define

$$\omega(t) = \int_t^{+\infty} \int_{t_2}^{+\infty} \ldots \int_{t_m}^{+\infty} \frac{h(t_{m+1})}{(1 + t_{m+1}^2)^m}\, dt_{m+1} \ldots dt_2, \qquad t \geqslant 0.$$

The function ω is well defined since

$$\omega(t) \leqslant \int_t^{+\infty} \int_{t_2}^{+\infty} \ldots \int_{t_{m-1}}^{+\infty} \left[\frac{h(t_m)}{(1 + t_m^2)^{m-1}} \int_{t_m}^{+\infty} \frac{dt_{m+1}}{1 + t_{m+1}^2} \right] dt_m \ldots dt_2$$
$$\leqslant \left(\int_t^{+\infty} \int_{t_2}^{+\infty} \ldots \int_{t_{m-1}}^{+\infty} \frac{h(t_m)}{(1 + t_m^2)^{m-1}}\, dt_m \ldots dt_2 \right) \cdot \tfrac{1}{2}\pi$$
$$\leqslant \ldots \leqslant (\tfrac{1}{2}\pi)^m h(t).$$

Moreover, ω is $C^{(\infty)}$ on $[0, +\infty)$ and its m first derivatives are

$$\omega^{(j)}(t) = (-1)^j \int_t^{+\infty} \int_{t_{j+2}}^{+\infty} \ldots \int_{t_m}^{+\infty} \frac{h(t_{m+1})}{(1 + t_{m+1}^2)^m}\, dt_{m+1} \ldots dt_{j+2},$$

$t \geq 0$, $0 \leq j \leq m-1$ (with $t_1 = t$),

$$\omega^{(m)}(t) = (-1)^m \frac{h(t)}{(1+t^2)^m}, \quad t \geq 0.$$

Using the same method to majorize $\omega^{(j)}$, $0 \leq j \leq m$, as was used for ω we obtain the bounds

$$|\omega^{(j)}(t)| \leq (\tfrac{1}{2}\pi)^{m-j} \frac{h(t)}{(1+t^2)^j}, \quad t \geq 0,\ 0 \leq j \leq m.$$

Now, set $v(x) = \omega(\|x\|^2)$, $x \in \boldsymbol{R}^p$. It follows that, for $k = (k_1, \ldots, k_p) \in N^p$ and $\|x\|^2 = t$,

$$\begin{aligned}\|F(x)\,\partial^k_\varphi v(x)\|_A &\leq \|F(x)\|_A \sum_{0 \leq l \leq |k|} |\omega^{(l)}(t)|\,|P_l(x)| \\ &\leq \sum_{0 \leq l \leq |k|} (\tfrac{1}{2}\pi)^{m-l} \frac{h(t)}{(1+t^2)^l} |P_l(x)|\,\|F(x)\|_A \\ &\leq \sum_{0 \leq l \leq |k|} (\tfrac{1}{2}\pi)^{m-l} (1+\|x\|^4)^{-l} |P_l(x)|\, \acute{g}(x)\,\|F(x)\|_A \\ &\leq \alpha(x) \quad \text{for every } x \in \boldsymbol{R}^p\end{aligned}$$

(by a suitable choice of β), where P_l is a polynomial of degree l.

(ii) It suffices to apply part (i) to the function

$$\prod_{0 \leq |k| \leq m} [1 + \|\partial^k F(x)\|_A].$$

(iii) In the case where $\sup_x \|F(x)\|_A \leq 1$ we note that whenever $0 \leq |k| \leq m$ and $n > m$, then $\partial^k(F^n)$ is a finite combination of expressions of the form

$$P(n)\,F^{q_0} \prod_{1 \leq |j| \leq |k|} (\partial^j F)^{q_j},$$

with $P(n)$ a polynomial in n whose degree does not exceed $|k|$ and $(q_j)_{0 \leq |j| \leq |k|}$ nonnegative integers such that $0 \leq q_0 \leq n$, $0 \leq q_j \leq |k|$, $|j| > 0$. Therefore to prove this part it is enough to apply part (i) to the function

$$\prod_{\substack{0 \leq |j| \leq m \\ 1 \leq q \leq m}} [1 + \|\partial^j F(x)\|_A^q], \quad x \in \boldsymbol{R}^p. \ \blacksquare$$

We denote by $C_\infty(\boldsymbol{R}^p)$ the space of bounded continuous functions on $\boldsymbol{R}^p$.

Theorem (2.4). *The correspondence*

$$F \in C^{(m)}(\boldsymbol{R}^p) \mapsto T_F \in QM(C_0^{(m)}(\boldsymbol{R}^p))$$

where $T_F(f) = Ff$ for every $f \in C_0^{(m)}(\boldsymbol{R}^p)$ is bijective. It also induces a bijection between $C^{(m)}(\boldsymbol{R}^p) \cap C_\infty(\boldsymbol{R}^p)$ and $QM_r(C_0^{(m)}(\boldsymbol{R}^p))$.

Proof. If T is a quasimultiplier of $C_0^{(m)}(\boldsymbol{R}^p)$, T is a quotient f/g with f, g in $C_0^{(m)}(\boldsymbol{R}^p)$ and g is zero-free on $\boldsymbol{R}^p$. It follows that $T \in C^{(m)}(\boldsymbol{R}^p)$. The injection of $C_0^{(m)}(\boldsymbol{R}^p)$ into $C_0(\boldsymbol{R}^p)$ can be extended to the respective quasimultiplier algebras as an injection too. Since $QM_r(C_0(\boldsymbol{R}^p)) = C_\infty(\boldsymbol{R}^p) - C_0(\boldsymbol{R}^p)$ is uniform — we have $QM_r(C_0^{(m)}(\boldsymbol{R}^p))$ contained in $C^{(m)}(\boldsymbol{R}^p) \cap C_\infty(\boldsymbol{R}^p)$. Now, if F is m times continuously differentiable on $\boldsymbol{R}^p$ we can write $F = (Fv)/v$, where $v \in C_0^{(m)}(\boldsymbol{R}^p)$, $v(x) \neq 0$ for every $x \in \boldsymbol{R}^p$ and $Fv \in C_0^{(m)}(\boldsymbol{R}^p)$ by Lemma (2.3)(ii) with $A = \boldsymbol{C}$. This proves the first part of the theorem.

