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Complemented kernels of partial differential operators in weighted spaces of (generalized) functions

by

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Abstract. It is shown that in a large class of weighted function spaces, every partial differential equation with constant coefficients may be solved by means of a linear and continuous operator.

Introduction. The existence of a (continuous linear) solution operator for continuous linear equations has been studied by several authors ([8]–[13], [15], [17], [18]). D. Vogt ([18]) showed that hypoelliptic partial differential operators (with constant coefficients) have no right inverses in $C^\infty(\mathbf{R}^N)$, thus improving a classical result of Grothendieck for elliptic equations. Ideals in weighted spaces of entire functions were studied in [10]–[13]. In [8] it was proved that a large class of hypoelliptic operators have no right inverses in the classical weighted spaces $(W_{M,\infty})'_b$ and $\mathscr{E}_0(M)$ (introduced by Gelfand–Shilov and Palamodov, resp.).

In this paper, a complementary result will be proved: We will show that a suitable choice of the system of weight functions will lead to so-called general splitting spaces, i.e. every system of partial differential equations with constant coefficients will have a (continuous linear) right inverse in these spaces. We will consider spaces of (ultra)distributions and (ultra)differentiable functions determined by the weight systems $\{\exp(W(x)+nV(x)) \mid n\in \mathbf{N}\}$ (and $\{\exp(W(x)-nV(x)) \mid n\in \mathbf{N}\}$, resp.), where $W(x)=\sum_{i\leqslant N} W_i(|x_i|)$ and $V(x) = \sum_{i\leqslant N} V_i(|x_i|)$ and W_i, $V_i\in C^1(\mathbf{R})$.

Let $w_i := (W_i)'$, $v_i := (V_i)'$ and $\delta = 1$ or $\delta = -1$, according to the choice of the sign of nV in the above weight system. We mainly need the following conditions on the weight functions:

1. $w_i+\delta n v_i$ is strictly increasing and unbounded for any $n\in \mathbf{N}$.
2. $2w_i\circ V_i^{-1}(t) \leqslant w_i\circ V_i^{-1}(Ct)$ for large t.
3. $v_i = o(w_i)$ and $t = O(V_i(t))$.
4. $w_i(t) \leqslant \exp(CV_i(t))$ for large t.

The last condition is used for weighted spaces of distributions and C^∞-functions. A stronger assumption is needed when ultradistributions and ultradifferentiable functions are treated (see (1.5) and Remark 1.3).

Simple examples satisfying these conditions are $W_i(t) = t^{\alpha}$ and $V_i(t) = t^{\alpha_1}$ for $\alpha > \alpha_1$, or $W_i(t) = \exp(\gamma t^{\alpha})$ and $V_i(t) = \exp(\gamma_1 t^{\alpha_1})$ for $\alpha_1 < \alpha$, or $\alpha_1 = \alpha$ and $\gamma_1 < \gamma$.

The conditions are in a sense stable for taking compositions (see the remark after 3.3). Notice that the usual condition that the weighted space should be stable for shifts is not needed. This would imply that the weights are bounded by $\exp(Ct)$, while no a priori bound is implied by the conditions of this paper. In fact, the whole space of distributions of finite order is filled with weighted spaces satisfying our conditions.

The paper is divided into three parts: We first show that the weighted spaces of (test) functions are isomorphic to power series spaces of infinite type. This extends the corresponding results in [16] and proves the linear topological invariants (DN) and (Ω) (see [15]) for these spaces. The second section contains a Paley–Wiener Theorem and an existence theorem for suitable (pluri)subharmonic functions. In the third section, the Fundamental Principle of Ehrenpreis and the general splitting theorem of D. Vogt (Theorem 7.1 in [15]) are used to prove the final result.

Partial differential operators in weighted Gevrey spaces of ultradifferentiable functions or ultradistributions of Roumieu type are considered in a forthcoming paper ([9]). These spaces correspond to power series spaces of finite type and one has to use the notion of graded spaces, tame linear maps and tame splitting theorems, as there is no general splitting theorem for continuous exact sequences of power series spaces of finite type ([19]).

1. Sequence space representations. The general splitting theorem of D. Vogt (Theorem 7.1 in [15]) is based on certain linear topological invariants ((DN) and (Ω), see [15]), which characterize the (nuclear) subspaces and quotients of (s). We will prove these invariants in this section for the spaces of (test) functions of this paper. More precisely, we will show that the spaces are isomorphic to power series spaces of infinite type. We will need essentially weaker conditions than those normally used in the literature (see [16], § 3).

We will only consider functions defined on the real line, leaving the case of several variables to the reader (see Section 3).

1.1. Definition. Let $W, V \in C^1(\boldsymbol{R})$ and let V be positive.

(a) $C^{\infty}(W, V) := \{f \in C^{\infty}(\boldsymbol{R}) \mid p_k(f) := \sup_{j\le k} \|f^{(j)} e^{W+kV}\|_{\infty} < \infty$ for any $k \in \boldsymbol{N}\}$.

(b) Let (M_j) be a sequence of positive numbers satisfying $M_0 = 1$ and the conditions (M.1), (M.2)′ and (M.3)′ of Komatsu ([6]). Then

$$C^{\infty}_{(M_j)}(W, V) := \{f \in C^{\infty}(\boldsymbol{R}) \mid p'_k(f) := \sup_j \|f^{(j)} e^{W+kV}\|_{\infty} k^j/M_j < \infty \quad \text{for any } k \in \boldsymbol{N}\}.$$

$C^{\infty}_{\cdot}(W, V)$ will sometimes be used as a common notation for $C^{\infty}(W, V)$ and $C^{\infty}_{(M_j)}(W, V)$ (similarly, $p^{\cdot}_k$ for p_k and p'_k).

$C^{\infty}(W, V)$ is a $K\{M_p\}$-space (see [2]). In Section 3, partial differential operators in $C^{\infty}_{\cdot}(W, V)'_b$ and in $C^{\infty}_{\cdot}(-W, V)$ will be studied for positive and even functions W and V.

We may suppose that

$$V(t) \geq 1, \quad |v|+|w| \notin L^1(\boldsymbol{R}), \tag{1.1}$$

where $v := V'$ and $w := W'$. Indeed, we may consider $V(t)+1$ (or $V(t)+2+\cos t$) instead of V (if $|v| \in L^1(\boldsymbol{R})$), and $C^{\infty}_{\cdot}(W, V)$ is not changed.

With $U(t) := \int_0^t (|w(\tau)|+|v(\tau)|)\,d\tau$ and $y \neq 0$ let $\tilde{y}$ solve the equation

$$U(\tilde{y}) = U(y) + \operatorname{sgn}(y)\tfrac{1}{2} V(y). \tag{1.2}$$

Then $\operatorname{sgn}(\tilde{y}) = \operatorname{sgn}(y)$ and $\varepsilon(y) := |\tilde{y}| - |y|$ is strictly positive.

Let $0 \leq \varphi \in D(\boldsymbol{R})$ be fixed. Let φ_y be the convolution of a characteristic function with $\varphi(\cdot/\varepsilon(t))/\varepsilon(t)$, where $t \in \operatorname{sgn}(y)[|y|, |\tilde{y}|] =: I_y$ or $t \in I_{\zeta}$ for some $\zeta \in I_y$, or let φ_y be the sum of two such functions. Moreover, we suppose that

$$\operatorname{supp} \varphi_y \subset I_{\zeta}. \tag{1.3}$$

Cut-off functions of the φ_y-type and Lemma 1.2 below will frequently be used in this paper (see 1.5, 2.2 and 2.3).

The *Young conjugate* F^* of a convex function F is defined by

$$F^*(y) := \sup_x (xy - F(x)).$$

Let $M(t)$ be the function associated with (M_j) in the sense of Komatsu ([6]), i.e.

$$M(t) = \ln\big(\sup_j (|t|^j/M_j)\big) \quad \text{for } t \in \boldsymbol{C}.$$

Let $\mathscr{F}$ or $\hat{}$ be the Fourier transform: $\hat{f}(z) := \int f(t) e^{-izt}\,dt$.

1.2. Lemma. *Let W and V satisfy* (1.1). *Let φ_y and ζ be defined as above.*

(a) *Let*

$$(|w|+|v|)(t) \leq \exp(D_1 V(t)) \quad \textit{for any } t. \tag{1.4}$$

Then for any $k \in \boldsymbol{N}$ there are $C_i > 0$ such that for any $f \in C^{\infty}(\boldsymbol{R})$ and any $y \neq 0$

$$\sup_{z\in \boldsymbol{C}} |(f\varphi_y)^{\wedge}(z)|(1+|z|)^k \leq C_1 \sup_{\substack{x\in I_{\zeta}\\ j\le k}} |f^{(j)}(x)|\, e^{C_2 V(\zeta)} \int_{I_{\zeta}} e^{x \operatorname{Im} z}\,dx.$$

(b) *Let g_j and h_j be increasing sequences such that $(1/g_j)_{j\in \boldsymbol{N}} \in l^1$ and $m_j := M_j/M_{j-1} \geq h_j g_j$. Let $G_j := \prod_{i\le j} g_i$ and $H_j := \prod_{i\le j} h_i$ and let H be the* *function associated to* (H_j). *Suppose that*

$$(|w|+|v|)(t) \leqslant D_2 H^{-1}(D_2 V(t)) V(t) \quad \textit{for any } t. \tag{1.5}$$

Then there is $(\tilde{G}_j)_{j\in N}$ *satisfying* (M.1) *and* (M.3)′ *such that for any* $k \in N$ *there are* $C_i > 0$ *and* $k_1 \in N$ *such that for any* $f \in C^\infty_{(M_j)}(\boldsymbol{R})$ *and any* $y \neq 0$

$$\sup_{z\in C} |(f\varphi_y)\hat{}(z)|\, e^{M(kz)} \leqslant C_1 \big(\sup_{\substack{x\in I_\zeta\\ j}} |f^{(j)}(x)|\, k_1^j/M_j\big) e^{C_2 V(\zeta)} \int_{I_\zeta} e^{x \operatorname{Im} z}\, dx,$$

if φ_y *is defined by* $\varphi \in D_{(\tilde{G}_j)}(\boldsymbol{R})$.

Proof. I. We may choose increasing unbounded sequences $\tilde{h}_j$ and $\tilde{g}_j$ such that $(1/\tilde{g}_j) \in l^1$ and $m_j \geqslant \tilde{h}_j \tilde{g}_j$ and $h_j = o(\tilde{h}_j)$. So $(\tilde{G}_j)$ satisfies (M.1) and (M.3)′. With $h(t) := \max\{j \mid h_j \leqslant t\}$ we get (see (3.11) in [6])

$$H(\varepsilon t) = \int_0^{\varepsilon t} \frac{h(\lambda)}{\lambda}\, d\lambda = \int_0^t \frac{h(\varepsilon\lambda)}{\lambda}\, d\lambda \geqslant \int_0^t \frac{\tilde{h}(\lambda)}{\lambda}\, d\lambda + C_1$$
$$= \tilde{H}(t) + C_1 \geqslant \tfrac{1}{2}\tilde{H}(t) \quad \text{for large } t.$$

With $\varepsilon < 1/(2AD_2)$ we get for large $V(t)$

$$D_1 H^{-1}(D_2 V(t)) V(t) \leqslant \frac{1}{2A} \tilde{H}^{-1}(2D_2 V(t)) V(t), \tag{1.6}$$

and (1.5) implies that

$$(|w|+|v|)(t) \leqslant \frac{1}{4A} \tilde{H}^{-1}(D_3 V(t)) V(t) \quad \text{for any } t \text{ and } D_3 = D_3(A). \tag{1.5*}$$

II. For $x \neq 0$ and $\eta \in I_x$ we get

$$\begin{gathered} |V(\eta) - V(x)| \leqslant \operatorname{sgn}(x) \int_x^{\tilde{x}} |v(\tau)|\, d\tau \leqslant \operatorname{sgn}(x) \big(U(\tilde{x}) - U(x)\big) = \tfrac{1}{2} V(x), \\ \tfrac{1}{2} V(x) \leqslant V(\eta) \leqslant \tfrac{3}{2} V(x). \end{gathered} \tag{1.7}$$

By the mean value theorem, this implies

$$\begin{aligned} \frac{1}{\varepsilon(t)} &= \frac{V(t)\operatorname{sgn}(t)}{\tilde{t}-t\ \ V(t)} = \frac{U(\tilde{t}) - U(t)}{\tilde{t}-t} \cdot \frac{2}{V(t)} \\ &= \frac{2(|w|+|v|)(\tau_t)}{V(t)} \leqslant 4(|w|+|v|)(\tau_t)/V(\tau_t) \end{aligned} \tag{1.8}$$

for some $\tau_t \in I_t$.

