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On a subalgebra of the algebra $C([0, 1])$ whose maximal ideal space is a torus

by

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Abstract. A subalgebra of $C([0, 1])$ whose maximal ideal space is a two-dimensional torus is constructed.

1. Introduction. In this paper we give an answer to one of the questions formulated by Gelfand [3] some time ago. Namely, we construct an analytic antisymmetric subalgebra of the algebra $C([0, 1])$ of all the continuous complex-valued functions on the interval $[0, 1]$ whose maximal ideal space is a two-dimensional torus. Our construction essentially follows that used by Hoffman and Singer [5] to give an example of a subalgebra of $C([0, 1])$ whose maximal ideal space is a two-dimensional sphere. We select a nowhere dense arc γ on the torus T and show that Wermer's [2] algebra of functions continuous on T and analytic on $T\setminus\gamma$ is nontrivial and moreover its space of maximal ideals is T.

2. Plane integral of the derivative of the elliptic sine. The automorphism group of the torus T contains a torus. Namely: to every point $\tilde{\xi}\in T$ there corresponds an automorphism $\varphi_{\tilde{\xi}}\colon T\to T$, $\varphi_{\tilde{\xi}}(\tilde{z}) = (z+\xi)^{\sim}$. In the following instead of $\varphi_{\tilde{\xi}}(\tilde{z})$ we will write $z+\xi$. Consider a conformal mapping Φ of the square $[-1, 0]\times[0, 1]$ on the upper half-plane $\{\operatorname{Im} w \geqslant 0\}$ such that $\Phi(0) = 0$, $\Phi(-1/2) = \infty$, $\Phi(-1) = -1$, $\Phi\colon [(-1, 0), (-1/2, 0)] \to [(-1, 0), (-\infty, 0)]$, $\Phi\colon [(-1/2, 0), (0, 0)] \to [(0, 0), (\infty, 0)]$. $\Phi(z)$ can be continued by symmetry to a meromorphic function on the rectangle $[-1, 0]\times[-1, 1]$ and further to a meromorphic function on the plane. The Riemann surface of this function is a torus. On the torus T obtained by the factorization $z_1 \sim z_2 \Leftrightarrow z_1 - z_2 = 2m+2ni$, where m and n are integers, it has two first order poles at the points $(-1/2, 0)$ and $(1/2, 0)$; we will denote it as before by $\Phi(z)$. For the construction of the function $\Phi(z)$ called the elliptic sine see e.g. [6].

Consider the expression

$$\iint_{E_s^1\cup E_s^2} \Phi'(z-\delta)\, dz \wedge \overline{dz}, \tag{1}$$

where $\operatorname{Im}\delta = 0$, $\delta > 0$, E_s^1 and E_s^2 are two squares with sides of length s

parallel to the coordinate axes and with centers at $(-1/2, 0)$ and $(1/2, 0)$ respectively. For $\delta > s/2$ the integral (1) exists since the poles of $\Phi'(z-\delta)$ are in this case outside $E_s^1 \cup E_s^2$. Let $s/2 < \delta < 1/8$. From the construction of $\Phi(z)$ it follows that

$$\iint_{E_s^1} \Phi'(z-\delta)\, dz \wedge \overline{dz} = \iint_{{}_\delta E_s^1} \Phi'(z-\delta)\, dz \wedge \overline{dz},$$

where ${}_\delta E_s^1$ is the square symmetric to E_s^1 with respect to the line $\operatorname{Re} z = \delta$. In the rectangle $[0, 1]\times[-1, 1]$ we have

$$\Phi(z) = f(z) + \frac{A}{z-1/2},$$

where $f(z)$ is regular in the rectangle and A is a nonzero constant. Therefore

$$\Phi'(z) = f'(z) - \frac{A}{(z-1/2)^2}$$

and we have

$$\iint_{{}_\delta E_s^1 \cup E_s^2} \Phi'(z-\delta)\, dz \wedge \overline{dz}$$

$$= \iint_{{}_\delta E_s^1 \cup E_s^2} f'(z-\delta)\, dz \wedge \overline{dz} - A \iint_{{}_\delta E_s^1 \cup E_s^2} \frac{dz \wedge \overline{dz}}{(z-\delta-1/2)^2} = Q(s, \delta).$$