If, further, F is bounded on $\boldsymbol{R}^p$ and $M = \sup_{x \in \boldsymbol{R}^p} |F(x)|$ we can apply to F/M part (iii) of Lemma (2.3), and so there exists $v \in C_0^{(m)}(\boldsymbol{R}^p)$, which nowhere vanishes on $\boldsymbol{R}^p$, such that $(F/M)^n v \in C_0^{(m)}(\boldsymbol{R}^p)$ for every $n \geq 1$; furthermore,

$$\|F^n v\| = M^n \|(F/M)^n v\| = M^n \sup_{\substack{x \in \boldsymbol{R}^p \\ 0 \leq |k| \leq m}} |\partial^k [(F/M)^n]\, v(x)| = M^n O(n^m)$$

for $n \geq 1$. If $\lambda = 1/(2(M+1))$, then $\|\lambda^n F^n v\| \to 0$ as $n \to \infty$, and F is regular as a quasimultiplier. $\blacksquare$

Corollary (2.5). *The character space of $QM_r(C_0^{(m)}(\boldsymbol{R}^p))$ coincides topologically with $\beta\boldsymbol{R}^p$, the Stone–Čech compactification of $\boldsymbol{R}^p$.*

Proof. Let $A = QM_r(C_0^{(m)}(\boldsymbol{R}^p))$ and $B = C_\infty(\boldsymbol{R}^p)$. By Lemma (2.2), A is dense in B. Suppose that $T \in A$ and that $\hat{T}$ is invertible in B. By Theorem (2.4), T is invertible in A, and so, by Lemma (2.1), $\hat{A} = \hat{B}$. Clearly $\hat{B} = \beta\boldsymbol{R}^p$. $\blacksquare$

§ 3. On the regular quasimultipliers of $L^1(G)$, G noncompact. Throughout this section we denote by Σ the extended spectrum of $L^1(G)$, where G is a nondiscrete metrizable noncompact locally compact abelian group. If T is a quasimultiplier of $L^1(G)$ we denote by $\hat{T}$ the extended Fourier transform of T. Note that $\hat{T}$ is a continuous function on Γ, the dual group of G, and $\hat{T}$ is bounded if T is a regular quasimultiplier.

The main result of this section is the following:

If G is compactly generated, then Σ contains homeomorphically $\beta\Gamma$, the Stone–Čech compactification of Γ.

This assertion is obtained as a corollary of Theorem (3.3) which we will state and prove below. For the proof we need some partial results that we state as lemmas. They also give additional information.

Let A be a semisimple unital Banach algebra. We denote by $L^1(\boldsymbol{R}^p; A)$ the convolution Banach algebra of all A-valued Bochner integrable functions on $\boldsymbol{R}^p$. Recall that the tensor product $L^1(\boldsymbol{R}^p) \otimes A$ is dense in $L^1(\boldsymbol{R}^p; A)$ and $L^1(\boldsymbol{R}^p; A)^\wedge = \boldsymbol{R}^p \times \hat{A}$ ([21], p. 473; [3], p. 236). If $h \in L^1(\boldsymbol{R}^p; A)$, $\hat{h}$ denotes the vector Fourier transform of h and $\check{h}(x) = \hat{h}(-x)$, $x \in \boldsymbol{R}^p$. Let

$$C^{(m)}(\boldsymbol{R}; A) = \{F\colon \boldsymbol{R}^p \to A\colon F \text{ is } C^{(m)}\}, \quad m = 0, 1, 2, \ldots$$

LEMMA (3.1). *Let m, p be integers such that $m > p/2$. If $F \in C^{(2m)}(\mathbf{R}^p; A)$ and F is bounded then there exists $v \in C_0^{(2m)}(\mathbf{R}^p) \cap C^{(\infty)}(\mathbf{R}^p)$ such that $F^n v \in L^1(\mathbf{R}^p; A)$, $(F^n v)\check{} \in L^1(\mathbf{R}^p; A)$ for every $n \geqslant 0$ and $T = (Fv)\check{}/\check{v}$ is a regular quasimultiplier of $L^1(\mathbf{R}^p; A)$ such that $\hat{T} = F$.*

Proof. Without loss of generality we can assume that F is bounded by 1. According to Lemma (2.3) there exists a zero-free function $v \in C_0^{(2m)}(\mathbf{R}^p) \cap C^{(\infty)}(\mathbf{R}^p)$ such that $\partial^k v \in L^1(\mathbf{R}^p)$ and $\partial^k(F^n v) \in L^1(\mathbf{R}^p; A)$, $k \in \mathbf{N}^p$, $0 \leqslant |k| \leqslant 2m$, $n \geqslant 1$. If $f_n = (F^n v)\check{}$, $n \geqslant 1$, then

$$x_1^{k_1} \ldots x_p^{k_p} (F^n v)\check{}(x) = i^{|k|} \left[\frac{\partial^k (F^n v)}{\partial x^k} \right]\check{}(x),$$

$x = (x_1, \ldots, x_p) \in \mathbf{R}^p$, $k = (k_1, \ldots, k_p) \in \mathbf{N}^p$, $0 \leqslant |k| \leqslant 2m$, $n \geqslant 1$, by an argument similar to the one of the scalar case. Therefore

$$(1 + \|x\|^2)^m f_n(x) = \check{D}_n(x), \qquad x \in \mathbf{R}^p,\ n \geqslant 1,$$

where D_n is the corresponding finite combination of partial derivatives of $F^n v$. It follows that $f_n \in L^1(\mathbf{R}^p; A)$; in fact,

$$\|f_n\|_1 \leqslant \int_{\mathbf{R}^p} \frac{\|\check{D}_n(x)\|_A}{(1 + \|x\|^2)^m} dx \leqslant O(n^m) \int_{\mathbf{R}^p} \frac{dx}{(1 + \|x\|^2)^m} = O(n^m)$$

(the last integral is finite because $m > p/2$).