III. Let $t \in I_y \cup I_\zeta$ for some $\zeta \in I_y$. Then

$$\begin{aligned} \sup_j A^j/(\varepsilon(t)^j \tilde{H}_j) &\leqslant \exp\big(\tilde{H}(A/\varepsilon(t))\big) \leqslant \exp\big(\tilde{H}(4A(|w|+|v|)(\tau_t)/V(\tau_t))\big) \\ &\leqslant \exp\big(D_3 V(\tau_t)\big) \leqslant \exp\big(3D_3(V(\zeta)+V(y))\big) \leqslant \exp\big(9D_3 V(\zeta)\big) \end{aligned} \tag{1.9}$$

where we have used (1.8), (1.5*) and (1.7). (1.8) and (1.9) imply

$$\begin{aligned} \sup_j \|\varphi_y^{(j)}\|_\infty (2k)^j/M_j &\leqslant \sup_j \frac{\|\varphi^{(j)}\|_\infty (4k)^j}{\tilde{G}_j} \big(\varepsilon(t_1)^{-j} + \varepsilon(t_2)^{-j}\big)/\tilde{H}_j \\ &\leqslant C_1 \exp\big(C_2 V(\zeta)\big). \end{aligned} \tag{1.10}$$

1.2(b) is now an easy consequence of (1.3) and (1.10).

IV. 1.2(a) may be proved as above, taking the estimates with finite j. Indeed, (1.8), (1.1) and (1.7) give the following estimate:

$$\varepsilon(t)^{-j} \leqslant C^j (|w|+|v|)(\tau_t)^j \leqslant \exp\big(C_4 V(\tau_t)\big) \leqslant \exp\big(C_5 V(\zeta)\big). \tag{1.11}$$

(1.4) (and (1.5)) do not restrict the global growth of the weight functions, as does the assumption

$$\sup_{|\zeta|\leqslant 1} \big(W(\zeta+t) + nV(\zeta+t)\big) \leqslant m + W(t) + mV(t) \quad \text{for some } m(n). \tag{*}$$

Indeed, for positive functions W and V this implies that

$$W(t) + V(t) \leqslant C_1 \exp(C_1 t) \quad \text{for some } C_1 > 0.$$

(*) is usually used in the literature (see e.g. [16]). It means that $C_*^\infty(W, V)$ is stable for shifts.

(1.5) may often be given in an explicit form:

1.3. Remark. (a) *Let* (M_j) *satisfy* (M.2) (see [6]) *and let* $m_j/(j(\ln j)^\alpha)$ *be increasing for some* $\alpha > 1$. *Then* (1.5) *follows from*

$$(|w|+|v|)(t) \leqslant C_1\, m_{[C_1 V(t)]}/(\ln V(t))^\beta \quad \textit{for some } \beta > 1 \textit{ and large } t.$$

(b) *For* $M_j = \exp(Aj^{\alpha+1}/(\alpha+1))$, $0 < \alpha \leqslant 1$, (1.5) *follows from*

$$(|w|+|v|)(t) \leqslant C_1 \exp\big(C_1 V(t)^{\alpha/(\alpha+1)}\big) \quad \textit{for some } C_1 > 0. \tag{1.12}$$

Proof. (a) Let $h_j := m_j/(j(\ln j)^{\tilde{\beta}})$ with $1 < \tilde{\beta} < \beta\ (<\alpha)$. Then h_j is strictly increasing, H_j satisfies (M.1), (M.2) and

$$\lim_{j\to\infty} H_j^{1/j} \geqslant \lim_{j\to\infty} h_{[j/2]}^{1/2} = \infty.$$

From Lemma 1.4(c) in [14] we get for large t

$$(|w|+|v|)(t) \leqslant h_{[C_2 V(t)]}[C_2 V(t)] \leqslant C_3 H^{-1}(C_3 V(t)) V(t).$$

As $(1/g_j) := \big(1/(j(\ln j)^{\beta})\big) \in l^1$, the proof is complete.

(b) Let $h_j := \exp\big(\tilde{A}(j-1)^{\alpha}\big)$ with $\tilde{A} < A$ and $g_j := \exp\big((A-\tilde{A})(j-1)^{\alpha}\big)$. Then $m_j \geqslant h_j g_j$ and $(1/g_j) \in l^1$. By (3.11) in [6] we get

$$H(t)-H(1) = \int_1^t \frac{h(\lambda)}{\lambda}\,d\lambda = \int_0^{\ln t} h(e^{\lambda})\,d\lambda \leqslant \int_0^{\ln t} \big((\lambda/\tilde{A})^{1/\alpha}+1\big)\,d\lambda \leqslant C_2 (\ln t)^{(\alpha+1)/\alpha}.$$

(1.12) now implies (1.5).

The Gevrey sequence $M_j = (j!)^{\alpha}$, $\alpha > 1$, is a special case of 1.3(a). (1.5) holds in this case if

$$(|w|+|v|)(t) \leqslant C_1\, V(t)^{\alpha}/(\ln V(t))^{\beta} \quad \text{for some } \beta > 1.$$

The sequence $M_j = \exp(Aj^2)$ is maximal in (M_j) satisfies (M.2)′. So the minimal restriction given by (1.5) is (by 1.3(b))

$$(|w|+|v|)(t) \leqslant \exp\big(C_1\, V(t)^{1/2}\big) \quad \text{for large } t,$$

while only (1.4) is needed for C^{∞}-functions.

We now inductively define a partition of the real line by using (1.2), assuming that (1.1) holds. Let $x_0 := 0$, $x_{\pm 1} := \pm 1$ and let

$$x_{r+\operatorname{sgn}(r)} := \tfrac{1}{2}(x_r+\tilde{x}_r) = x_r + \tfrac{1}{2}\varepsilon(r)\operatorname{sgn}(r),$$

where $\tilde{x}_r$ solves the equation

$$U(\tilde{x}_r) = U(x_r) + \tfrac{1}{2}\operatorname{sgn}(x_r)\, V(x_r).$$

A solution $\tilde{x}_r$ may be chosen by (1.1). (x_r) is strictly increasing (by (1.1)) and

(1.13) $\quad (x_r)$ is unbounded from above and below.

Otherwise, e.g. $x := \lim_{r\to\infty} x_r$ would exist and $\lim \tilde{x}_r = \lim(2x_{r+1}-x_r) = x$. Hence $U(x) = \lim U(\tilde{x}_r) = U(x) + \frac{1}{2}V(x)$ and $V(x) = 0$, contradicting (1.1).

The following definition depends on W and V through the definition of x_r. Let $\varepsilon(0) := x_2 - x_{-2}$.

1.4. Definition.

(a) $\quad \Lambda := \{(c_{sr}) \in \boldsymbol{C}^{\boldsymbol{Z}\times\boldsymbol{Z}} \mid$

$$q_k(c_{sr}) := \sum_{s,r} |c_{sr}|\,\big(1+|s/\varepsilon(r)|\big)^k \exp\big(kV(x_r)\big) < \infty \text{ for any } k \in \boldsymbol{N}\}.$$

(b) $\quad \Lambda_{(M_j)} := \{(c_{sr}) \in \boldsymbol{C}^{\boldsymbol{Z}\times\boldsymbol{Z}} \mid$

$$q_k'(c_{sr}) := \sum_{s,r} |c_{sr}| \exp\big(M(ks/\varepsilon(r)) + kV(x_r)\big) < \infty \text{ for any } k \in \boldsymbol{N}\}.$$

We will sometimes use $\Lambda.$ as a common notation.

1.5. Proposition. *Let W and V satisfy* (1.1) *and let*

(1.14) $$\exp\big(-CV(t)\big) \in L^1 \quad \textit{for some } C > 0.$$

Then $C_.^{\infty}(W, V)$ and $\Lambda.$ contain each other as complemented subspaces if W and V satisfy (1.4) (*for $C^{\infty}(W, V)$*) *or* (1.5) (*for $C^{\infty}_{(M_j)}(W, V)$*).

Proof. (i) For $r \neq 0$ let $t \in I_r = \operatorname{sgn}(x_r)\,[|x_r|, |\tilde{x}_r|]$, where $\tilde{x}_r$ solves (1.2). Then

(1.15) $$|W(t)+V(t)-W(x_r)-V(x_r)| \leqslant \operatorname{sgn}(x_r)\big(U(\tilde{x}_r)-U(x_r)\big) = \tfrac{1}{2}V(x_r).$$

This implies by (1.7)

(1.15)′ $$\begin{array}{l} W(t)+nV(t) \leqslant W(x_r)+2nV(x_r) \\ W(x_r)+nV(x_r) \leqslant W(t)+2nV(t) \end{array} \quad \text{for } t \in I_r.$$

(ii) Let $0 \leqslant \varphi \in D(\boldsymbol{R})$ be such that $\int \varphi(t)\,dt = 1$ and $\operatorname{supp}\varphi \subset \{x \mid |x| \leqslant 1/64\}$. For $r \neq 0$ let $\tilde{\varphi}_r := \chi_{B_r} * \varphi\big(\cdot/\gamma(r)\big)/\gamma(r)$, where χ_{B_r} is the characteristic function of $B_r := \big(-\infty, |x_r|+\gamma(r)/2\big)\operatorname{sgn}(x_r)$ and $\gamma(r) := \min\big(\varepsilon(r), \varepsilon(r-\operatorname{sgn}(r))\big)$. For $r \neq 0$ let $\varphi_r := \tilde{\varphi}_{r+\operatorname{sgn}(r)} - \tilde{\varphi}_r$. Let $\varphi_0(t) := \tilde{\varphi}_{\operatorname{sgn}(t)}(t)$. Then $\{\varphi_r \mid r \in \boldsymbol{Z}\}$ is a resolution of the identity subordinate to $\{I_r \mid r \in \boldsymbol{Z}\}$ $(I_0 := (x_{-2}, x_2))$. φ_r is a function as considered in 1.2 (for $|r| \geqslant 2$; $y = x_{r-\operatorname{sgn}(r)}$ and $\zeta = x_r$). This also holds for

$$\psi_r := -\chi_{\tilde{B}_r} * \varphi\big(\cdot/\gamma(r)\big)/\gamma(r) + \chi_{\tilde{\tilde{B}}_r} * \varphi\big(\cdot/\gamma(r\pm 1)\big)/\gamma(r\pm 1)$$

with $\tilde{B}_r := \big(-\infty, \pm(|x_r|+\gamma(r)/16)\big)$ and $\tilde{\tilde{B}}_r := \big(-\infty, \pm(|x_{r\pm 1}|+7\gamma(r\pm 1)/16)\big)$, where the sign is chosen as $\operatorname{sgn}(r)$. Let $\psi_0 \in D.(x_{-2}, x_2)$ be 1 on $\operatorname{supp}\varphi_0$.

