STUDIA MATHEMATICA, T. LXXXIX. (1988)

# On the convergence of lacunary polynomials *

by

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**Abstract.** It is proved that if $\{p_n\}$ is a sequence of positive numbers such that $p_{n+1}/p_n \geqslant 1+a$ for each $n = 1, 2, \ldots$, where $a$ is a solution of the inequality $2a > (1+a)^{1+1/a}$, then on the linear span of the functions $t^{p_n}$, $n = 1, 2, \ldots$, considered on the interval $[0, 1)$ the topology of convergence in Lebesgue measure coincides with the topology of uniform convergence on compact subintervals of $[0, 1)$.

Let $\{p_n\}$ be a sequence of positive numbers such that $p_{n+1}/p_n \geqslant 1+a$ for $n = 1, 2, \ldots$, where $a$ satisfies the inequality $2a > (1+a)^{1+1/a}$ $(a > 3.403\ldots)$, let $p_0 = 0$ and let $c_{k,n}$, $k = 1, 2, \ldots$, $n = 1, 2, \ldots$, be numbers such that for each $k$, $c_{k,n} = 0$ for large $n$. Let

$$f_k(t) = \sum_{n=0}^{\infty} c_{k,n} t^{p_n}.$$

The aim of this paper is to prove the following

THEOREM. *If the sequence of functions $\{f_k\}$ converges almost everywhere on the interval $[0, 1)$ then*:

(i) *For each $n$ the sequence $c_{k,n}$, $k = 1, 2, \ldots$, converges to a limit $c_n$.*

(ii) *The series $\sum_{n=0}^{\infty} c_n t^{p_n}$ converges to a function $f$ on the interval $[0, 1)$.*

(iii) *For each $0 < r < 1$ the sequence $\{f_k\}$ converges uniformly to $f$ on the interval $[0, r]$.*

Proof. The validity of the above theorem is a simple consequence of the following two lemmas.

LEMMA 1. *Let $0 < s < 1$. Assume that*:

(a) $\limsup_{k\to\infty} \sum_{n=0}^{\infty} |c_{k,n}| s^{p_n} < \infty$.

(b) *For each $n = 1, 2, \ldots$ the limit $\lim_k c_{k,n}$ exists and equals $c_n$.*

*Then for each $r < s$ the sequence of functions $\{f_k\}$ converges uniformly on the interval $[0, r]$.*

* This paper is based upon posthumous notes of S. Mazur. It contains a result presented by him at the Conference on Functional Analysis which took place in Warsaw in September of 1960. The result gave a solution to a problem posed by S. Mazur in 1949 and published in Colloquium Mathematicum 2 (1951), p. 152 (cf. the end of the paper).

** Prepared by W. Smoleński from the notes taken by W. Kołodziej.

Proof. For $0<t<r$ and $m=1,2,\ldots$ we have

$$|f_k(t)-f(t)| \leqslant \sum_{n=0}^{m} |c_{k,n}-c_n|\, r^{p_n} + \sum_{n=m+1}^{\infty} (|c_{k,n}|+|c_n|)\, r^{p_n}.$$

Let $M$ be such that $\sum_{n=0}^{\infty} |c_{k,n}|\, s^{p_n} \leqslant M$. Then $|c_{k,n}|\, s^{p_n} \leqslant M$ and $|c_n|\, s^{p_n} \leqslant M$. Consequently,

$$\sum_{n=m+1}^{\infty} (|c_{k,n}|+|c_n|)\, r^{p_n} \leqslant 2M \sum_{n=m+1}^{\infty} (r/s)^{p_n}.$$

Given $\varepsilon>0$ there exists $m_0$ such that

$$\sum_{n=m_0+1}^{\infty} (r/s)^{p_n} \leqslant \varepsilon/(4M)$$

and there exists $k_0$ such for each $k>k_0$

$$\sum_{n=0}^{m_0} |c_{k,n}-c_n|\, r^{p_n} < \varepsilon/2.$$

Hence $|f_k(t)-f(t)|<\varepsilon$ for $0\leqslant t\leqslant r$ and $k>k_0$. ∎

Lemma 2. *Let $0<s<1$. If*

$$\limsup_{k\to\infty} \sum_{n=0}^{\infty} |c_{k,n}|\, s^{p_n} = \infty$$

*then the Lebesgue measure of the set*

$$Z = \{t\in(s,1)\colon \limsup_{k\to\infty} |f_k(t)| < \infty\}$$

*is equal to* 0.