It is easy to see that $\hat{f}_n = F^n v$ by using continuous functionals on A. Analogously, there is $g \in L^1(\mathbf{R}^p)$ with $\hat{g} = v$. The ideal generated by g is dense in $L^1(\mathbf{R}^p; A)$ since $L^1(\mathbf{R}^p; A) = [L^1(\mathbf{R}^p) \otimes A]^-$ and $[g * L^1(\mathbf{R}^p)]^- = L^1(\mathbf{R}^p)$, and the unicity for $n \geqslant 1$ of f_n/g as a quasimultiplier of $L^1(\mathbf{R}^p; A)$ follows from the semisimplicity of A.

Set $T = f_1/g$, so that $T \in QM(L^1(\mathbf{R}^p; A))$. Since

$$(T^n * g)\hat{}(x, \Phi) = \frac{\hat{f}_1(x, \Phi)^n}{\hat{g}(x)^n} \hat{g}(x) = F^n(x)\hat{}(\Phi) v(x) = \hat{f}_n(x, \Phi)$$

for all $x \in \mathbf{R}^p$, $\Phi \in \hat{A}$ (the power T^n refers to the convolution) and A is semisimple, $T^n * g = f_n$ follows, whence g belongs to the domain of definition of T^n. Further,

$$\|T^n * g\|_{L^1(\mathbf{R}^p; A)} = \|f_n\|_1 = O(n^m),$$

whence $\sup_n \|(\lambda T)^n * g\|_1 < +\infty$ for every $\lambda \in [0, 1)$ and T is regular. ∎

Note that if $F \in C^{(2m)}(\mathbf{R}^p, A)$ is not bounded, then $T = f_1/g$, where $f_1 = (Fv)\check{}$, is in $QM(L^1(\mathbf{R}^p; A))$ and $\hat{T} = F$, but in general T is not regular.

Let $p, q \in \mathbf{N}$ and let $\mathbf{T}^q$ be the q-dimensional torus. We identify $\mathbf{T}^q$ with $\underset{(q)}{[0, 2\pi) \times \ldots \times [0, 2\pi)} = [0, 2\pi)^q$ and we denote by $C^{(r,s)}$ the set of all functions F on $\mathbf{R}^p \times \mathbf{T}^q$ which are of class $C^{(r)}$ on $\mathbf{R}^p$ and $C^{(s)}$ on $\mathbf{T}^q$.

LEMMA (3.2). *Assume $G = \mathbf{R}^p \times \mathbf{Z}^q$ and $m > p/2$ fixed. If $F \in C^{(2m+1, 2q)} \cap C_\infty(\mathbf{R}^p \times \mathbf{T}^q)$ there exists $v \in C_0^{(2m)}(\mathbf{R}^p) \cap C^{(\infty)}(\mathbf{R}^p)$ and, for each $n \geqslant 0$, a unique $h_n \in L^1(G)$ such that $\hat{h}_n = F^n v$ and $\|h_n\|_1 \leqslant M n^{2(m+q)}$, where M is a constant depending only on m and q.*

Proof. If we consider in $C(\mathbf{T}^q)$ the topology of uniform convergence on $\mathbf{T}^q$ a simple argument based upon the compactness of $\mathbf{T}^q$ shows that any complex continuous function on $\mathbf{R}^p \times \mathbf{T}^q$ can be viewed as a function belonging to $C(\mathbf{R}^p; C(\mathbf{T}^q))$. Actually, the function F of the statement is in $C^{(2m)}(\mathbf{R}^p; C(\mathbf{T}^q))$ (via Taylor's formula). Assume F is bounded by 1. According to Lemma (3.1) applied to F and Lemma (2.3)(i) applied to the function

$$\prod_{\substack{0 \leqslant |l| \leqslant 2(m+q) \\ 0 \leqslant |r| \leqslant 2q,\, 0 \leqslant |s| \leqslant 2m \\ r+s=l,\, 1 \leqslant j \leqslant 2(m+q)}} [1 + \|(\partial^l/\partial t^r\, \partial x^s) F\|_{C(\mathbf{T}^q)}^j]$$

regarded as a member of $C(\mathbf{R}^p; C(\mathbf{T}^q))$, we obtain $v \in C_0^{(2m)}(\mathbf{R}^p) \cap C^{(\infty)}(\mathbf{R}^p)$ such that $F^n v, f_n = (F^n v)\check{} \in L^1(\mathbf{R}^p; C(\mathbf{T}^q))$ for $n \geqslant 0$ and

$$\sup_{y \in \mathbf{R}^p} \left\| \frac{\partial^{2q}}{\partial t_1^2 \ldots \partial t_q^2} D_n(y, \cdot) \right\|_{C(\mathbf{T}^q)} \alpha(y)^{-1} \leqslant C_{m,q} n^{2(m+q)}, \qquad n \geqslant 1,$$

where $D_n(x, \cdot)$ is such that

$$\check{D}_n(x, \cdot) = (1 + \|x\|^2)^m (F^n v)\check{}(x, \cdot) \qquad \text{for } x \in \mathbf{R}^p,$$

$\alpha \in L^1(\mathbf{R}^p)$ and $C_{m,q}$ is a constant depending only on m and q. We next observe that

$$f_n(x, t) = \frac{\check{D}_n(x, t)}{(1 + \|x\|^2)^m}, \qquad (x, t) \in \mathbf{R}^p \times \mathbf{T}^q,$$

whence

$$\sup_{t \in \mathbf{T}^q} \left| \frac{\partial^{2q}}{\partial t_1^2 \ldots \partial t_q^2} f_n(x, t) \right| = \sup_{t \in \mathbf{T}^q} \frac{1}{(1 + \|x\|^2)^m} \left| \int_{\mathbf{R}^p} \frac{\partial^{2q}}{\partial t_1^2 \ldots \partial t_q^2} D_n(y, t) e^{ixy} dy \right|$$

$$\leqslant \frac{1}{(1 + \|x\|^2)^m} C_{m,q} n^{2(m+q)} \|\alpha\|_1 \qquad \text{for } x \in \mathbf{R}^p,\ n \geqslant 1.$$