(iii) Let $S_x f := f(\cdot - x)$ for $f \in C^{\infty}(\boldsymbol{R})$. Let $\varkappa_1\colon C^{\infty}(\boldsymbol{R}) \to s\hat{\otimes}_{\pi}\omega$ and $\varkappa_2\colon s\hat{\otimes}_{\pi}\omega \to C^{\infty}(\boldsymbol{R})$ be defined by

$$\varkappa_1(f) := \Big(\frac{1}{\varepsilon(r)}\big(S_{-x_r}(f\varphi_r)\big)^{\wedge}\big(2\pi s/\varepsilon(r)\big)\, e^{W(x_r)}\Big)_{(s,r)\in\boldsymbol{Z}^2},$$

$$\varkappa_2(c_{sr})(t) := \sum_r \psi_r S_{x_r}\Big(\sum_s c_{sr}\, e^{2\pi i s t/\varepsilon(r)}\, e^{-W(x_r)}\Big).$$

$\varkappa_1$ is a mapping into $s\hat{\otimes}_{\pi}\omega$, since $f\varphi_r \in D(\boldsymbol{R})$. $\varkappa_2$ is defined, since $(c_{sr})_{s\in\boldsymbol{Z}} \in (s)$ and the supports of $\{\psi_r\}$ are locally finite. We have

$$\begin{aligned} \varkappa_2 \circ \varkappa_1(f)(t) &= \sum_r \psi_r S_{x_r}\Big(\sum_s \frac{1}{\varepsilon(r)}\big(S_{-x_r}(f\varphi_r)\big)^{\wedge}\big(2\pi s/\varepsilon(r)\big)\, e^{2\pi i s t/\varepsilon(r)}\Big) \\ &= \sum_r \psi_r S_{x_r}\big(S_{-x_r}((f\varphi_r)^{\sim})(t)\big) \\ &= \sum_r \psi_r (f\varphi_r)^{\sim}(t) = \sum_r \varphi_r f(t) = f(t), \end{aligned}$$

where $(f\varphi_r)^{\sim} := \sum_{\nu\in \mathbf{Z}} S_{\varepsilon(r)\nu}(f\varphi_r)$. We have used the fact that

(1.17) $$\psi_r|_{\operatorname{supp}\varphi_r} = 1 \quad \text{for } r\in \mathbf{Z},$$
$$\operatorname{supp}(S_{-x_r}(f\varphi_r)) \subset \pm(0,\, \varepsilon(r)) \quad \text{for } r\neq 0$$

by (1.8), and that $\operatorname{supp}(S_{-x_0}(f\varphi_0)) = \operatorname{supp}(f\varphi_0) \subset (x_{-2},\, x_2)$ and $\varepsilon(0) = x_2 - x_{-2}$.

So $\varkappa_2 \circ \varkappa_1$ is the identity on $C^\infty(\mathbf{R})$.

(iv) For $g\in D(\mathbf{R})$ we have $(S_{-x_r} g)^\wedge(z) = \hat{g}(z)\, e^{izx_r}$. Lemma 1.2(a) now shows the following estimate for $f\in C^\infty(W, V)$:

$$q_k(\varkappa_1(f)) = \sum_{s,r} \frac{1}{\varepsilon(r)} |(f\varphi_r)^\wedge(2\pi s/\varepsilon(r))| \big(1+|s/\varepsilon(r)|\big)^k e^{W(x_r)+kV(x_r)}$$
$$\leqslant C_1 \Big(\sum_{s,r} (2\pi)^{-k}\big(1+|s/\varepsilon(r)|\big)^{-2} e^{-C_3 V(x_r)}\Big) \sup_{\substack{r,x\in I_r\\ j\leqslant k+2}} |f^{(j)}(x)|$$
$$\times \exp\big(W(x_r)+(C_2+C_3+k)\, V(x_r)\big) \leqslant C_4\, p_{C_5}(f)$$

by (1.15′) and (1.11), if we choose C_3 so large that

(1.18) $$\sum_r \varepsilon(r)^2 e^{-C_3 V(x_r)} \leqslant C\Big(1+\sum_{|r|\geqslant 1} (|x_{r+1}|-|x_r|)\, e^{-C_2 V(x_r)/2}\Big)^2$$
$$\leqslant C'\Big(\int e^{-C_3 V(x)/4}\, dx\Big)^2 < \infty$$

by (1.7) and (1.14).

For $f\in C^\infty_{(M_j)}(W, V)$, Lemma 1.2(b) implies (with $k' = k/(2\pi)$)

$$q'_k(\varkappa_1(f)) = \sum_{s,r} \frac{1}{\varepsilon(r)} |(f\varphi_r)^\wedge(2\pi s/\varepsilon(r))|\, e^{M(k' 2\pi s/\varepsilon(r)) + W(x_r)+kV(x_r)}$$
$$\leqslant C_1 \sum_{s,r} e^{M(k' 2\pi s/\varepsilon(r)) - M(k'' 2\pi s/\varepsilon(r)) - C_2 V(x_r)}\, p'_{C_3}(f) \leqslant C_4\, p'_{C_5}(f),$$

for suitable k'' and C_2. For $s\neq 0$ this follows from (1.18) and

(1.19) $$M(\varrho t) - M(\varrho) \geqslant B \ln(\varrho/A) \ln t \quad \text{for any } \varrho > 0 \text{ and } t\geqslant 1$$

(see Prop. 3.4 in [6]). For $s = 0$ we use (1.9) and (1.18):

(1.19′) $$\exp(-C_2 V(x_r)) \leqslant \varepsilon(r) \exp\big((-C_2 + C'_2)\, V(x_r)\big).$$

So $\varkappa_1$ is a continuous linear mapping from $C^\infty_{\cdot}(W, V)$ into $\Lambda_{\cdot}$.

(v) The proof of Lemma 1.2(b) shows the following estimate:

$$\sup_j \|(\psi_r e^{2\pi i s(x-x_r)/\varepsilon(r)})^{(j)}\|_\infty k^j/M_j \leqslant \big(\sup_j \|\psi_r^{(j)}\|_\infty (2k)^j/M_j\big) \sup_j \big(4\pi s k/\varepsilon(r)\big)^j/M_j$$
$$\leqslant C_1 \exp\big(C_2 V(x_r) + M(4\pi k s/\varepsilon(r))\big).$$

This implies by (1.15′)

$$p'_k(\varkappa_2(c_{sr})) = \sup_{j,x} \Big|\Big(\sum_r \psi_r S_{x_r}\Big(\sum_s c_{sr}\, e^{2\pi i s x/\varepsilon(r) - W(x_r)}\Big)\Big)^{(j)} k^j/M_j\Big|\, e^{W(x)+kV(x)}$$
$$\leqslant C_1 \sum_{s,r} |c_{sr}| \exp\big((C_2+C_3)\, V(x_r) + M(4\pi k s/\varepsilon(r))\big) \leqslant C_4\, q'_{C_5}(c_{sr}).$$

The corresponding estimate for the seminorms p_k and q_k follows similarly. So $\varkappa_2$ is a continuous linear mapping from $\Lambda_{\cdot}$ into $C^\infty_{\cdot}(W, V)$.

(vi) $\varkappa_1 \circ \varkappa_2$ is a continuous projection in $\Lambda_{\cdot}$ (by the results of (iii)–(v)) onto a subspace which is isomorphic to $C^\infty_{\cdot}(W, V)$ via $\varkappa_1$.

(vii) It is sufficient for the purposes of this paper that $C^\infty_{\cdot}(W, V)$ is isomorphic to a complemented subspace of $\Lambda_{\cdot}$. So we will only sketch the remaining part of the proof.

Let $\varkappa_3\colon s\hat{\otimes}_\pi \omega \to C^\infty(\mathbf{R})$ and $\varkappa_4\colon C^\infty(\mathbf{R}) \to s\hat{\otimes}_\pi \omega$ be defined by

$$\varkappa_3(c_{sr})(t) := \sum_r \Psi_r \Big(S_{x_r}\Big(\sum_s c_{sr}\, e^{16\pi i s t/\varepsilon(r)}\, e^{-W(x_r)}\Big)\Big),$$

$$(\varkappa_4(f))_{sr} := \frac{8}{\varepsilon(r)} (S_{-x_r}(\Phi_r f))^\wedge (16\pi s/\varepsilon(r))\, e^{W(x_r)},$$

where for $r\neq 0$,

$$\Psi_r := \chi_{B_r} * \varphi(\cdot/\varepsilon(r))/\varepsilon(r), \qquad B_r := \pm\big(|x_r|+\varepsilon(r)/8,\, |x_r|+\varepsilon(r)/4\big),$$
$$\Phi_r := \chi_{\tilde{B}_r} * \varphi(\cdot/\varepsilon(r))/\varepsilon(r), \qquad \tilde{B}_r := \pm\big(|x_r|+\varepsilon(r)/16,\, |x_r|+5\varepsilon(r)/16\big)$$

where the sign is chosen as $\operatorname{sgn}(r)$. By this choice,

$$\operatorname{supp}\Phi_r \subset I_r, \qquad \Phi_r|_{\operatorname{supp}\Psi_r} = 1, \qquad \sum_{\nu\in\mathbf{Z}} S_{\nu\varepsilon(r)/8} \Psi_r = 1.$$

The choice of $\varepsilon(0)$ and $\Psi_0,\, \Phi_0 \in D_{\cdot}(0, 1)$ is clear.

It is proved as above that $\varkappa_3 \circ \varkappa_4$ is a continuous projection in $\Lambda_{\cdot}$ onto a subspace which is isomorphic to $C^\infty_{\cdot}(W, V)$ (see also [16], § 5).

Let $\alpha = (\alpha_j)$ be an increasing unbounded sequence of positive numbers. A *power series space of infinite type* is defined as follows:

$$\Lambda_\infty(\alpha) := \Big\{(c_r)\,\Big|\, \sum_{r=1}^{\infty} |c_r|\, e^{n\alpha_r} < \infty \text{ for any } n\in \mathbf{N}\Big\}.$$

The main result of this section now follows from 1.5 by Pełczyński's trick (see [17]):

1.6. THEOREM. *For $W, V\in C^1(\mathbf{R})$ let $w := W'$ and $v := V'$. Let $V\geqslant 0$ and*

(1.14) $$e^{-CV(\cdot)} \in L^1(\mathbf{R}) \quad \textit{for some } C > 0.$$

(a) *$C^\infty(W, V)$ is isomorphic to (s) if*

$$(1.4') \qquad (|w|+|v|)(t) \leqslant \exp\big(D_1\big(V(t)+1\big)\big) \quad \textit{for any } t.$$

(b) *$C^\infty_{(M_j)}(W, V)$ is isomorphic to a nuclear and stable power series space $\Lambda_\alpha(\alpha)$ if*

$$(1.5') \qquad (|w|+|v|)(t) \leqslant D_2 H^{-1}\big(D_2\big(V(t)+1\big)\big)\big(V(t)+1\big) \quad \textit{for any } t, \textit{ and}$$

$$(1.20) \qquad m(2t) \leqslant Cm(t) \quad \textit{for large } t.$$

Here H is chosen as in (1.5) *and $m(t) := \max\{j \mid m_j = M_j/M_{j-1} \leqslant t\}$.*

$\alpha = (\alpha_n)$ is defined by $\alpha_n := \min\{t \mid m_{[t]}h(t) \geqslant n\}$, where $h(t)$ is the L-measure of $\{x \mid V(x) \leqslant t\}$.

If $V(t)$ and $V(-t)$ are nondecreasing for $t > 0$, then $C^\infty_{(M_j)}(W, V)$ is isomorphic to

$$\tilde{\Lambda}_{(M_j)} := \Big\{(c_{sr}) \in C^{Z\times Z} \,\Big|\, \sum_{s,r} |c_{sr}|\, e^{k(m(s)+V(r))} < \infty \textit{ for any } k \in N\Big\}.$$

Proof. We may assume that (1.1), (1.4), (1.5) (and (1.14)) hold (see (1.1)). So $\Lambda.$ is defined.