Assume that for every s and δ that satisfy $s/2 < \delta < 1/8$ we have $Q(s, \delta) = 0$. Then $\lim_{s\to 0} Q(s, \delta)/s^2 = 0$. But this limit is equal to

$$\lim_{s\to 0} \frac{Q(s, \delta)}{s^2} = -2i\left(f'(1/2-\delta) + f'(1/2+\delta) - A\left(\frac{1}{(-\delta)^2} + \frac{1}{\delta^2}\right)\right).$$

This means that for any $\delta < 1/8$

(2) $$2A\frac{1}{\delta^2} = f'(1/2-\delta) + f'(1/2+\delta),$$

which gives a contradiction, because the limit as $\delta \to 0$ of the left-hand side of (2) is infinite while the limit of the right-hand side is finite and is equal to $2f'(1/2)$.

It follows therefore that there exist δ_0 and s_0 such that

$$\iint_{E_{s_0}^1 \cup E_{s_0}^2} \Phi'(z-\delta_0)\, dz \wedge \overline{dz} \neq 0.$$

Let

$$\Big|\iint_{E_{s_0}^1 \cup E_{s_0}^2} \Phi'(z-\delta_0)\, dz \wedge \overline{dz}\Big| = a, \qquad \max_{z\in E_{s_0}^1 \cup E_{s_0}^2} |\Phi'(z-\delta_0)| = M.$$

Let $\gamma_1 \subset E_{s_0}^1$ be a simple Jordan curve with $\overline{E_{s_0}^1 \setminus \gamma_1} = E_{s_0}^1$ and $\mu(E_{s_0}^1 \setminus \gamma_1) < \varepsilon/2$, where $\varepsilon = a/(4M)$, and μ is the Lebesgue measure on the plane. Here the bar denotes closure. Such curves are known to exist. Let γ_2 be a translation of γ_1 by the vector $(1, 0)$, so that $\gamma_2 \subset E_{s_0}^2$, $\overline{E_{s_0}^2 \setminus \gamma_2} = E_{s_0}^2$ and $\mu(E_{s_0}^2 \setminus \gamma_2) < \varepsilon/2$. We have

$$\Big|\iint_{\gamma_1 \cup \gamma_2} \Phi'(z-\delta_0)\, dz \wedge \overline{dz}\Big|$$

$$= \Big|\iint_{E_{s_0}^1 \cup E_{s_0}^2} \Phi'(z-\delta_0)\, dz \wedge \overline{dz} - \iint_{(E_{s_0}^1 \cup E_{s_0}^2)\setminus(\gamma_1\cup\gamma_2)} \Phi'(z-\delta_0)\, dz \wedge \overline{dz}\Big|$$

$$\geqslant \Big|a - \Big|\iint_{(E_{s_0}^1 \cup E_{s_0}^2)\setminus(\gamma_1\cup\gamma_2)} \Phi'(z-\delta_0)\, dz \wedge \overline{dz}\Big|\Big|.$$

But

$$\Big|\iint_{(E_{s_0}^1 \cup E_{s_0}^2)\setminus(\gamma_1\cup\gamma_2)} \Phi'(z-\delta_0)\, dz \wedge \overline{dz}\Big| \leqslant 2M\mu[(E_{s_0}^1 \cup E_{s_0}^2)\setminus(\gamma_1\cup\gamma_2)] < 2M\varepsilon = a/2.$$

Therefore

(3) $$\Big|\iint_{\gamma_1\cup\gamma_2} \Phi'(z-\delta_0)\, dz \wedge \overline{dz}\Big| \geqslant a/2.$$

3. Function continuous on T and analytic on $T\setminus\gamma$. Consider the function

$$F(z) = \iint_{\gamma_1\cup\gamma_2} \Phi(\zeta - z)\, d\zeta \wedge \overline{d\zeta}.$$

It is defined and analytic at all the points of the torus except at the points of the three curves $\gamma_{0,0}$, $\gamma_{-1,0}$, $\gamma_{1,0}$ obtained from the curve γ_1 by the translations $(1/2, 0)$, $(-1/2, 0)$ and $(3/2, 0)$ respectively. $F(z) \neq$ constant, since $F(z)$ is analytic at $z = \delta_0$ and $|F'(\delta_0)| \geqslant a/2$ by the inequality (3).