Thus if $e_{n,k}(x)$ are the Fourier coefficients of $f_n(x)$ ($k = (k_1, \ldots, k_q) \in \mathbf{Z}^q$, $n \geqslant 1$, $x \in \mathbf{R}^p$), we have

$$|e_{n,k}(x)| = \left| \int_{\mathbf{T}^q} f_n(x, t) e^{-ikt} dt \right| = \left| \frac{1}{k_1^2 \ldots k_q^2} \int_{\mathbf{T}^q} \left[\frac{\partial^{2q}}{\partial t_1^2 \ldots \partial t_q^2} f_n(x, t) \right] e^{-ikt} dt \right|$$

$$\leqslant C_{m,q} \frac{n^{2(m+q)} \|\alpha\|_1}{k_1^2 \ldots k_q^2} \cdot \frac{1}{(1 + \|x\|^2)^m}, \qquad k_1, \ldots, k_q \neq 0.$$

It follows that the function

$$h_n\colon (x, k)\in \boldsymbol{R}^p\times \boldsymbol{Z}^q \mapsto h_n(x, k) = e_{n,k}(x)\in \boldsymbol{C}, \qquad n\geqslant 1,$$

is in $L^1(\boldsymbol{R}^p\times \boldsymbol{Z}^q)\cong L^1(\boldsymbol{R}^p; l^1(\boldsymbol{Z}^q))$ since for every $n\geqslant 1$,

$$\|h_n\|_1 = \int_{\boldsymbol{R}^p\times \boldsymbol{Z}^q} |h_n(x, k)|\,dx\,dk = \sum_{k\in \boldsymbol{Z}^q} \int_{\boldsymbol{R}^p} |e_{n,k}(x)|\,dx \leqslant C_{m,q}\, n^{2(m+q)},$$

where $C_{m,q}$ is a constant depending only on m and q. Therefore if $T = h_1/g$ $(\hat{g} = v)$ we have

$$(T^n * g)^{\wedge}(x, t) = \hat{T}^n(x, t)\hat{g}(x, t) = \frac{\hat{h}_1^n(x, t)}{\hat{g}^n(x, t)}\hat{g}(x, t)$$

$$= \frac{[\hat{f}_1(x)(t)]^n}{v^n(x)}v(x) = \frac{F^n(x, t)\,v^n(x)}{v^n(x)}v(x)$$

$$= (F^n v)(x, t) = \hat{f}_n(x)(t) = \hat{h}_n(x, t),$$

i.e. $T^n * g = h_n$ and T is a regular quasimultiplier of $L^1(G)$ with $\hat{T} = F$. The proof for the other cases is clear. ∎

Let K be a metrizable compact abelian group with dual group Δ. Consider $G = \boldsymbol{R}^p\times \boldsymbol{Z}^q\times K$ where $p, q\in \boldsymbol{N}$. The dual group of G is $\Gamma = \boldsymbol{R}^p\times \boldsymbol{T}^q \times \Delta$. If F is a function on Γ we put $F_\delta(x, t) = F(x, t, \delta)$, $(x, t, \delta)\in\Gamma$.

Theorem (3.3). *Let $G = \boldsymbol{R}^p\times \boldsymbol{Z}^q\times K$ be as above and $m > p/2$. Then for every bounded function F on Γ such that $F_\delta\in C^{(2m+1,2q)}$, $\delta\in\Delta$, there exists a unique $T\in QM_r(L^1(G))$ with $\hat{T} = F$. Consequently, $\beta\Gamma\subset\Sigma$.*

Proof. Note that to give a bounded continuous function F on $\Gamma = \boldsymbol{R}^p\times \boldsymbol{T}^q\times\Delta$ is equivalent to giving a family $(F_\delta)_{\delta\in\Delta}$ of uniformly bounded continuous functions on $\boldsymbol{R}^p\times \boldsymbol{T}^q$, where $F_\delta(\cdot) = F(\cdot, \delta)$. Let F in $C_\infty(\Gamma)$ be $C^{(2m+1)}$ on $\boldsymbol{R}^p$ and $C^{(2q)}$ on $\boldsymbol{T}^q$, where $m > p/2$. Assume F is bounded by 1. By Lemma (3.2), for any $\delta\in\Delta$ and $n\geqslant 0$ we can obtain a zero-free function $v_\delta\in C_0^{(2m)}(\boldsymbol{R}^p)\cap C^{(\infty)}(\boldsymbol{R}^p)$ and $g_\delta, h_{\delta,n}\in L^1(\boldsymbol{R}^p\times \boldsymbol{Z}^q)$ such that $\hat{g}_\delta = v_\delta$, $\hat{h}_{\delta,n} = F_\delta^n v_\delta$ and $\|h_{\delta,n}\|_1 \leqslant Mn^{2(m+q)}$. Let $(\beta_\delta)_{\delta\in\Delta}$ be a family in $L^1(\boldsymbol{R}^p\times \boldsymbol{Z}^q)$ satisfying

$$\sum_{\delta\in\Delta}\|\beta_\delta\|_1 < +\infty, \qquad \hat{\beta}_\delta(x, t)\neq 0, \qquad (x, t, \delta)\in \boldsymbol{R}^p\times \boldsymbol{Z}^q\times\Delta.$$

It is clear that

$$h = \sum_{\delta\in\Delta}(\beta_\delta * h_\delta)\,\delta \qquad g = \sum_{\delta\in\Delta}(\beta_\delta * g_\delta)\,\delta$$

belong to $L^1(G)$, the ideal of $L^1(G)$ generated by g is dense in $L^1(G)$ and if $T = h/g$ we have

$$\hat{T}(x, t, \delta) = \frac{\hat{h}(x, t, \delta)}{\hat{g}(x, t, \delta)} = \frac{\hat{\beta}_\delta(x, t)\,\hat{h}_\delta(x, t)}{\hat{\beta}_\delta(x, t)\,\hat{g}_\delta(x)} = F_\delta(x, t) = F(x, t, \delta)$$

for all $(x, t, \delta)\in\Gamma$.