(a) $\big(\varepsilon(r)e^{-C_3V(x_r)}\big) \in l^2$ for suitable C_3 by (1.18). So $e^{C_3V(x_r)}/\varepsilon(r)$ tends to ∞ for $|r| \to \infty$, and Λ is isomorphic to a power series space $\Lambda_\infty(\beta)$ of infinite type (via an increasing rearrangement of $\{C_3 V(x_r)+\ln\big(1+|s/\varepsilon(r)|\big) \mid (s, r) \in Z \times Z\}$). Also, Λ is nuclear by the Grothendieck–Pietsch criterion. So Λ is isomorphic to a complemented subspace of (s) ([15], Th. 1.5). On the other hand, Λ contains (s) as a complemented subspace via the projection $(c_{sr}) \to (c_{s0})$. So Λ is isomorphic to (s) by [17], and $C^\infty(W, V)$ is isomorphic to (s) by 1.5 and [17] again.

(b)(i) As m is increasing, we get from (1.20) and (3.11) in [6]

$$\left.\begin{aligned} M(kt)-M(t) &= \int_{\ln t}^{\ln(kt)} m(e^\lambda)\,d\lambda \leqslant \ln k\, m(kt) \leqslant C_k m(t) \\ M(kt)-M(t) &\geqslant \ln k\, m(t) \end{aligned}\right\} \quad \text{for } t \geqslant 1.$$

So $\Lambda_{(M_j)}$ is isomorphic to

$$\Big\{(c_{sr}) \,\Big|\, \sum_{s,r} |c_{sr}|\, e^{M(s/\varepsilon(r))+km(s/\varepsilon(r))+kV(x_r)} < \infty \text{ for any } k \in N\Big\}.$$

Again by (1.20), $\Lambda_{(M_j)}$ is isomorphic to

$$\tilde{\Lambda} := \Big\{(c_{sr}) \,\Big|\, \sum_{s,r} |c_{sr}|\, e^{kf(s,r)} < \infty \text{ for any } k \in N\Big\}$$

with $f(s, 0) := m(|s|+1)$ and $f(s, \pm r) := m\big((|s|+1)/(|x_{\pm(r+1)}-x_{\pm r}|)\big)$ for $r > 0$, since $\varepsilon(\pm r) = 2(|x_{\pm(r+1)}-x_{\pm r}|)$ and since by (1.9)

$$(1.21) \qquad f(0, r) = m\big(2/\varepsilon(r)\big) \leqslant M\big(2e/\varepsilon(r)\big)+C_1 \leqslant \tilde{H}\big(2e/\varepsilon(r)\big)+C_2 \leqslant C_3\, V(x_r).$$

(ii) Let β_n be an increasing arrangement of $\{f(s, r) \mid (s, r) \in Z \times Z\}$ and let

$$\tilde{K}_f(t) := |\{(s, r) \in Z \times Z \mid f(s, r) \leqslant t\}|,$$
$$K_f(t) := |\{(s, r) \in Z \times Z \mid f(s, r) < t\}|.$$

Then

$$\min\{t \mid \tilde{K}_f(t) \geqslant n\} \leqslant \beta_n \leqslant \min\{t \mid K_f(t) \geqslant n\},$$

$$\begin{aligned} \tilde{K}_f(t) &\leqslant 2m_{[t]+1}+2\,|\{(s, r) \in N \times (Z\setminus\{0\}) \mid V(x_r) \leqslant t,\ s \leqslant \mu(\tilde{I}_r)\, m_{[t]+1}-1\}| \\ &\leqslant 2m_{[t]+1}\Big(1+\sum_{V(x_r)\leqslant t} \mu(\tilde{I}_r)\Big) \leqslant 2m_{[t]+1}\big(1+h(2t)\big), \end{aligned}$$

since by (1.7), $\tilde{I}_r := \operatorname{sgn}(r)\,[|x_r|, |x_{r+\operatorname{sgn}(r)}|]$ is contained in $\{x \mid V(x) \leqslant 2t\}$ if $V(x_r) \leqslant t$. Next,

$$\begin{aligned} K_f(t) &\geqslant \big|\{(s, r) \in N \times (Z\setminus\{0\}) \mid V(x_r) \leqslant t/2,\ |s|+1 \leqslant (m_{[t/2]}-1)\,\mu(\tilde{I}_r)\}\big| \\ &\geqslant \Big(\sum_{V(x_r)\leqslant t/4} \mu(\tilde{I}_r)\Big)(m_{[t/2]}-1) \geqslant \tfrac{1}{2}h(t/8)\, m_{[t/2]}, \end{aligned}$$

since $V(x_r) \leqslant t/4$ if $V(\zeta) \leqslant t/8$ for some $\zeta \in \tilde{I}_r$. So we have proved that

$$(1.22) \qquad \tfrac{1}{2}\alpha_{[n/4]} \leqslant \beta_n \leqslant 8\alpha_{2n}$$

(in particular, β_n increases to ∞).

(iii) $\Lambda_{(M_j)}$ is nuclear by the Grothendieck–Pietsch criterion, since

$$\sum_{s,r} e^{-C_1 m(s/\varepsilon(r))-C_2V(x_r)} \leqslant C_3+\sum_{\substack{s\neq 0\\ r}} (1+|s/A|)^{-BC_1}\, \varepsilon(r)^{BC_1}\, e^{-C_2V(x_r)} < \infty$$

by (3.12) in [6], (1.19′) and (1.18).

(iv) $\Lambda_{(M_j)} \times \Lambda_{(M_j)}$ is isomorphic to $\Lambda_{(M_j)}$ via the mapping $(c_{sr}, d_{sr}) \to (a_{i,j})$, where $a_{2s+1,r} := c_{sr}$ and $a_{2s,r} := d_{sr}$ (for $s = 0$ see (1.21)). So the assumptions of [17] are satisfied and $C^\infty_{(M_j)}(W, V)$ is isomorphic to $\Lambda_\infty(\beta)$. As $\Lambda_\infty(\beta)$ is stable, we have

$$\beta_{Cn} \leqslant C_1\beta_n, \qquad \beta_n \leqslant C_1\beta_{[n/C]} \qquad \text{for any } C \in N \text{ and some } C_1(C).$$

So (1.22) shows that $\Lambda_\infty(\beta)$ is isomorphic to $\Lambda_\infty(\alpha)$.

(v) To prove the last statement in 1.6, we may assume that $V_+ := V|_{R_+}$ and $V_- := V(-\,\cdot)|_{R_+}$ are strictly increasing to ∞ (by adding $\arctan t$). Then $h(t) = V_+^{-1}(t)+V_-^{-1}(t)$ for large t.

Let γ_n be an increasing arrangement of $\{g(s, r) := m(s)+V(r) \mid (s, r) \in Z \times Z\}$ and let $\tilde{K}_g$ and K_g be defined as above. Then

$$\tilde{K}_g(t) \leqslant 2(m_{[t]}+1)\big(2+V_+^{-1}(t)+V_-^{-1}(t)\big) \leqslant 4m_{[t]}\,h(t),$$
$$K_g(t) \geqslant (m_{[t/4]}-2)\big(V_+^{-1}(t/4)+V_-^{-1}(t/4)-2\big) \geqslant \tfrac{1}{2}m_{[t/4]}\,h(t/4).$$

As also α_n is stable, this shows as above that α_n and γ_n are equivalent, i.e. $\Lambda_\infty(\gamma) \simeq \Lambda_\infty(\alpha) \simeq C^\infty_{(M_j)}(W, V)$.

This completes the proof of the theorem.

Theorem 1.6(a) extends Theorem 3.4 of [16], where the weighted space is assumed to be shift stable (see the remark after 1.2). Sequence space representations for weighted spaces of ultradifferentiable functions seem to appear here for the first time. $\tilde{\Lambda}_{(M_j)}$ is the sequence space, which is also obtained in the shift invariant case.

The isomorphism class of $C^\infty(W, V)$ is independent of W and V, while the isomorphism type of $C^\infty_{(M_j)}(W, V)$ is independent of W, but depends on V.

The proof of 1.6(b) implies a sequence space representation for periodic ultradifferentiable functions, which was also given in [16], Lemma 8 (with a longer proof). (1.20) holds iff

$$2m_j \leqslant m_{Cj} \quad \text{for some } C \in N,$$

which is satisfied if (M.1) and (M.3) hold ([16]), or if $M_j/j!$ satisfies (M.1), since then m_j/j is increasing and

$$2m_j = 2j\,(m_j/j) \leqslant 2j\,(m_{2j}/(2j)) = m_{2j}.$$

2. The Fourier transform on $C^\infty_\cdot(-W, V)'_b$ and $C^\infty_\cdot(W, V)$. We will check the assumptions of the Fundamental Principle of Ehrenpreis (resp. the Division and Extension Theorem, see [3]) in this section for $C^\infty_\cdot(-W, V)$ and $C^\infty_\cdot(W, V)'_b$, where W, $V \in C^1(R)$ are even functions larger than 1. Again, $w := W'$ and $v := V'$.

Let F^* be the Young conjugate of F ($F^*(x) := \sup_t (xt - F(t))$). For $k \geqslant 0$ let $M_k(z) := k \ln(1+|z|^2)$, $M'_k(z) = M(kz)$ and $M^\cdot_{-k} := -M^\cdot_k$, where $M^\cdot_k$ is the common notation for M_k and M'_k.

2.1. Definition. Let $L^\cdot_k(z) := (W + kV)^*(\operatorname{Im} z) - M^\cdot_k(z)$ for $k \in R$.

(a) $\mathscr{H}^+_\cdot := \{f \in \mathscr{H}(C) \mid \|f\|_k := \|f e^{-L^\cdot_k}\|_\infty < \infty$ for any $k \in N\}$.

(b) $\mathscr{H}^-_\cdot := \{f \in \mathscr{H}(C) \mid \|f\|_k < \infty$ for some $k \in -N\}$.

The spaces carry their natural projective (and inductive, resp.) topologies.

2.2. Paley–Wiener Theorem. *For $\delta = 1$ (in (a)) or $\delta = -1$ (in (b)) suppose that*

(2.1) $\quad (w + \delta k v)(t)$ *is increasing to ∞ for any $k \in N$ and large t,*

(2.1′) $\quad V(t) \to \infty \quad$ *for $t \to \infty$.*

(a) *The Fourier transform $\mathscr{F}$ (or $\hat{\ }$) is an isomorphism of $C^\infty_\cdot(W, V)$ onto $\mathscr{H}^+_\cdot$.*

(b) *If W, V (and (M_j)) satisfy (1.4) (resp. (1.5)), then $\mathscr{F}$ is an isomorphism of $C^\infty_\cdot(-W, V)'_b$ onto $\mathscr{H}^-_\cdot$.*

Proof. (a) $C^\infty_\cdot(W, V)$ is contained in $\mathscr{S}(R)$. We may use the Fourier inversion formula and shift the path of integration into the complex plane (by (M.2)′) to show that $\mathscr{F}$ is surjective.

(b)(i) For $T \in C^\infty_\cdot(-W, V)'_b$ there is $k \in N$ such that

$$|\hat{T}(z)| = |\langle {}_xT, e^{-ixz}\rangle| \leqslant C p^\cdot_k(e^{-xz}) \quad \text{for any } z \in C.$$

This shows (by (2.1)) that $\mathscr{F}$ is a (continuous) mapping from $C^\infty_\cdot(-W, V)'_b$ into $\mathscr{H}^-_\cdot$.

(ii) Let $C^d_\cdot$ denote the d-periodical functions in $C^\infty_\cdot(-W, V)$. The Fourier series expansion of $f \in C^d_\cdot$ converges in $C^\infty_\cdot(-W, V)$ (by (2.1) and (M.2)′). So if $\mathscr{F}(T) = 0$ for $T \in C^\infty_\cdot(-W, V)'$, then T vanishes on $C^d_\cdot$ for $d > 0$, and $T = 0$, since $\bigcup_{d>0} C^d_\cdot$ is dense in $C^\infty_\cdot(-W, V)$. So $\mathscr{F}$ is injective.