Proof. Without loss of generality we may assume that

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} |c_{k,n}|\, s^{p_n} = \infty.$$

Assume, on the contrary, that $|Z|>0$, where $|A|$ denotes the Lebesgue measure of a set $A$. Then there exists $M$ such that

$$|\{t\in(s,1)\colon |f_k(t)|<M \text{ for each } k=1,2,\ldots\}| > 0.$$

Denote the above set by $W$ and choose $r\in W$ such that $W$ has density 1 at $r$. If we put

$$S_{k,n} = \sum_{i=0}^{n} c_{k,i}\, r^{p_i}, \qquad S_k^* = \sup_n |S_{k,n}|$$

then $\lim_{k\to\infty} S_k^* = \infty$. Indeed, otherwise $S_k^*<N$ for infinitely many $k$. Consequently, for those $k$, $|c_{k,n}|\, r^{p_n} < 2N$ for all $n$ and hence



$$\sum_{n=0}^{\infty} |c_{k,n}|\, s^{p_n} \leqslant 2N \sum_{n=0}^{\infty} (s/r)^{p_n} < \infty$$

which is impossible since we have assumed that $\lim_{k\to\infty} \sum_{n=0}^{\infty} |c_{k,n}|\, s^{p_n} = \infty$.

Now, we will show that

$$\lim_{k\to\infty} S_{k,n}/S_k^* = 0 \quad \text{for each } n.$$

Indeed, in view of the estimate $|c_{k,n} r^{p_n}/S_k^*| \leqslant 2$, if the above convergence did not hold there would exist an increasing sequence of positive integers $\{k_i\}$ such that

$$\lim_{i\to\infty} c_{k_i,n}\, r^{p_n}/S_{k_i}^* = b_n \quad \text{for each } n=1,2,\ldots$$

and $b_n\neq 0$ for some $n$. By Lemma 1 it follows that

$$\lim_{i\to\infty} f_{k_i}(t)/S_{k_i}^* = \lim_{i\to\infty} \sum_{n=0}^{\infty} (c_{k_i,n}\, r^{p_n}/S_{k_i}^*) \left(\frac{t}{r}\right)^{p_n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{b_n}{r^{p_n}}\right) t^{p_n} = g(t) \quad \text{for } 0\leqslant t<r.$$

On the other hand, $\lim_{k\to\infty} f_k(t)/S_k^* = 0$ for $t\in Z$. Therefore the analytic function $g$ is equal to 0 on $[0,r)$. This yields $b_n=0$ for all $n$, which proves the above claim.

Let $b=(1+a)^{-1/a}$. Then $b-b^{1+a} = a(1+a)^{-1-1/a} > 1/2$. Hence there exists $\alpha>1$ such that $b^{\alpha}-b^{1+a}>1/2$. Also we have

$$\lim_{n\to\infty} (1-b^{1/p_n})/(1-b^{\alpha/p_n}) = 1/\alpha < 1.$$

Since $r$ is a point of density 1 for $W$ we get

$$\lim_{n\to\infty} |W\cap J_n|/|I_n| \geqslant \lim_{n\to\infty} |W\cap I_n|/|I_n| - \lim_{n\to\infty} |I_n\setminus J_n|/|I_n|$$
$$= 1-1/\alpha > 0,$$

where $I_n=[rb^{\alpha/p_n}, r]$ and $J_n=[rb^{\alpha/p_n}, rb^{1/p_n}]$. So there exists $n_0$ such that for each $n>n_0$ we can find $t_n\in J_n$ such that $|f_k(t_n)|\leqslant M$ for all $k$.

We have

$$S_k^* \pm f_k(t) = \sum_{n=0}^{\infty} (S_k^* \pm S_{k,n}) \left[\left(\frac{t}{r}\right)^{p_n} - \left(\frac{t}{r}\right)^{p_{n+1}}\right]$$

for $0<t<r$, $k$, $n=1,2,\ldots$ Since each term in the last series is nonnegative

we get

$$S_k^* \pm f_k(t) \geqslant (S_k^* \pm S_{k,n})\left[\left(\frac{t}{r}\right)^{p_n} - \left(\frac{t}{r}\right)^{p_{n+1}}\right] \quad \text{for } 0 < t < r.$$

Hence we obtain easily

$$|f_k(t)| \geqslant |S_{k,n}|\left[\left(\frac{t}{r}\right)^{p_n} - \left(\frac{t}{r}\right)^{p_{n+1}}\right] - S_k^*\left(1 - \left[\left(\frac{t}{r}\right)^{p_n} - \left(\frac{t}{r}\right)^{p_{n+1}}\right]\right)$$

for $0 < t < r$ and all $k$, $n = 1, 2, \ldots$

Let $c$ be a number such that $0 < c < 1$ and $(1+c)(b^\alpha - b^{1+\alpha}) > 1$. For each $k$, choose $n_k$ such that $|S_{k,n_k}| > cS_k^*$. Since $\lim_{k\to\infty} S_{k,n}/S_k^* = 0$ we obtain $\lim_{k\to\infty} n_k = \infty$. If $n_k > n_0$ then

$$\begin{aligned} M \geqslant |f_k(t_{n_k})| &\geqslant |S_{k,n_k}|\left[\left(\frac{t_{n_k}}{r}\right)^{p_{n_k}} - \left(\frac{t_{n_k}}{r}\right)^{p_{n_k+1}}\right] \\ &- S_k^*\left(1 - \left[\left(\frac{t_{n_k}}{r}\right)^{p_{n_k}} - \left(\frac{t_{n_k}}{r}\right)^{p_{n_k+1}}\right]\right) \geqslant S_k^*\left\{(1+c)\left[\left(\frac{t_{n_k}}{r}\right)^{p_{n_k}} - \left(\frac{t_{n_k}}{r}\right)^{p_{n_k+1}}\right] - 1\right\} \\ &\geqslant S_k^*\{(1+c)[b^\alpha - b^{p_{n_k+1}/p_{n_k}}] - 1\} \geqslant S_k^*[(1+c)(b^\alpha - b^{1+\alpha}) - 1], \end{aligned}$$

because $rb^{\alpha/p_n} \leqslant t_n \leqslant rb^{1/p_n}$ for each $n$.