Finally, the function $f_n = \sum_{\delta\in\Delta}(\beta_\delta * h_{n,\delta})\,\delta$, $n\geqslant 1$, is integrable on G since

$$\|f_n\|_1 \leqslant \sum_{\delta\in\Delta}\|\beta_\delta * h_{n,\delta}\|_1 \leqslant Mn^{2(m+q)}\sum_{\delta\in\Delta}\|\beta_\delta\|_1 < +\infty.$$

By using Fourier transforms, we have $(h/g)^n * g = f_n$ so $\|T^n * g\|_1 = \|f_n\|_1 = O(n^{2(m+q)})$, which implies that T is a regular quasimultiplier. Now, by means of a partition of unity on $\boldsymbol{R}^p$, it is easy to verify that each continuous (bounded) function on $\boldsymbol{R}^p\times \boldsymbol{T}^q\times\Delta$ can be approximated by some continuous (bounded) function F on Γ such that $F_\delta\in C^{(2m+1,2q)}$, $\delta\in\Delta$. Then it is enough to recall Lemma (2.1) to conclude that $\beta\Gamma\subset\Sigma$ homeomorphically. ∎

Note again that if the function F of Theorem (3.3) is not assumed to be bounded then the same argument as in the above proof for $n = 1$ yields $T\in QM(L^1(G))$ with $\hat{T} = F$.

Corollary (3.4). *If G is a nondiscrete metrizable compactly generated abelian group (in particular, if G is a metrizable connected abelian group) with dual group Γ, then the extended spectrum Σ of $L^1(G)$ contains homeomorphically $\beta\Gamma$, the Stone–Čech compactification of Γ.*

Proof. By a well-known structure theorem (see [7], p. 90) G is homeomorphic as a group to $\boldsymbol{R}^p\times \boldsymbol{Z}^q\times K$ for some $p, q\in \boldsymbol{N}$ and K a compact metrizable group (if G is connected then $G = \boldsymbol{R}^p\times K$; [7], p. 390). Therefore the assertion follows from Theorem (3.3). ∎

Another consequence of Lemma (2.1) is that $\beta\Gamma = \Sigma$ if and only if every regular quasimultiplier T of $L^1(G)$ whose Fourier transform $\hat{T}$ is bounded below on Γ is invertible in $QM_r(L^1(G))$. This condition on every $T\in QM_r(L^1(G))$ is a "Wiener problem". Unfortunately, we do not know whether it holds or not. We recall that the same problem for the algebra $M(G)$ of finite Borel measures has a negative answer: there exists $\mu\in M(G)$ satisfying $|\hat{\mu}(\gamma)|\geqslant c$ for every $\gamma\in\Gamma$ and some $c > 0$, but not invertible in $M(G)$ ([18], p. 107). Because of this we may think that such a measure serves perhaps to show that $\Sigma\neq\beta\Gamma$. Nevertheless, we have

Theorem (3.5). *Let G be a nondiscrete metrizable compactly generated abelian group. If $\mu\in M(G) = \mathrm{Mul}(L^1(G))$ then $\sigma(\mu) = \{\hat{\mu}(\gamma)\colon \gamma\in\Gamma\}^-$. In particular, if $|\hat{\mu}(\gamma)|\geqslant c > 0$ for every $\gamma\in\Gamma$ and some $c > 0$, then μ is invertible in $QM_r(L^1(G))$.*

We recall that the *spectrum* $\sigma(\mu)$ of μ in $QM_r(L^1(G))$ is by definition the set of complex numbers λ such that $\mu-\lambda$ is not invertible in $QM_r(L^1(G))$. As is well known $\sigma(\mu)$ coincides also with $\{\hat{\mu}(\chi)\colon \chi\in\Sigma\}$ (see [1], p. 62).

Proof of the theorem. First, assume that μ has a compact support. Then $\hat{\mu}$ is an entire function on $\boldsymbol{R}^p\times\boldsymbol{T}^q$ if $G=\boldsymbol{R}^p\times\boldsymbol{Z}^q\times K$ as above, and so if λ is a complex number such that $|\hat{\mu}(\gamma)-\lambda|\geqslant c$ for every $\gamma\in\Gamma$ and some $c>0$ the function $\gamma\in\Gamma\mapsto(\hat{\mu}(\gamma)-\lambda)^{-1}$ is bounded on Γ and $C^{(\infty)}$ on $\boldsymbol{R}^p\times\boldsymbol{T}^q$. Then, by Theorem (3.3), there is a unique regular quasimultiplier T of $L^1(G)$ such that $T=(\mu-\lambda)^{-1}$. We have showed that $\lambda\notin\sigma(\mu)$ and the theorem is true in that case.

Now, for any $\mu\in M(G)$ we can take a sequence $(\mu_n)_{n=1}^{\infty}$ of measures in $M(G)$ with compact support such that $\lim_{n\to\infty}\mu_n=\mu$ in norm since G is compactly generated, hence σ-compact. If $\lambda\in\sigma(\mu)$ there is $\chi\in\Sigma$ such that $\mu(\chi)=\lambda$. On the other hand, $(\hat{\mu}_n)_{n=1}^{\infty}$ converges to $\hat{\mu}$ uniformly on Γ and $\lim_{n\to\infty}\hat{\mu}_n(\chi)=\hat{\mu}(\chi)$. For $\varepsilon>0$ given we choose n_0 such that

$$\sup_{\gamma\in\Gamma}|\hat{\mu}_{n_0}(\gamma)-\hat{\mu}(\gamma)|<\varepsilon/3,\qquad |\hat{\mu}_{n_0}(\chi)-\hat{\mu}(\chi)|<\varepsilon/3.$$

Since $\hat{\mu}_{n_0}(\chi)\in\sigma(\mu_{n_0})$ and μ_{n_0} has compact support there exists $\gamma_0\in\Gamma$ such that $|\hat{\mu}_{n_0}(\gamma_0)-\hat{\mu}_{n_0}(\chi)|<\varepsilon/3$. Finally, the inequalities

$$|\hat{\mu}(\chi)-\hat{\mu}(\gamma_0)|\leqslant|\hat{\mu}(\chi)-\hat{\mu}_{n_0}(\chi)|+|\hat{\mu}_{n_0}(\chi)-\hat{\mu}_{n_0}(\gamma_0)|$$
$$+|\hat{\mu}_{n_0}(\gamma_0)-\hat{\mu}(\gamma_0)|<\varepsilon$$

show that $\sigma(\mu)\subset\{\hat{\mu}(\gamma)\colon \gamma\in\Gamma\}^{-}$, and the theorem is proved. ■

Remark. As a consequence of Theorem (3.5) we deduce that the bounded structure induced by $QM_r(L^1(G))$ on $M(G)$ is strictly weaker than the one associated to the norm of the total variation. If not, $M(G)$ would be a complete subalgebra of $QM_r(L^1(G))$; hence any character χ belonging to the Shilov boundary of the carrier space of $M(G)$ would be extendable to a character of $QM_r(L^1(G))$ ([8], p. 38). Because of Theorem (3.5), χ would be approximable in Σ by elements of Γ and Γ would be dense in the Shilov boundary of $M(G)\hat{}$. But this is false ([19], p. 234).