(iii) For $g \in \mathscr{H}^-_\cdot$ we define a linear functional T on $D_\cdot(R)$ by

$$\langle T, \check{\psi}\rangle := (2\pi)^{-1} \int_R g(x)\,\hat{\psi}(x)\,dx. \tag{2.2}$$

T is defined (by (M.2)′), since

$$|\hat{\psi}(z)| \leqslant C_k\, e^{C|\operatorname{Im} z| - M^\cdot_k(z)} \quad \text{for some } C > 0 \text{ and any } k > 0, \tag{2.3}$$

if $\psi \in D_\cdot(R)$.

If T is continuous on $D_\cdot(R)$ for the topology induced by $C^\infty_\cdot(-W, V)$, then T may be uniquely extended to $\tilde{T} \in C^\infty_\cdot(-W, V)'$, since $D_\cdot(R)$ is dense in $C^\infty_\cdot(-W, V)$. So $\mathscr{F}(\tilde{T})$ is defined and is an entire function. If

$$\int \mathscr{F}(\tilde{T})(x)\,\psi(x)\,dx = \int g(x)\,\psi(x)\,dx \quad \text{for any } \psi \in D(R),$$

then $\mathscr{F}(\tilde{T}) = g$.

Choose $\varphi \in D_\cdot(R)$ such that $\varphi = 1$ near 0. Then $\varphi_n \tilde{T} := \varphi(\cdot/n)\,\tilde{T} \to \tilde{T}$ in $C^\infty_\cdot(-W, V)'_b$, and $\mathscr{F}(\varphi_n \tilde{T}) \to \mathscr{F}(\tilde{T})$ uniformly on compact sets by (b)(i). On the other hand,

$$\begin{aligned}\langle \mathscr{F}(\varphi_n \tilde{T}), \psi\rangle &= \langle\langle {}_x\tilde{T}, \varphi_n e^{-ix\zeta}\rangle, \psi(\zeta)\rangle = \langle\langle {}_xT, (\check{\varphi}_n e^{ix\zeta})^\vee\rangle, \psi(\zeta)\rangle\\ &= (2\pi)^{-1} \iint g(x)\,\hat{\check{\varphi}}_n(x-\zeta)\,dx\,\psi(\zeta)\,d\zeta\\ &= (2\pi)^{-1} \int g(x)\,(\hat{\check{\varphi}}_n * \psi)(x)\,dx \to \int g(x)\,\psi(x)\,dx\end{aligned}$$

by (M.2)′ and the theorem of dominated convergence.

So $\mathscr{F}$ is surjective if T (as defined in (2.2)) is continuous on $D_\cdot(R)$ for the topology of $C^\infty_\cdot(-W, V)$.

(iv) We choose φ_r as in part (ii) of the proof of Proposition 1.5. Then there are k_i such that for any $\psi \in D_\cdot(R)$ and any $|y_r| \geqslant 1$

$$|\langle T,\check{\psi}\rangle| \leqslant \sum_r |\langle T,\check{\psi}\check{\varphi}_r\rangle| \leqslant (2\pi)^{-1}\sum_r \Big|\int g(x+iy_r)(\psi\varphi_r)\hat{\ }(x+iy_r)\,dx\Big|$$

$$\leqslant C_1\sum_r e^{(W-kV)^*(y_r)}\int \exp\big(M_k^{\cdot}(x+iy_r)-M_{k_1}^{\cdot}(x+iy_r)\big)\,dx$$

$$\times\sup_x \big|\exp\big(M_{k_1}^{\cdot}(x+iy_r)\big)(\psi\varphi_r)\hat{\ }(x+iy_r)\big|$$

$$\leqslant C_2 \sup_{r,x\in I_r} e^{(W-kV)(x_{|r|})+xy_r+(W-kV)^*(y_r)}\, p_{k_2}^{\cdot}(\psi)\sum_r\int_{I_r} 1\,dx\, e^{-CV(x_r)}$$

by (M.2)′, Lemma 1.2 and (1.15′). By (1.18), the last sum is finite. We now choose

$$y_r = -\operatorname{sgn}(r)(w-kv)(x_{|r|}) \quad \text{for large } |r|,$$

$$y_r = 1 \quad \text{otherwise}.$$

This implies

$$\sup_{x\in I_r} xy_r + (W-kV)^*(y_r) \leqslant -x_{|r|}\,|y_r| + (W-kV)^*(y_r) + C_3$$

$$\leqslant -(W-kV)(x_{|r|}) + C_4.$$

So T is continuous for the topology of $C_\cdot^\infty(-W, V)$ and the proof is complete.

2.3. Theorem. *Let $L_k^{\cdot}$ be defined as in 2.1. Let W and V satisfy* (2.1) *and* (1.4) *(resp.* (1.5)*) and let*

(2.4) $$t = O(V(t)).$$

(a) *For any $k>0$ there are k', $C_1>0$ (resp. for any $k'<0$ there are $k<0$ and $C_1>0$) such that*

$$\sup_{|\zeta|\leqslant 1} L_{k'}^{\cdot}(z+\zeta) + 2\ln(1+|z|) \leqslant L_k^{\cdot}(z) + C_1.$$

(b) *$L_k^{\cdot}(z)$ is subharmonic (sh.) for $k<0$.*

(c) *For any $k>0$ there are k', $C>0$ and sh. functions Φ_k such that*

$$-C + L_{k'}^{\cdot}(z) \leqslant \Phi_k(z) \leqslant L_k^{\cdot}(z).$$

Proof. (a) For large $|k'|$ and $|\zeta|\leqslant 1$ we get by (2.4)

$$(W+k'V)^*(\operatorname{Im}\zeta + \operatorname{Im} z) \leqslant C_1 + (W+k'V-\mathrm{Id})^*(\operatorname{Im} z)$$

$$\leqslant C_1 + \big(W+(k'-C_2)V\big)^*(\operatorname{Im} z).$$

(a) is now trivial (by (1.19)).

(b) follows from [4], Section 1.6.

(c)(i) The construction in some sense uses formula (1.2) twice: For fixed $k>0$ and $k'>k$ (to be determined later) we define

(2.5) $$(W+k'V)^*(y_r) =: \big(W+(k'+1)V\big)^*(y_{r+1}), \qquad r\geqslant 1, \qquad y_1 = 1.$$

y_r is strictly increasing to ∞, since V is larger than 1 by assumption. Let

(2.6) $$x_0 := 0, \qquad x_r := t(y_r) \quad \text{for } r\geqslant 1, \quad \text{where}$$

$$(W+k'V)^*(y_r) = y_r t(y_r) - (W+k'V)\big(t(y_r)\big).$$

(ii) Let $0\leqslant\varphi\in D_\cdot(0, 1/4)$ and $\int\varphi(t)\,dt = 1$. For $r\geqslant 1$ let

$$\psi_r := \varepsilon(r)^{-2}\varphi\big(\cdot/\varepsilon(r)\big) * \chi_{B_r} e^{-W(x_r)-k'V(x_r)}$$

with $B_r := \big(x_r+\varepsilon(r)/4,\ x_r+3\varepsilon(r)/4\big)$ and $\varepsilon(r) := \tilde{x}_r - x_r$ (see (1.2)). Lemma 1.2, (1.8) and (1.15′) show for large $k'>k$ and $r\geqslant 1$

(2.7) $$|\hat{\psi}_r(z)\, e^{M_k^{\cdot}(z)}| \leqslant C_1\exp\big((C_2+1)V(\tilde{x}_r) + |\tilde{x}_r \operatorname{Im} z|$$

$$-W(x_r) - k'V(x_r)\big)\exp\big(-V(x_r)\big)$$

$$\leqslant C_3\exp(W+kV)^*(\operatorname{Im} z)\exp\big(-V(x_r)\big).$$

(iii) Let

(2.8) $$|\operatorname{Re} z| \leqslant 1/\varepsilon(r), \qquad y_{r+1} \geqslant \operatorname{Im} z \geqslant y_r \qquad \text{for } r\geqslant 1.$$

Then

$$|\hat{\chi}_{B_r}(z)| \geqslant e^{y_r(x_r+\varepsilon(r)/4)}\Big|\int_0^{1/2}\operatorname{Re} e^{-ixz}\,dx\Big|\,\varepsilon(r)/2$$

$$\geqslant \varepsilon(r)\cos(1/2)\, e^{y_r x_r}/2,$$

$$|\varphi(\cdot/\varepsilon(r))\hat{\ }(z)| \geqslant \Big|\int_0^{1/4}\varphi(t)\operatorname{Re} e^{-iz\varepsilon(r)t}\,dt\Big|\,\varepsilon(r) \geqslant \varepsilon(r)\cos(1/4)$$

by (2.8). The choice of x_r and (2.8) now imply for large r

(2.9) $$|\hat{\psi}_r(z)| \geqslant C_4 e^{x_r y_r - W(x_r) - k'V(x_r)} \geqslant C_5 e^{(W+k'V)^*(y_r)}$$

$$= C_5 e^{(W+(k'+1)V)^*(y_{r+1})} \geqslant C_5 e^{(W+(k'+1)V)^*(\operatorname{Im} z)}.$$

(iv) Let k' be so large that (by (M.2)′)

$$-M_{k'}^{\cdot}(t) \leqslant -M_k^{\cdot}(t) - \ln(1+|t|) + C_6$$

and let $\psi_{rs}(z) := \ln|\hat{\psi}_r(z - s/\varepsilon(r))| - M_{k'}^{\cdot}(s/\varepsilon(r))$. Then

(2.10) $$|\psi_{rs}(z)| \leqslant C_7 + (W+kV)^*(\operatorname{Im} z) - M_{k/2}^{\cdot}(z) - \big(\ln(1+|s/\varepsilon(r)|) + V(x_r)\big).$$

Let

$$\tilde{\Phi}_k(z) := \sup_{\substack{r\in \mathbf{N}\\ s\in \mathbf{Z}}}\{\psi_{rs}(z),\ \psi_{rs}(-z)\}.$$

(2.10) shows that this supremum locally is a maximum and that

$$\tilde{\Phi}_k(z) \leqslant L^{\cdot}_{k/2}(z) + C_7.$$

So $\tilde{\Phi}_k$ is continuous and sh., since ψ_{rs} is sh. ([4], Sect. 1.6).

If $|\operatorname{Im} z| \geqslant \tilde{C}$, then there are $r \in N$ and s such that for $\eta = 1$ or $\eta = -1$

$$y_{r+1} \geqslant \eta \operatorname{Im} z \geqslant y_r, \qquad |\eta \operatorname{Re} z - s/\varepsilon(r)| \leqslant 1/(2\varepsilon(r)).$$

So we get by (2.9)

$$\tilde{\Phi}_k(z) \geqslant C_8 + \big(W + (k'+1)V\big)^* (\operatorname{Im} z) - M^{\cdot}_{2k'}(z) \geqslant C_8 + L^{\cdot}_{2k'}(z).$$

$\tilde{\Phi}_k(z)$ has the desired properties for large $|\operatorname{Im} z|$. Taking the maximum of $\tilde{\Phi}_k$ and finitely many functions of the form

$$\sup_{s \in Z} \{\psi_{1s}(z - i\gamma),\ \psi_{1s}(-z + i\gamma)\}, \qquad \gamma \in R,$$

for suitable γ, the resulting function Φ_k shows 2.3(c) for $k/4$.