$F(z)$ is uniformly continuous on $T\setminus(\gamma_{0,0}\cup\gamma_{-1,0}\cup\gamma_{1,0})$. Indeed, $\Phi(z)$ has two simple poles at $z = \pm 1/2$, therefore

$$\Phi(z) = \frac{A}{z-1/2} + \frac{B}{z+1/2} + g(z),$$

where A, B are constants and $g(z)$ is regular in the square $[-1, 1]\times[-1, 1]$. Therefore for a point z in the vicinity of the arc $\gamma_{0,0}$ we have

$$\iint_{\gamma_1\cup\gamma_2} \Phi(\zeta - z)\, d\zeta \wedge \overline{d\zeta} = \iint_{\gamma_1\cup\gamma_2} g(\zeta - z)\, d\zeta \wedge \overline{d\zeta}$$

$$+ \iint_{\gamma_1} \frac{A}{\zeta - z - 1/2}\, d\zeta \wedge \overline{d\zeta} + \iint_{\gamma_2} \frac{B}{\zeta - z + 1/2}\, d\zeta \wedge \overline{d\zeta}$$

$$+ \iint_{\gamma_2} \frac{A}{\zeta - z - 1/2}\, d\zeta \wedge \overline{d\zeta} + \iint_{\gamma_1} \frac{B}{\zeta - z + 1/2}\, d\zeta \wedge \overline{d\zeta}.$$

The first three integrals on the right-hand side of this equality are uniformly continuous for z close to $\gamma_{0,0}$ because $g(z)$ is analytic, and $1/(\zeta-z-1/2)$ and $1/(\zeta-z+1/2)$ are analytic on γ_1 and γ_2 respectively for any z in the vicinity of $\gamma_{0,0}$. The fourth and fifth integrals have the same property by the Denjoy lemma [1]. (See also Arens [4].) The same argument applies to the neighborhoods of the arcs $\gamma_{-1,0}$ and $\gamma_{1,0}$. This fact combined with the analyticity of $F(z)$ outside $\gamma_{0,0}\cup\gamma_{-1,0}\cup\gamma_{1,0}$ proves the uniform continuity of $F(z)$ on the set $T\setminus(\gamma_{0,0}\cup\gamma_{-1,0}\cup\gamma_{1,0})$. Thus $F(z)$ can be extended by continuity to the entire torus T. We will denote this extended function by the same notation $F(z)$.

It is easy to see that $F(z_1)=F(z_2)$ whenever $z_1-z_2=1$. We have

$$\begin{aligned} F(z_1) &= \iint_{\gamma_1\cup\gamma_2} \Phi(\zeta-z_1)\,d\zeta\wedge\overline{d\zeta} = \iint_{\gamma_1\cup\gamma_2} \Phi(\zeta-z_2-1)\,d\zeta\wedge\overline{d\zeta} \\ &= \iint_{\gamma_1\cup\gamma_2} \Phi((\zeta-1)-z_2)\,d\zeta\wedge\overline{d\zeta} \\ &= \iint_{\gamma_1} \Phi((\zeta-1)-z_2)\,d\zeta\wedge\overline{d\zeta} + \iint_{\gamma_2} \Phi((\zeta-1)-z_2)\,d\zeta\wedge\overline{d\zeta} \\ &= \iint_{\gamma_2} \Phi(\zeta-z_2)\,d\zeta\wedge\overline{d\zeta} + \iint_{\gamma_1} \Phi(\zeta-z_2)\,d\zeta\wedge\overline{d\zeta} = F(z_2), \end{aligned}$$