Another approach to the problem "$\Sigma=\beta\Gamma$" consists in considering the regularity of $QM_r(L^1(G))$ in Shilov's sense. The definition of this concept is very natural: a pseudo-Banach algebra B is said to be regular in Shilov's sense or *regular*, for short, if for every closed subset X of the character space of B and every character χ of B such that $\chi\notin X$ there exists $b\in B$ satisfying $b(X)=0$, $b(\chi)\neq0$. If B is a regular pseudo-Banach algebra, if A is a semisimple Banach algebra and if there is a bounded injective homomorphism $\psi\colon B\to A$ with dense range then the character space of A is dense in the character space of B (the proof is routine). This implies that whenever $QM_r(L^1(G))$ is regular then $\beta\Gamma=\Sigma$ since $C_\infty(\Gamma)$ is semisimple, but we do not dispose of any criterion which permits us to assert the regularity of $QM_r(L^1(G))$. However, is $QM_r(L^1(G))$ regular on a representative part of Σ, for instance on $\beta\Gamma$? Yes (G is always compactly generated):

PROPOSITION (3.6). *Let $\varphi_0\in\beta\Gamma$ and let X be a closed subset of $\beta\Gamma$ such that $\varphi_0\notin X$. Then there exists $T\in QM_r(L^1(G))$ such that $\hat{T}(\varphi_0)=1$ and $\hat{T}(X)=0$. In fact, T can be chosen so that $\hat{T}$ is $C^{(\infty)}$ on $\boldsymbol{R}^p\times\boldsymbol{T}^q$. (We suppose $G=\boldsymbol{R}^p\times\boldsymbol{Z}^q\times K$.)*

Proof. Assume X is the complement in $\beta\Gamma$ of an open neighborhood of φ_0 of the type

$$V_{f_1,\ldots,f_n;2\varepsilon}=\{\varphi\in\beta\Gamma\colon |f_j(\varphi)-f_j(\varphi_0)|<2\varepsilon,\ j=1,\ldots,n\}$$

where $f_1,\ldots,f_n\in C_\infty(\Gamma)$ since $\beta\Gamma$ is the spectrum of $C_\infty(\Gamma)$. If $\Gamma=\boldsymbol{R}^p\times\boldsymbol{T}^q\times\varDelta$, as in Theorem (3.3), for every $j=1,\ldots,n$ there exists $g_j\in C_\infty(\Gamma)$ which is also infinitely differentiable on $\boldsymbol{R}^p\times\boldsymbol{T}^q$ such that

$$\sup_{\varphi\in\beta\Gamma}|f_j(\varphi)-g_j(\varphi)|=\sup_{\gamma\in\Gamma}|f_j(\gamma)-g_j(\gamma)|<\varepsilon/2.$$

If we suppose $|g_j(\varphi)-g_j(\varphi_0)|<\varepsilon$ for some $\varphi\in\beta\Gamma$ it follows that

$$|f_j(\varphi)-f_j(\varphi_0)|\leqslant|f_j(\varphi)-g_j(\varphi)|+|g_j(\varphi)-g_j(\varphi_0)|+|g_j(\varphi_0)-f_j(\varphi_0)|<2\varepsilon.$$

Hence we have $V_{g_1,\ldots,g_n;\varepsilon}\subset V_{f_1,\ldots,f_n;2\varepsilon}$. Put $h_j=g_j-g_j(\varphi_0)$, $j=1,\ldots,n$, and $h=\sum_{j=1}^{n}|h_j|^2$; if $g=(1/\varepsilon^2)h$ and $V=\{\varphi\in\beta\Gamma\colon g(\varphi)<1\}$ then $\varphi_0\in V\subset V_{g_1,\ldots,g_n;\varepsilon}$. Choose an infinitely differentiable function ψ on $\boldsymbol{R}$ satisfying $\psi(0)=1$, $\psi(r)=0$ for $|r|\geqslant1$. The function $f=\psi\circ g$ satisfies $f(\varphi_0)=1$, $f(V^c)=0$ and it is bounded on Γ and $C^{(\infty)}$ on $\boldsymbol{R}^p\times\boldsymbol{T}^q$; therefore there exists $T\in QM_r(L^1(G))$ such that $\hat{T}=f$, i.e. $\hat{T}(\varphi_0)=1$, $\hat{T}(V^c)=0$. ■

COROLLARY (3.7). *If G is a nondiscrete metrizable compactly generated abelian group then $QM_r(L^1(G))$ is regular if and only if $\Sigma=\beta\Gamma$.*

To conclude this section we establish a positive result. We denote by $C_\delta^{(r,s)}$ the space of functions F on $\Gamma=\boldsymbol{R}^p\times\boldsymbol{T}^q\times\varDelta$ such that $F_\delta\in C^{(r,s)}$ for every $\delta\in\varDelta$ $(r,s\in\boldsymbol{N})$. We also consider the subalgebra

$$\mathscr{B}=\{T\in QM_r(L^1(G))\colon \hat{T}\in C_\delta^{(2m+1,2q)}\}$$

and its Mackey adherence $\mathscr{A}$ in $QM_r(L^1(G))$ (as usual from Theorem (3.3), $G=\boldsymbol{R}^p\times\boldsymbol{Z}^q\times K$).

PROPOSITION (3.8). *If G is nondiscrete metrizable and compactly generated, then $\mathscr{A}$ is a full pseudo-Banach subalgebra of $QM_r(L^1(G))$ which contains the measure algebra $M(G)$ and whose character space is $\beta\Gamma$. Moreover, $\mathscr{A}$ is regular.*

Proof. For the proof we summarize previous arguments:

1° $M(G) \subset \mathscr{A}$ since every $\mu \in M(G)$ is Mackey approximable by measures with compact support.