3. Solution operators for systems of partial differential equations. The structural results of the preceding sections may now be applied to prove the existence of continuous linear solution operators for systems of partial differential equations with constant coefficients. For the convenience of the reader, we first restate the conditions on the weight functions (now defined on R^N) which were used so far, and which will generally be assumed in this section:

Let $W(x) = \sum_{i \leqslant N} W_i(|x_i|)$ and $V(x) := \sum_{i \leqslant N} V_i(|x_i|)$ and let $(W_i)' =: w_i$ and $(V_i)' =: v_i$ be continuous. For $i \leqslant N$ and large t we assume that:

1. W_i and V_i are strictly increasing and $t = O\big(V_i^{\cdot}(t)\big)$.
2. $w_i + \delta k v_i$ is strictly increasing and unbounded for any $k \in N$. Here and below δ may be 1 or -1.

For ultradifferentiable functions $C^\infty_{(M_j)}$ we assume that $M_j = \prod_{i \leqslant N} M_{i,j_i}$, where $(M_{i,j})_{j \in N}$ satisfy (M.1), (M.2)′ and (M.3)′ (see [6]) and that

$$m_i(2t) \leqslant C\big(m_i(t) + 1\big) \tag{1.20}$$

for $m_i(t) := \max\{j \mid m_{i,j} := M_{i,j}/M_{i,j-1} \leqslant t\}$.

The spaces $C^\infty_{\cdot}(\delta W, V)$ and $\mathscr{H}^\delta_{\cdot}$ are now defined as in 1.1 and 2.1.

The weight functions $L^{\cdot}_k$ now have the form

$$L^{\cdot}_k(z) = (W + kV)^* (\operatorname{Im} z) - M^{\cdot}_k(z),$$

where

$$M^{\cdot}_k(z) := k \sum_{i \leqslant N} \ln(1 + |z_i|) \qquad (\text{for } C^\infty(\delta W, V)),$$

$$M^{\cdot}_k(z) := \sum_{i \leqslant N} M_i(k|z_i|) \operatorname{sgn}(k) \qquad (\text{for } C^\infty_{(M_j)}(\delta W, V)).$$

Let $R(D)$ be an $r \times s$ system of partial differential operators with constant coefficients and let $\{(\partial_j, V_j) \mid j = 1, \ldots, J\}$ be a Noetherian operator for ${}^tR(-z)$ (see [3]), with linear differential operators $\partial_j(z, D_z)$ (of size $1 \times s$) with polynomial coefficients and algebraic varieties V_j contained in the characteristic variety of R,

$$V_R := \{z \in C^N \mid \operatorname{rank} {}^tR(-z) < s\}.$$

Let

$$\varrho(f) := (\partial_j f|_{V_j})_{j \leqslant J} \quad \text{ for } f \in \mathscr{H}(C^N)^s,$$

$$\mathscr{H}(V_R) := \{(f_j) \in \prod_{j \leqslant J} \mathscr{H}(V_j) \mid \varrho(f) = (f_j) \text{ for some } f \in \mathscr{H}(C^N)^s\},$$

$$\mathscr{H}^+_{\cdot R} := \{(f_j) \in \mathscr{H}(V_R) \mid \sup_{z \in V_j} |f_j(z)| \leqslant C e^{L^{\cdot}_k(z)} \text{ for any } k \in N\},$$

$$\mathscr{H}^-_{\cdot R} := \{(f_j) \in \mathscr{H}(V_R) \mid \sup_{z \in V_j} |f_j(z)| \leqslant C e^{L^{\cdot}_k(z)} \text{ for some } k \in -N\}.$$

$\mathscr{H}^+_{\cdot R}$ (and $\mathscr{H}^-_{\cdot R}$) carry their natural projective (resp. inductive) topology.

3.1. Proposition. *Let W and V (and (M_j)) satisfy the general assumptions of this section and* (1.4) *(resp.* (1.5)*). Then ϱ is a topological isomorphism*

(a) *from $(\mathscr{H}^+_{\cdot})^s / {}^tR(-z)(\mathscr{H}^+_{\cdot})^r$ onto $\mathscr{H}^+_{\cdot R}$,*

(b) *from $(\mathscr{H}^-_{\cdot})^s / {}^tR(-z)(\mathscr{H}^-_{\cdot})^r$ onto $\mathscr{H}^-_{\cdot R}$,*

and $\mathscr{H}^-_{\cdot R}$ is defined by a compact (injective) spectrum.

Proof. (b) For any $k \in -N$ there is $j \in -N$ (by Th. 2.3(a)) such that

$$L^{\cdot}_k(z) - L^{\cdot}_j(z) \to -\infty \qquad \text{for } |z| \to \infty.$$

So

$$\mathscr{H}^-_{\cdot} \text{ is defined by a compact (injective) spectrum,} \tag{3.1}$$

i.e. $\mathscr{H}^-_{\cdot}$ is a (DFS)-space. For $k \in R$ let

$$(\mathscr{H}_k)^s := \{f \in \mathscr{H}(C^N)^s \mid \|f e^{-L^{\cdot}_k}\|_\infty < \infty\},$$

$$\mathscr{H}_{kR} := \{(f_j) \in \mathscr{H}(V_R) \mid \|f_j e^{-L^{\cdot}_k}\|_\infty < \infty\}.$$

Let B_k and B_{kR} be the respective unit balls. Then

$$\varrho\colon (\mathscr{H}_k)^s \to \mathscr{H}_{nR} \text{ is continuous for some } n(k). \tag{3.2}$$

The Division and Extension Theorem (D/E-Th., see p. 240 in [3]) and Theorem 2.3(a)(b) show that for any $k \in -N$ there are $n \in -N$ and $C > 0$ such that the equation

$$\varrho f = (f_j)$$

may be solved with $f \in CB_n$ if $(f_j) \in B_{kR}$. So $\varrho\colon (\mathscr{H}_{\cdot}^{-})^s \to \mathscr{H}_{\cdot R}^{-}$ is surjective.

$\mathscr{H}_{\cdot R}^{-}$ is defined by a compact (injective) spectrum, since $B_{kR} \subset \varrho(CB_n)$, which is (relatively) compact in $\mathscr{H}_{mR}$ for some m by (3.1) and (3.2).

$\varrho\colon (\mathscr{H}_{\cdot}^{-})^s \to \mathscr{H}_{\cdot R}^{-}$ is open by the open mapping theorem for (DFS)-spaces. Finally, ${}^tR(-z)(\mathscr{H}_{\cdot}^{-})^r = \operatorname{Ker} \varrho \cap (\mathscr{H}_{\cdot}^{-})^s$ by the D/E-Theorem and Theorem 2.3 (a) (b).

(a) (i) $\varrho\colon (\mathscr{H}_{\cdot}^{+})^s \to \mathscr{H}_{\cdot R}^{+}$ is continuous by Theorem 2.3 (a).

For any $k \in N$ there is $n \in N$ such that

$${}^tR(-z)(\mathscr{H}_{\cdot}^{+})^r \textit{ is dense in } {}^tR(-z)(\mathscr{H}_n)^r \textit{ for the topology of } (\mathscr{H}_k)^s.$$

As ${}^tR(-z)$ is continuous by Th. 2.3 (a) and $\mathscr{F}$ is an isomorphism, we only have to show that $C_{\cdot}^{\infty}(W, V)$ is dense in $C_{\cdot n(k)}^{\infty}(W, V)$ for the topology of $C_{\cdot k}^{\infty}(W, V)$, where $C_{\cdot k}^{\infty} := \{f \in C^{\infty}(\boldsymbol{R}^N) \mid p_k^{\cdot}(f) < \infty\}$ (see 1.1). This is trivially proved using cut-off functions and convolution. By the D/E-Theorem and Th. 2.3 (a) (c), ${}^tR(-z)(\mathscr{H}_{\cdot}^{+})^r$ is contained in $\operatorname{Ker} \varrho \cap (\mathscr{H}_{\cdot}^{+})^s$, and $\operatorname{Ker} \varrho \cap (\mathscr{H}_{n(k)})^s$ is contained in ${}^tR(-z)(\mathscr{H}_k)^r$. So $\operatorname{Ker} \varrho \cap (\mathscr{H}_{\cdot}^{+})^s$ is dense in $\operatorname{Ker} \varrho \cap (\mathscr{H}_{n(k)})^s$ for the topology of $(\mathscr{H}_k)^s$ for any $k \in N$ and some $n(k) \in N$.

Again by the D/E-Theorem and Th. 2.3 (a) (c), for any $k \in N$ there are C and $n \in N$ such that the equation

$$\varrho g = (f_j) \tag{3.3}$$

has a solution $g \in B_k$ if $(f_j) \in CB_{nR}$. Hence (3.3) has a solution $g \in (\mathscr{H}_{\cdot}^{+})^s \cap B_k$ if $(f_j) \in C(k) B_{n(k)R} \cap (\mathscr{H}_{\cdot}^{+})^r$ (by the classical Mittag-Leffler argument).

So $\varrho\colon (\mathscr{H}_{\cdot}^{+})^s \to \mathscr{H}_{\cdot R}^{+}$ is surjective and open.

(ii) $\operatorname{Ker} \varrho \cap (\mathscr{H}_{\cdot}^{+})^s = {}^tR(-z)(\mathscr{H}_{\cdot}^{+})^r$.

Proof. We have already noticed that $\operatorname{Ker} \varrho \cap (\mathscr{H}_{\cdot}^{+})^s$ contains ${}^tR(-z)(\mathscr{H}_{\cdot}^{+})^r$. To prove the opposite inclusion, we have to show that

$${}^tR(-z) g = f \tag{3.4}$$

is solvable with $g \in (\mathscr{H}_{\cdot}^{+})^r$ if $\varrho f = 0$ and $f \in (\mathscr{H}_{\cdot}^{+})^s$.

(α) If ${}^tR(-z) f = 0$ for some $0 \neq f \in C[z]^r$, then there is an $r_1 \times r$ matrix Q of polynomials such that

$${}^tR(-z) f = 0 \text{ for } f \in C[z]^r \quad \text{iff} \quad f = {}^tQ(-z) g \text{ for some } g \in C[z]^{r_1}.$$

Then $\{({}^t\check{R}_j, C^N) \mid j \leqslant s\}$ is a Noetherian operator for ${}^tQ(-z)$, where the R_j are the columns of R. The solvability of (3.4) now follows as in (b) (i) by the D/E-Theorem and Th. 2.3 (a) (c) (with $\{({}^t\check{R}_j, C^n)\}$ instead of ϱ and ${}^tQ(-z)$ instead of ${}^tR(-z)$).

(β) If ${}^tR(-z) f = 0$ for $f \in C[z]^r$ iff $f = 0$, then ${}^tR(-z) f = 0$ for $f \in \mathscr{H}(C^N)^r$ iff $f = 0$. So the solutions $g = g_k \in (\mathscr{H}_k)^r$ of (3.4) (existing by the D/E-Theorem) are in fact unique and therefore contained in $(\mathscr{H}_{\cdot}^{+})^r$.

Let $P(D)$ be an $r \times s$ system of partial differential operators with constant coefficients. Let N_P be the kernel of $P(D)$ (in the hyperfunctions, say) and let $Q(D)$ be the matrix of relations implied by $P(D)$ (i.e. ${}^tP(-z) f = 0$ for $f \in C[z]^r$ iff $f = {}^tQ(-z) g$ for some $g \in C[z]^{r_1}$). $Q(D)$ may be 0.