where we have used the fact that the real period of $\Phi(z)$ is equal to 2, from which it follows that

$$\Phi((\zeta-1)-z)|\gamma_1 = \Phi(\zeta-z)|\gamma_2, \qquad \Phi((\zeta-1)-z)|\gamma_2 = \Phi(\zeta-z)|\gamma_1$$

since $\{\gamma_1\}=\{\gamma_2\}-1$. It follows that $F(z)$ may be considered as a function on a torus which is obtained by the factorization of the plane given by $z_1\sim z_2 \Leftrightarrow z_1-z_2=m+2ni$, where m and n are integers. This factorization identifies the arcs γ_1 and γ_2.

4. Wermer's family of functions on a torus. We have constructed, therefore, a nonconstant function $F(z)$ continuous on a torus and analytic outside some nowhere dense arc. Since in the construction of the function Φ at the beginning of Section 2 the rectangle $[-2,0]\times[0,1]$ instead of the square $[-1,0]\times[0,1]$ could have been used, we may assume that the torus T is the square $[-1,1]\times[-1,1]$ with the corresponding identification of the sides. Also, we may assume that the arc γ outside which $F(z)$ is analytic is located in some small neighborhood of the center of the square.

Following Wermer [2] consider the family of functions

$$(4)\qquad \begin{cases} F(z) \\ [F(z)-F(z_0+1/2)]\,[F(z)-F(z_0-1/2)]\,\Phi(z-z_0) \\ \text{where } z_0+1/2\notin\gamma,\ z_0-1/2\notin\gamma. \end{cases}$$

Note that in line two of (4) the function $\Phi(z)$ is identical to that constructed in Section 2. All the functions of the family are continuous on T and analytic on $T\setminus\gamma$, since $\Phi(z)$ is meromorphic on T with two simple poles at $z=\pm 1/2$.

Let us show that this family separates points of the torus. Assume on the contrary that there are two points $z_1, z_2\in T$, $z_1\neq z_2$, such that

$$(5)\qquad F(z_1)=F(z_2)$$

and that for any z_0 that satisfies $z_0+1/2\notin\gamma$, $z_0-1/2\notin\gamma$ we have

$$(6)\qquad \begin{aligned} &[F(z_1)-F(z_0+1/2)]\,[F(z_1)-F(z_0-1/2)]\,\Phi(z_1-z_0) \\ &\quad = [F(z_2)-F(z_0+1/2)]\,[F(z_2)-F(z_0-1/2)]\,\Phi(z_2-z_0). \end{aligned}$$

From (5) and (6) it follows that

$$(7)\qquad [F(z_1)-F(z_0+1/2)]\,[F(z_1)-F(z_0-1/2)]\,[\Phi(z_1-z_0)-\Phi(z_2-z_0)]=0.$$

For $|z_0|<\varepsilon$ with ε small enough, $z_0+1/2\notin\gamma$ and $z_0-1/2\notin\gamma$ implying that (7) holds in the entire neighborhood of the origin in which the first two factors of the left-hand side of (7) are analytic functions and the third factor is a meromorphic function with at most four simple poles. Therefore there is an open subset of the set $\{|z_0|<\varepsilon\}$ on which all the factors of (7) are analytic. From the uniqueness theorem it follows that either $F(z_0+1/2)$ or $F(z_0-1/2)$ is constant on the torus, or the equality

$$\Phi(z_1-z_0)=\Phi(z_2-z_0)$$

holds on the torus identically for all z_0. The first possibility is an obvious contradiction. From the second possibility it follows that $\Phi(z)\equiv\Phi(z+(z_2-z_1))$ which can be true only if $z_1=z_2$ on the torus, in contradiction to the assumption that $z_1\neq z_2$.