2° $\mathscr{B}$ is clearly full: if $T \in \mathscr{B}$ is invertible in $QM_r(L^1(G))$ the extended Fourier transform of its inverse belongs to $C_\delta^{(2m+1,2q)}$ and is bounded on Γ. Now, since $\mathscr{A}$ is the Mackey adherence of $\mathscr{B}$, $\mathscr{A}$ is full ([8], p. 32).

3° For every $T \in \mathscr{A}$, $\sigma(T) = \{\hat{T}(\gamma): \gamma \in \Gamma\}^-$. Indeed, for $T \in \mathscr{B}$ this is proved as for measures with compact support in Theorem (3.5). For any $T \in \mathscr{A}$ the proof is based on an approximation of T by elements of $\mathscr{B}$, again as in Theorem (3.5).

4° From 3°, if $T \in \mathscr{A}$ and $|\hat{T}(\gamma)| \geqslant c$ for all $\gamma \in \Gamma$ and some $c > 0$, there exists $S \in QM_r(L^1(G))$ such that $S * T = I$. Actually, S is in $\mathscr{A}$ since $\mathscr{A}$ is full. Since $\mathscr{A}$ is also clearly dense in $C_\infty(\Gamma)$ the conditions of Lemma (2.1) are satisfied and so $\beta\Gamma$ is the character space of $\mathscr{A}$.

5° $\mathscr{A}$ is regular on $\beta\Gamma$ because it contains $\mathscr{B}$, which is regular on $\beta\Gamma$ according to Proposition (3.6). ■

QUESTION: Does $\mathscr{A}$ equal $QM_r(L^1(G))$?

§ 4. **Quasimultipliers as pseudomeasures.** Now, let G be a locally compact abelian group and Γ its dual group. We denote by (x, γ) the action of $\gamma \in \Gamma$ on $x \in G$. Let $A(G) = \{\hat{g}: g \in L^1(\Gamma)\}$; $A(G)$ is isometric to $L^1(\Gamma)$ under the norm $\|\hat{f}\|_{A(G)} = \int_\Gamma |f(\gamma)|\,d\gamma$, where $d\gamma$ is the normalized Haar measure on Γ. The space of continuous linear functionals on $A(G)$ is denoted by $P(G)$ and its elements are called *pseudomeasures* on G. If $\sigma \in P(G)$ one defines the *Fourier transform* $\hat{\sigma}$ of σ as the unique element in $L^\infty(\Gamma)$ such that $\langle \sigma, \hat{f} \rangle = \langle \hat{\sigma}, f \rangle$ for every $f \in L^1(\Gamma)$. The correspondence $\sigma \mapsto \hat{\sigma}$ is an isometry. If $\sigma_1, \sigma_2 \in P(G)$, then $\sigma_1 * \sigma_2$ is defined as the inverse image of $\hat{\sigma}_1 \cdot \hat{\sigma}_2$ by "^". Thus $P(G)$ is a Banach algebra isometric to $L^\infty(\Gamma)$.

Recall that $M(G) \subset P(G)$ ([14], p. 99). Moreover, for G nondiscrete metrizable, any quasimultiplier T of $L^1(G)$ with $\hat{T}$ bounded on Γ is a pseudomeasure on G (obvious). If G is also compact then $QM_r(L^1(G)) = l^\infty(\Gamma)$ (see § 1) and therefore *Esterle's regular quasimultipliers of $L^1(G)$ are exactly the pseudomeasures on G* in this case.

The action of $T = f/g \in QM_r(L^1(G))$, considered as a pseudomeasure on G, on the elements in $A(G)$ is given by

$$\langle T, \hat{a} \rangle = \sum_{\gamma \in \Gamma} \frac{\hat{f}(\gamma)}{\hat{g}(\gamma)} \alpha(\gamma), \qquad \alpha = (\alpha(\gamma))_{\gamma \in \Gamma} \in l^1(\Gamma).$$

Let p be a trigonometric polynomial on G, i.e. $p(x) = \sum_{\gamma \in \Phi} a_\gamma (x, \gamma)$, $x \in G$, with $a_\gamma \in \boldsymbol{C}$ and Φ a finite subset of Γ. We write $\tilde{p}$ for the polynomial $\tilde{p}(x) = p(-x)$, $x \in G$. Then we have

$$\langle T, p \rangle = \sum_{\gamma \in \Phi} \frac{\hat{f}(\gamma)}{\hat{g}(\gamma)} a_\gamma = \sum_{\gamma \in \Phi} \frac{a_\gamma}{\hat{g}(\gamma)} \int_G f(-y)(-y, \gamma)\,dy$$
$$= \int_G f(-y)\Big(\sum_{\gamma \in \Phi} (a_\gamma/\hat{g}(\gamma))(-y, \gamma)\Big)\,dy = (f * \tilde{q})(0),$$

where $q(x) = \sum_{\gamma \in \Phi} (a_\gamma/\hat{g}(\gamma))(x, \gamma)$, $x \in G$. But

$$(f * \tilde{q})\hat{}\,(\gamma) = \hat{f}(\gamma)\hat{\tilde{q}}(\gamma) = \hat{f}(\gamma)\frac{a_\gamma}{\hat{g}(\gamma)} = \hat{T}(\gamma)\hat{\tilde{p}}(\gamma) = (T * \tilde{p})\hat{}\,(\gamma),$$

for all γ, i.e. $f * \tilde{q} = T * \tilde{p}$ and so $\langle T, p \rangle = (T * \tilde{p})(0)$. Since the trigonometric polynomials are dense in $A(G)$, this last equality characterizes T as a pseudomeasure (note that we use additive notation for the group operation in G and denote its identity by 0).

We return to a noncompact group G. Clearly, $QM_r(L^1(G))$ does not coincide with $P(G)$ since $L_\infty(\Gamma) \neq C_\infty(\Gamma)$. Thus the theory is not so rotund as in the compact case although some interesting facts are available if G is compactly generated. We recall that a pseudomeasure $\sigma \in P(G)$ is zero on an open subset $\Omega \subset G$ if $\langle \sigma, \hat{f} \rangle = 0$ for all $\hat{f} \in A(G)$ with support in Ω. Then the *support* of σ is defined as the complement of the largest open set on which σ vanishes ([6], p. 463). We will say that a quasimultiplier T of $L^1(G)$ such that $\hat{T}$ is bounded has a *compact support* if it has compact support as a pseudomeasure.