We may now prove the main result of this paper, concerning the following two sequences:

$$0 \to N_P \cap C_{\cdot}^{\infty}(-W, V)^s \to C_{\cdot}^{\infty}(-W, V)^s \xrightarrow{P(D)} N_Q \cap C_{\cdot}^{\infty}(-W, V)^r \to 0, \tag{3.5}$$

$$0 \to N_P \cap (C_{\cdot}^{\infty}(W, V)_b')^s \to (C_{\cdot}^{\infty}(W, V)_b')^s \xrightarrow{P(D)} N_Q \cap (C_{\cdot}^{\infty}(W, V)_b')^r \to 0. \tag{3.6}$$

3.2. Theorem. *Let W and V (and (M_j)) satisfy the general assumptions of this section and (1.4) (resp. (1.5)). Let moreover*

$$v_i = o(w_i), \tag{3.7}$$

$$2w_i \circ V_i^{-1}(t) \leqslant w_i \circ V_i^{-1}(Ct) \quad \textit{for large } t. \tag{3.8}$$

Then the sequences (3.5) and (3.6) are exact and split, i.e. $P(D)$ has continuous linear solution operators

$$L_1\colon C_{\cdot}^{\infty}(-W, V)^r \cap N_Q \to C_{\cdot}^{\infty}(-W, V)^s,$$

$$L_2\colon (C_{\cdot}^{\infty}(W, V)_b')^r \cap N_Q \to (C_{\cdot}^{\infty}(W, V)_b')^s.$$

Proof. (a) The spaces in (3.5) (resp. (3.6)) are (FS)-spaces (resp. (DFS)-spaces). So the Paley–Wiener Theorem 2.2 implies that (3.5) (and (3.6)) are exact iff

$${}^tP(-z)\colon (\mathscr{H}_{\cdot}^{\delta})^r / \overline{{}^tQ(-z)(\mathscr{H}_{\cdot}^{\delta})^{r_1}} \to (\mathscr{H}_{\cdot}^{\delta})^s$$

is injective with closed range for $\delta = 1$ (resp. $\delta = -1$). But ${}^tP(-z)(\mathscr{H}_{\cdot}^{\delta})^r$ is closed by 3.1, being the kernel of the continuous mapping ϱ. We have already noticed that ${}^tP(-z)$ is injective if Q vanishes, and that $\{({}^t\check{P}_j, C^N) \mid j \leqslant s\}$ is a Noetherian operator for ${}^tQ(-z)$ if $Q \neq 0$. So the injectivity follows from 3.1 (applied to $R = Q$) in this case.

The exactness of (3.5) and (3.6) is thus proved.

(b) The splitting of (3.5) will be proved by the general splitting theorem of D. Vogt ([15], Th. 7.1). All spaces are nuclear by Theorem 1.6. We now have to check the linear topological invariants (DN) and (Ω) (see [15]):

$C_{\cdot}^{\infty}(-W, V)^r$ and the subspace $C_{\cdot}^{\infty}(-W, V)^r \cap N_Q$ have (DN), since $C_{\cdot}^{\infty}(-W, V)^r$ is isomorphic to a power series space of infinite type by Theorem 1.6. $(C_{\cdot}^{\infty}(-W, V)^s \cap N_P)_b'$ is isomorphic to $\mathscr{H}_{\cdot P}^{-}$ by Th. 2.2 and 3.1(a).

$\mathscr{H}_{\cdot P}^{-}$ may be defined by the equivalent inductive spectrum

(3.9) $\tilde{\mathcal{H}}^-_{kP} := \{(f_j)\in\mathcal{H}(V_P) \mid |f_j(z)| \leqslant C e^{\Sigma(L_i(z_i)+k\tilde{L}_i(z_i))}\}, \quad k\in N,$

where

$$L_i(\zeta) = (W_i - V_i)^*(\operatorname{Im}\zeta) \quad \text{for} \quad C^\infty(-W, V)$$

(resp. $L_i(\zeta) = (W_i - V_i)^*(\operatorname{Im}\zeta) + M_i(\zeta)$ for $C^\infty_{(M_j)}(-W, V)$),

$$\tilde{L}_i(\zeta) = V_i \circ (w_i - v_i)^{-1}(\operatorname{Im}\zeta) + \ln(1+|\zeta|)$$

(resp. $\tilde{L}_i(\zeta) = V_i \circ (w_i - v_i)^{-1}(\operatorname{Im}\zeta) + m_i(\zeta)$).

Proof. (i) We have shown in the proof of 1.6 that for any $k > 0$ there is $C > 0$ such that (by (1.20))

$$M(kt) \leqslant M(t) + Cm(t), \qquad M(t) + km(t) \leqslant M(e^k t).$$

Here and below we have omitted the index i. This shows the claim for the functions $M_k^*(z)$.

(ii) $$(W-kV)^*(t) - (W-V)^*(t) \geqslant C_1 + \int_{C_2}^{(w-v)^{-1}(t)} (k-1)v(\zeta)\,d\zeta$$

$$\geqslant C_3 + (k-1)\, V\circ(w-v)^{-1}(t) \quad \text{for large } t,$$

$$(W-kV)^*(t) - (W-V)^*(t) \leqslant C_4 + (k-1)\, V\circ(w-kv)^{-1}(t)$$

$$\overset{(*)}{\leqslant} C_5 + C(k-1)\, V\circ(w-v)^{-1}(t) \quad \text{for large } t,$$

where $(*)$ holds iff

$$(w-kv)\circ V^{-1}(Ct) \geqslant (w-v)\circ V^{-1}(t) \quad \text{for large } t.$$

This is seen by (3.7) and (3.8):

$$(w-kv)\circ V^{-1}(Ct) \geqslant \tfrac{1}{2} w\circ V^{-1}(tC) \geqslant (w-v)\circ V^{-1}(t).$$

The spectrum (3.9) is compact by 3.1 (b). Hence it is regular and the unit balls $\tilde{B}_k$ of $\tilde{\mathcal{H}}^-_{kP}$ are a fundamental system of (absolutely convex closed) bounded sets, whose polars are a basis of 0-neighbourhoods in $C^\infty_*(-W, V)^s \cap N_P$. So this space has (Ω) (see the proof of 5.2 in [15]) if for any $p\in N$ there is $k \geqslant p$ such that for any $i\in N$ there are C, $n \geqslant 1$ such that

$$|f_j(z)| \leqslant e^{L(z)+k\tilde{L}(z)} \quad \text{if}$$

$$|f_j(z)| \leqslant \min\left\{\frac{1}{Ct^n} e^{L(z)+i\tilde{L}(z)},\ t e^{L(z)+p\tilde{L}(z)}\right\}, \qquad t > 0.$$

This is easily shown. So $C^\infty_*(-W, V)^s \cap N_P$ has (Ω) and the sequence (3.5) is split by the splitting theorem 7.1 in [15].

(c) $C^\infty_*(W, V)^r$ and the quotient $C^\infty_*(W, V)^r/{}^tQ(-D)\,C^\infty_*(W, V)^{r_1}$ have (Ω), since $C^\infty_*(W, V)^r$ is isomorphic to a power series space of infinite type by Th. 1.6.

$C^\infty_*(W, V)^s/{}^tP(-D)\,C^\infty_*(W, V)^r$ is isomorphic to $\mathcal{H}^+_{\cdot P}$ by 2.2 and 3.1, where the norms in $\mathcal{H}^+_{\cdot P}$ may be defined by

$$\|(f_j)\|_k := \|f_j e^{L+k\tilde{L}}\|_\infty$$

with L and $\tilde{L}$ defined as in (b) (with $(W_i+V_i)^*$ and w_i+v_i instead of $(W_i-V_i)^*$ and w_i-v_i). Indeed, this follows from (3.7) and (3.8) as above.

By using these new norms, (DN) is easily proved for $\mathcal{H}^+_{\cdot P}$ (and hence for $C^\infty_*(W, V)^s/{}^tP(-D)\,C^\infty_*(W, V)^r$).

So the dual sequence to (3.6) is split by the splitting theorem of D. Vogt and (3.6) also splits.

The conditions of Theorem 3.2 will be illustrated by some simple examples:

3.3. Examples. (a) Let $W(t) = t^\beta(\ln t)^\gamma$ and $V(t) = t^{\beta_1}(\ln t)^{\gamma_1}$. Then the conditions of 3.2 are satisfied:

(i) for $C^\infty(\delta W, V)$ if $1 \leqslant \beta_1 < \beta$, $0 \leqslant \gamma_1 < \infty$, or if $1 < \beta_1 = \beta$, $0 \leqslant \gamma_1 < \gamma$;

(ii) for $C^\infty_{(j!^\alpha)}(\delta W, V)$, $1 < \alpha$, if $1 \leqslant \beta_1$ and $(\beta-1)/\alpha < \beta_1 < \beta$, $0 \leqslant \gamma_1 < \infty$, or if $1 \leqslant (\beta-1)/\alpha = \beta_1$, $(\gamma+1)/\alpha < \gamma_1 < \infty$, or if $1 < \beta = \beta_1$, $0 \leqslant \gamma_1 < \gamma$.

(b) Let $W(t) = \exp(t^\beta)$ and $V(t) = \exp(\gamma_1 t^{\beta_1})$ for $0 < \gamma_1$, β, $\beta_1 < \infty$. Then the conditions of 3.2 are satisfied:

(i) for $C^\infty(\delta W, V)$ if $0 < \beta_1 < \beta$, $\gamma_1 > 0$, or if $\beta_1 = \beta$, $\gamma_1 < 1$;

(ii) for $C^\infty_{(j!^\alpha)}(\delta W, V)$, $\alpha > 1$, if $\beta_1 = \beta$, $1/\alpha < \gamma_1 < 1$.

The functions in 3.3 (a)(i) and (b)(i) also satisfy the conditions of 3.2 for $C^\infty_{(M_j)}(\delta W, V)$ with $M_j = \exp(Aj^{\alpha+1})$, $0 < \alpha \leqslant 1$.

Notice that the sequence space representation in Th. 1.6 holds for an essentially larger class of the above functions.

The conditions needed in 3.2 for $C^\infty(\delta W, V)$ (and for $C^\infty_{(M_j)}(\delta W, V)$ with $M_j = \exp(Aj^{\alpha+1})$, see (1.12)) are in a sense stable for compositions: If W and V satisfy these conditions, then they are also satisfied by $W\circ F$ and $V\circ F$ if $f := F'$ is nondecreasing, positive and

(3.10) $$f(t) \leqslant \exp(C_1 F(t)) \quad \text{for large } t$$

(3.10′) $$\text{(resp. } f(t) \leqslant \exp(C_1 F(t)^{\alpha/(\alpha+1)}) \quad \text{for large } t\text{)}.$$

So all conditions follow for $\delta = 1$ from (3.10) (and (3.10′)) for $W = F$ and $V = F^\beta$, $\beta < 1$, if $(F^\beta)'$ is increasing and positive.

The conditions of 3.2 hold for $C^\infty_{(j!^\alpha)}(\delta W, V)$ and $W(t) = e^{\tilde{W}(t)}$, $V(t) = e^{\gamma\tilde{W}(t)}$ with $1/\alpha < \gamma < 1$ if $\tilde{w} = \tilde{W}'$ is strictly increasing and

(3.11) $\tilde{w}(t) \leqslant \exp(\varepsilon \tilde{W}(t))$ for any $\varepsilon > 0$ and large t.

This essentially is (1.4) for $W = V$! So the conditions seem to get weaker for faster growing weight functions.

Indeed, Theorem 3.2 may be shown for $C_*^{\infty}(W, W)'_b$ in this case, i.e. we may omit the o-condition (3.7):

3.4. Theorem. *Let $W_i = \exp \tilde{W}_i$, where $\tilde{W}_i$ and $\tilde{w}_i := \tilde{W}_i'$ are strictly increasing from 0 to ∞ on $[0, \infty)$. Let*

(3.12) $\tilde{w}_i(t) \leqslant \exp\big(\varepsilon \exp(\tilde{W}_i(t))\big)$ *for any $\varepsilon > 0$ and large t*

for $C^{\infty}(W, W)'_b$

(3.13) *(resp.* $\tilde{w}_i(t) \leqslant C_1 H_i^{-1}\big(C_1 \exp(\tilde{W}_i(t))\big)$ *for large t*

for $C^{\infty}_{(M_j)}(W, W)'_b$, where H_i is chosen as in (1.5)).