It follows that the sequence $\{S_k^*\}$ is bounded, contrary to what was assumed at the beginning of the proof. Thus $|Z| = 0$. ■

Remark 1. The proof of the theorem gives a stronger result. Namely, the assertion of the theorem is valid under the assumption that the set $A$ of points of pointwise convergence of the sequence $\{f_k\}$ has the property: $|A \cap [1-\delta, 1]| > 0$ for each $\delta > 0$.

Remark 2. Let $\{p_n\}$ be a sequence of positive numbers as in the theorem. The space $F$ of continuous functions on the interval $[0, 1]$ which are sums of power series of the form $\sum_{n=0}^{\infty} c_n t^{p_n}$ on the interval $[0, 1)$ has the following properties:

(i) If $f$ is a continuous function and $f$ is the pointwise limit of a sequence $\{f_k\}$ on $[0, 1]$ with $f_k \in F$ for $k = 1, 2, \ldots$ then $f \in F$.

(ii) $F$ is a linear space different from $C[0, 1]$.

(iii) If $t_1, t_2, \ldots, t_k \in [0, 1]$ and $s_1, s_2, \ldots, s_k$ are real numbers then there exists a function $f \in F$ such that $f(t_i) = s_i$ for $i = 1, 2, \ldots, k$.

The above example solves Problem P80, Colloquium Mathematicum 2 (1951), p. 152.

*Received January 26, 1987* (2272)

STUDIA MATHEMATICA, T. LXXXIX. (1988)

# Factorizations of natural embeddings of $l_p^n$ into $L_r$, I

by

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**Abstract.** In this paper we study quantitative aspects of the local $\mathscr{L}_p$-structure and the uniform approximation property of $L_p$, $1 \leqslant p \leqslant \infty$. Let $K > 1$. Given a subspace $X$ of $L_p$ with $\dim X = n < \infty$, the parameters $m_p(X, K)$ and $k_p(X, K)$ denote, respectively, the smallest dimension $m$ of a superspace $Y$, $X \subset Y \subset L_p$, such that $d(Y, l_p^m) \leqslant K$ and the smallest rank of an operator $u$ on $L_p$ such that $\|u\| \leqslant K$ and $ux = x$ for $x \in X$.

We consider mainly the case $p = 1$. For some natural Euclidean subspaces $X \subset L_1$ we show that $m_1(X, K)$ and $k_1(X, K)$ are at least exponential in $n$, which in general cannot be improved. In fact, our lower estimates lead to new $L_1$-characterizations of Sidon sets (cf. Section 2). Analogous estimates are obtained in Section 3 in the case where $X \subset L_1$ is spanned by $n$ i.i.d. $r$-stable random variables, $1 < r < 2$.

The case $p = \infty$ is treated in Section 4. We prove that $k_\infty(X, K) \leqslant m_\infty(X, K) \leqslant \exp(A(K)n)$ and, if $n > 1$, we show cases where $k_\infty(X, K) \geqslant \exp(\delta K^{-2} n)$, for some $\delta > 0$ and each $K > 1$.

Our method depends on analysis of factorizations of the embedding map $X \hookrightarrow L_p$. In Section 5 we show that a similar scheme can be applied also in the case of quotient maps onto some subspaces of $L_1$.

**0. Introduction.** In this paper and its sequel [FJS] we investigate quantitative aspects of the local $\mathscr{L}_p$-structure and the uniform approximation property of $L_p$. The two most basic questions can be phrased as follows:

($\mathscr{L}$) Given a subspace $X$ of $L_p$ (or $C(S)$, when $p = \infty$), $\dim X = n$, and a constant $K > 1$, estimate the smallest $m = m_p(X, K)$ such that there is a subspace $Y$ of $L_p$ with $X \subset Y$ and $d(Y, l_p^m) \leqslant K$. In particular, estimate $m_p(n, K) = \sup\{m_p(X, K): \dim X = n\}$. (Here $d(Y, Z)$ is the Banach–Mazur distance coefficient

$$\inf\{\|T\|\,\|T^{-1}\|;\ T\colon Y \to Z \text{ is an onto isomorphism}\}.)$$

($\mathscr{U}$) Given a subspace $X$ of $L_p$ (or $C(S)$, when $p = \infty$), $\dim X = n$, and a constant $K > 1$, estimate the smallest $k = k_p(X, K)$ such that there is an

[1] Supported in part by NSF DMS-8316627.
[2] Supported in part by NSF DMS-8200584 and the U.S.-Israel BSF.
[3] Supported in part by NSF DMS-8500764.