PROPOSITION 4.1. *Let G be a nondiscrete metrizable compactly generated abelian group. Then the regular quasimultipliers of $L^1(G)$ with compact support are exactly the pseudomeasures on G with compact support.*

Proof. Let σ be a pseudomeasure on G with compact support $\operatorname{supp}\sigma$. Consider a relatively compact neighborhood U of $\operatorname{supp}\sigma$ in G. Since G is compactly generated, $\bar{U}$ is contained in $Q \times \mathscr{F} \times K$ for some compact subset Q of $\boldsymbol{R}^p$ and a finite subset $\mathscr{F}$ of $\boldsymbol{Z}^q$ (as in § 3 we assume $G = \boldsymbol{R}^p \times \boldsymbol{Z}^q \times K$). Choose a function ψ in $C^{(\infty)}(\boldsymbol{R}^p)$ such that $\hat{\psi}$ has a compact support in $\boldsymbol{R}^p$, $\hat{\psi} \equiv 1$ on Q, and consider the function h in $L^1(\Gamma)$ given by

$$h(\gamma) = \psi(x)\Big(\sum_{n \in \mathscr{F}} \exp(int)\Big)\varepsilon_0(\delta)$$

where $\gamma = (x, t, \delta) \in \boldsymbol{R}^p \times \boldsymbol{T}^q \times \Delta = \Gamma$ and ε_0 is the unit in $L^1(\Delta)$. Since $\hat{h}(\gamma) = 1$ for all $\gamma \in \bar{U}$,

$$\langle \hat{\sigma}, f \rangle = \langle \sigma, \hat{f} \rangle = \langle \sigma, \hat{h}\hat{f} \rangle = \langle \sigma, (h * f)\hat{}\, \rangle = \langle \hat{\sigma}, h * f \rangle = \langle \hat{\sigma} * \tilde{h}, f \rangle$$

where $\tilde{h}(\gamma) = h(-\gamma)$, $\gamma \in \Gamma$, for every $f \in L^1(\Gamma)$; hence $\hat{\sigma} = \hat{\sigma} * \tilde{h}$ and $\hat{\sigma}$ is clearly

$C^{(\infty)}$ in the variables $\mathbf{R}^p \times \mathbf{T}^q$. We can apply Theorem (3.3) to choose $f, g \in L^1(G)$ such that $\hat{\sigma} = \hat{f}/\hat{g}$ and f/g belongs to $QM_r(L^1(G))$. The correspondence $\sigma \mapsto T = f/g$ is injective, and we have proved the proposition. ∎

Now note that for $G = \mathbf{R}^p$ each element σ in $P(\mathbf{R}^p)$ defines a unique distribution $d(\sigma)$ on $\mathbf{R}^p$ by

$$\langle d(\sigma), \psi \rangle = \int_{\mathbf{R}^p} \hat{\sigma}(t) \check{\psi}(t)\, dt$$

for every ψ in $C_{00}^{(\infty)}(\mathbf{R}^p) = \{\varphi \in C^{(\infty)}(\mathbf{R}^p)\colon \operatorname{supp}\varphi \text{ is compact}\}$ (here $\check{\psi}$ is the inverse Fourier transform of ψ). It is easily verifiable that a pseudomeasure on $\mathbf{R}^p$ has a compact support in the sense of the foregoing definition if and only if it has a compact support considered as a distribution. Therefore the regular quasimultipliers of $L^1(\mathbf{R}^p)$ with compact support and the distributions on $\mathbf{R}^p$ with compact support whose Fourier transforms are bounded on $\mathbf{R}^p$ are the same (Proposition (4.1)).

Note that if $T \in QM_r(L^1(\mathbf{R}^p))$ and $\hat{T}$ is $C^{(2m)}$ $(m > p/2)$ its action on $A(\mathbf{R}^p)$ is given by $\langle d(T), \psi \rangle = (T * \tilde{\psi})(0)$ where ψ is any function in $L^1(\mathbf{R}^p)$ such that $\check{\psi} \in C_{00}^{(\infty)}(\mathbf{R}^p)$. To see this, note that if $\hat{T} = \hat{f}/\hat{g}$ is $C^{(2m)}$ we may assume that $\hat{g}$ is $C^{(\infty)}$ (Lemma (3.1)). Then $\check{\psi}/\hat{g}$ belongs to $C_{00}^{(\infty)}(\mathbf{R}^p)$ and there exists a function φ (in $L^1(\mathbf{R}^p)$) satisfying $\hat{\varphi}\hat{g} = \check{\psi}$. It follows that

$$\langle d(T), \psi \rangle = \int_{\mathbf{R}^p} \hat{T}(x) \check{\psi}(x)\, dx$$
$$= \int_{\mathbf{R}^p} \hat{f}(x) (\check{\psi}(x)/\hat{g}(x))\, dx = \int_{\mathbf{R}^p} \hat{f}(x)\hat{\varphi}(x)\, dx = (f * \varphi)(0)$$

(this last equality is given in [11], p. 122). But $\hat{f}\hat{\varphi} = \hat{f}\check{\psi}/\hat{g} = \hat{T}\check{\psi} = (T * \tilde{\psi})\hat{}$ with $\tilde{\psi}(t) = \psi(-t)$, $t \in \mathbf{R}^p$. Therefore $\{\tilde{\psi}\colon \psi \in L^1(\mathbf{R}^p) \text{ and } \check{\psi} \in C_{00}^{(\infty)}(\mathbf{R}^p)\}$ is contained in the domain of T (as a quasimultiplier) and $\langle d(T), \psi \rangle = (T * \tilde{\psi})(0)$. This characterizes T as a pseudomeasure since $\{\psi \in L^1(\mathbf{R}^p)\colon \check{\psi} \in C_{00}^{(\infty)}(\mathbf{R}^p)\}$ is dense in $A(\mathbf{R}^p)$.

It would be desirable to characterize the regular quasimultipliers among the pseudomeasures on $\mathbf{R}^p$. Perhaps that would permit us to clarify the relation between $QM_r(L^1(\mathbf{R}^p))$ and the quasimultipliers T with bounded $\hat{T}$, a question making a part of the problem "$\Sigma = \beta\mathbf{R}^{p}$".

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