5. Maximal ideal space of Wermer's algebra on a torus. Let $A_\gamma(T)$ be the algebra of functions continuous on the torus T and analytic on the complement of the arc γ. By the maximum principle its restriction to γ is a subalgebra A_γ of the algebra $C([0,1])$ which is isometrically isomorphic to $A_\gamma(T)$. Let $B_\gamma(T)$ be the subalgebra of $A_\gamma(T)$ generated by the family (4) and let B_γ be its restriction to γ. Then $B_\gamma\subset C([0,1])$ and $B_\gamma(T)\cong B_\gamma$. Applying Wermer's argument [2], one may show that any function from $B_\gamma(T)$ maps the complement of γ and γ itself onto the same set, meaning, in particular, that the image of γ is a Peano curve. Another consequence of this is that the algebra B_γ is analytic. B_γ is antisymmetric since so is $B_\gamma(T)$. It is easy to see that the algebras $A_\gamma(T)$ and A_γ are also analytic and antisymmetric.

The algebra $A_\gamma(T)$ is isomorphic to a subalgebra of the algebra of functions analytic on $\{\operatorname{Im} z>0\}\setminus\tilde\gamma$ and continuous on $\{\operatorname{Im} z\geqslant 0\}$ where $\tilde\gamma$ is a nowhere dense arc which has nonzero two-dimensional Lebesgue measure.

Such isomorphism can be obtained by a conformal mapping of the rectangle onto the upper half-plane. Applying analytic continuation through $\{\operatorname{Im} z = 0\}$ by symmetry we may claim that $A_\gamma(T)$ is isomorphic to a subalgebra of the algebra of functions analytic on $C\setminus(\tilde{\gamma}\cup\tilde{\gamma}_c)$ and continuous on C where $\tilde{\gamma}_c$ is the reflection of $\tilde{\gamma}$ with respect to $\operatorname{Im} z = 0$, and C denotes the complex plane. Applying the method used by Hoffman and Singer [5] to prove Theorem 5, we arrive at the conclusion that the maximal ideal space of the algebra $A_\gamma(T)$, and also of the algebra A_γ, is a torus.

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Regular quasimultipliers of some semisimple Banach algebras

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Abstract. If A is a complex nonunital Banach algebra with dense principal ideals we denote by $QM_r(A)$ the pseudo-Banach algebra formed by Esterle's regular quasimultipliers of A. We study the character space $\hat{\hat{A}}$ of $QM_r(A)$ for several concrete algebras A. In particular, for every nondiscrete metrizable compactly generated abelian group G with dual group Γ we prove that $\beta\Gamma$ is homeomorphically embedded into $L^1(G)^{\hat{}}$ (if G is compact $\beta\Gamma$ equals $L^1(G)^{\hat{}}$). We also note that there is a relationship between $QM_r(L^1(G))$ and the space $P(G)$ of pseudomeasures on G. If G is compact, $QM_r(L^1(G)) = P(G)$.

Introduction. Let A be a complex nonunital commutative Banach algebra possessing dense principal ideals and such that $A^\perp = \{0\}$, where $A^\perp = \{a\in A:\ ab = 0 \text{ for all } b\in A\}$. A *quasimultiplier* T of A is an unbounded operator on A whose domain is a dense principal ideal; so T can be written as a quotient $T = a/b$ where $a, b\in A$ and $[bA]^- = A$. We put $QM(A) = \{T\colon T \text{ is a quasimultiplier of } A\}$. A quasimultiplier $T = a/b$ is said to be *regular* if there exist $\lambda > 0$ and $c\in\bigcap_{n=1}^\infty [b^n A]$ satisfying $\sup_n \|\lambda^n T^n c\| < +\infty$; let $QM_r(A) = \{T\in QM(A)\colon T \text{ regular}\}$. These notions and related ideas were introduced by Esterle in [5] to study the problem of existence of topologically simple radical Banach algebras.

The set $QM_r(A)$ is a pseudo-Banach algebra (see [1], [8]), i.e. it can be represented as an inductive limit of Banach algebras. To obtain this representation one needs the following definition. Two commutative Banach algebras A and B are said to be *similar* if there exist a commutative Banach algebra D with dense principal ideals and two continuous homomorphisms $\varphi\colon D\to A$, $\psi\colon D\to B$ such that $\varphi(D)$, $\psi(D)$ are dense ideals in A, B respectively. Then:

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