Then the sequence

$$0 \to N_P \cap (C_*^{\infty}(W, W)'_b)^s \to (C_*^{\infty}(W, W)'_b)^s \xrightarrow{P(D)} (C_*^{\infty}(W, W)'_b)^r \cap N_Q \to 0$$

is exact and split.

Proof. All the assumptions of 3.2 are valid except for (3.7). This was only used in 3.2 to prove (DN) for $\mathscr{H}_{*P}^{+}$. So we have to show the following: For any $p \in N$ there are $k \geqslant p$ and $C > 0$ such that

$$\|f_j e^{-(pW)^*(\operatorname{Im} z) + M_p^*(z)}\|^2 \leqslant C \|f_j e^{-(mW)^*(\operatorname{Im} z) + M_m^*(z)}\|_{\infty} \times \|f_j e^{-(kW)^*(\operatorname{Im} z) + M_k^*(z)}\|_{\infty}.$$

The assumptions on (M_j) being as before, this reduces to the following statement: For any $n \in N$ there are $k \geqslant p$ and C such that

(3.14) $$(mW)^*(t) + (kW)^*(t) \leqslant 2(pW)^*(t) + C.$$

Here again only one variable is considered. (3.14) holds iff

$$\Lambda := \{(c_j) \in C^N \mid \sum |c_j| e^{-(nW)^*(j)} < \infty \text{ for } j \in N\} \text{ has (DN)}$$

([15], Prop. 5.1). Λ is nuclear by the Grothendieck–Pietsch criterion, since

(3.15) $$(n_1 W)^*(t) - (n_2 W)^*(t) \geqslant (n_2 - n_1) W \circ w^{-1}(t/n_2) \geqslant (n_2 - n_1) \ln(t/n_2)/C.$$

The last estimate follows from (3.12). (3.12) is also implied by (3.13): $\ln(t/A) B \ln k \leqslant M(kt) \leqslant H(kt)$ by (1.19) and hence $\ln t = o(H(t))$.

(3.15) also implies that

$$\Lambda'_b = \{(c_j) \mid \sum |c_j| e^{(nW)^*(j)} < \infty \text{ for some } n \in N\}$$

and that Λ'_b is isomorphic to

$$\mathscr{H}_U := \{f \in \mathscr{H}(C) \mid |f(z)| \leqslant k e^{kU(z)} \text{ for some } k \in N\}$$

with $U(t) = W(\ln|t|) = \exp(\tilde{W}(\ln|t|))$ for $|t| \geqslant 1$ and $U(t) = 1$ otherwise. Theorem 3.4 now follows from

3.5. Proposition. *Let $\tilde{W}$ be convex and increasing and $\tilde{W}(t) = 0$ for $t \leqslant 0$. Let*

(3.12) $\tilde{w}(t) \leqslant \exp\big(\varepsilon \exp(\tilde{W}(t))\big)$ *for any $\varepsilon > 0$ and large t,*

where $\tilde{w} := \tilde{W}'$. Then $(\mathscr{H}_U)'_b$ has (DN) *for $U(t) = \exp(\tilde{W}(\ln|t|))$.*

Proof. Let $B_k := \{f \in \mathscr{H}(C) \mid |f(z)| \leqslant e^{kU(z)}\}$. By Lemma 2.1 in [15], we have to show the existence of $m \in N$ such that for any $k \in N$ there are p, C with

(3.16) $$B_k \subset e^{\varrho} B_m + C e^{-\varrho} B_{k+p} \quad \text{for } \varrho > 0.$$

We set $\psi(x) = (x - x_\varrho + k + 1)\varrho/k$ and $x_\varrho = \ln(\varrho/k)$. Then

(3.17) $$\begin{aligned} \psi(x) &\leqslant e^x + \varrho && \text{for } x \geqslant 0, \\ \psi(x) &\leqslant (k+1+p) e^x - \varrho && \text{for } x \geqslant 0,\ p = e^{2k} \end{aligned}$$

(consider only the unique point where the derivatives of the left and right hand sides are equal). Moreover,

(3.18) $$\psi(x) \geqslant (k+1) e^x \quad \text{for } \tilde{x}_\varrho := x_\varrho - k \leqslant x \leqslant x_\varrho.$$

Let $\varphi(z) := \psi(\tilde{W}(\ln|z|))$. Then φ is sh., since $\varphi(e^x) = \psi(\tilde{W}(x)) = \tilde{W}(x) C_1 + C_2$ is convex and increasing on $\boldsymbol{R}$ (see Th. 1.6.7 in [4]).

(3.17) and (3.18) give the following estimates:

(3.17′) $$\begin{aligned} \varphi(x) &\leqslant e^{\tilde{W}(\ln x)} + \varrho = U(x) + \varrho && \text{for } x \geqslant 0, \\ \varphi(x) &\leqslant (k+1+p) U(x) - \varrho && \text{for } p = e^{2k},\ x \geqslant 0, \end{aligned}$$

(3.18′) $$\varphi(x) \geqslant (k+1) U(x) \quad \text{for } \tilde{x}_\varrho \leqslant \tilde{W}(\ln x) \leqslant x_\varrho,$$

where we suppose that ϱ is so large that $x_\varrho = \ln(\varrho/k) > k$. Now,

$$1/\delta_\varrho := 1/\big(\exp(\tilde{W}^{-1}(x_\varrho)) - \exp(\tilde{W}^{-1}(\tilde{x}_\varrho))\big) \leqslant \tilde{w}(\tilde{W}^{-1}(\zeta))/k$$
$$\overset{(*)}{\leqslant} \exp\big(\tfrac{1}{2}\exp(x_\varrho)/\exp(k)\big) = \exp\big(\tfrac{1}{2}\exp(\tilde{x}_\varrho)\big) \leqslant \exp\big(\tfrac{1}{2} U(x)\big)$$

for $\tilde{x}_\varrho \leqslant \tilde{W}(\ln x) \leqslant x_\varrho$, where $(*)$ follows from (3.12) for large ϱ. We may choose $h \in D(\boldsymbol{R})$ such that $h(x) = 1$ for $\tilde{W}(\ln x) \leqslant \tilde{x}_\varrho$, and $h(x) = 0$ for $\tilde{W}(\ln x) \geqslant x_\varrho$, and

$$|h'(x)| \leqslant C_0/\delta_\varrho \leqslant C_0 \exp\big(\tfrac{1}{2} U(x)\big) \quad \text{for } \tilde{x}_\varrho \leqslant \tilde{W}(\ln x) \leqslant x_\varrho.$$

Let $f \in B_k$ and $H(z) := h(|z|)$. Then

$$|\bar\partial(Hf)(z)| \leqslant C_1 e^{(k+1/2)U(z)} \leqslant C_2 e^{\varphi(z)}(1+|z|)^{-2}.$$

So $\bar\partial(Hf)e^{-\varphi} \in L^2(C)$ and we may choose a solution G of

$$\bar\partial G = -\bar\partial(Hf)$$

such that for some C_3 independent of ϱ (see Th. 4.4.2 in [4])

$$\int |G(z)|^2 e^{-2\varphi(z)}(1+|z|^2)^{-2}\,dz \leqslant C_3.$$

Set $f_1 := G + Hf$ and $f_2 := (1-H)f - G$. Then $f = f_1 + f_2$ and $f_i \in H(C)$. For large ϱ we get

$$\Big(\int |f_1(z)|^2 e^{-4U(z)}\,dz\Big)^{1/2} \leqslant C_4\Big(\int |f_1(z)|^2 e^{-2U(z)}(1+|z|^2)^{-2}\,dz\Big)^{1/2} \leqslant C_5 e^{\varrho},$$

$$\Big(\int |f_2(z)|^2 e^{-2(k+2+p)U(z)}\,dz\Big)^{1/2}$$

$$\leqslant C_6\Big(\int |f_2(z)|^2 e^{-2(k+1+p)U(z)}(1+|z|^2)^{-2}\,dz\Big)^{1/2} \leqslant C_7 e^{-\varrho}.$$

The proof is completed by showing that the sup-norms may be estimated by the L^2-norms. For $f \in \mathscr{H}(C)$ we have

$$|f(0)| \leqslant \frac{C}{R}\Big(\int_{|z|\leqslant R} |f(z)|^2\,dz\Big)^{1/2} \quad \text{for any } R > 0.$$

For fixed z choose R_z such that $U(|z|+R_z) = 2U(|z|)$. Then

$$1/R_z = u(\zeta)/U(z) \leqslant Ce^{CU(\zeta)} \leqslant Ce^{2CU(z)}$$

by the mean value theorem and (3.12). This shows (for $|z| \geqslant 2$)

$$|f(z)| \leqslant \frac{C}{R_z}\Big(\int_{|\zeta|\leqslant R_z} |f(z+\zeta)|^2\,dz\Big)^{1/2}$$

$$\leqslant \tilde C e^{2CU(z)} e^{lU(|z|+R_z)}\Big(\int |f(\zeta)|^2 e^{-2lU(\zeta)}\,d\zeta\Big)^{1/2}$$

$$\leqslant \tilde C e^{(2C+2l)U(z)}\Big(\int |f(\zeta)|^2 e^{-2lU(\zeta)}\,d\zeta\Big)^{1/2} \quad \text{for } l \in N.$$

So the proposition is proved.

Property (DN) for $(\mathscr{H}_U)'_b$ was characterized in terms of U by R. Meise and B. A. Taylor ([12]), assuming, however, that $\mathscr{H}_U$ is invariant for shifts, which gives a priori estimates for $\tilde W$.

If $\tilde W + \ln \tilde w = \ln w$ is convex, then (3.14) may be seen directly: $w^{-1} \circ \exp$ is then concave and

$$w^{-1}(t/p) - w^{-1}(t/p^2) \geqslant w^{-1}(t) - w^{-1}(t/p).$$

This shows (3.14) for $k = p^2$, $m = 1$ and $C = 0$.

Theorem 3.4 does not hold for stable weight functions M (i.e. $M(2t) \leqslant CM(t)$). In this case, $C^\infty(M, M)$ is the space $W_{M,\infty}$, which was considered in [8]. A large class of hypoelliptic operators have no solution operators in $(W_{M,\infty})'_b$. In particular, semielliptic nonelliptic equations have no solution operators in $(W_{|x|^{1/\alpha},\infty})'_b$ for $0 < \alpha < 1$ (except for at most one value of α).

3.6. Remark. *Let $G \in C^1[0, \infty)$ and let $g := G'$ be strictly increasing from 0 to ∞ on $[0, \infty)$. Then there is $W \in C^1[0, \infty)$ such that $w := W'$ is increasing to ∞ and*

$$G(t) \leqslant W(t) \leqslant G(t+C)+1 \quad \textit{for some } C > 0 \textit{ and large } t,$$

$$w(t) \leqslant W(t)^\varepsilon \quad \textit{for any } \varepsilon > 1 \textit{ and large } t.$$

Proof. Let $c_n := G^{-1}(n)$ for $n \geqslant 0$. Then $\gamma_n := c_n - c_{n-1}$ is strictly decreasing to 0 and $\sum_n \gamma_n = \infty$. Let the graph of v_1 be defined by the line segments joining the points $(n, \Gamma_n)_{n\geqslant 1}$ (and $(0, \Gamma_1)$) for $\Gamma_n := 1/\big(n(\ln(n+1))^2\big) + \gamma_n$. Let V be the inverse function of $V_1(t) := \int_0^t v_1(x)\,dx$. Then $v := V'$ is strictly increasing and for large n we get for $t \in (n-1, n]$

$$G^{-1}(t) \leqslant c_n = \sum_{k=1}^{n} \gamma_k \leqslant \sum_{k=1}^{n} \Gamma_k \leqslant \int_0^n v_1(x)\,dx \leqslant V_1(t+1),$$

$$G^{-1}(t) \geqslant c_{n-1} = \sum_{k=1}^{n-1} \gamma_k \geqslant \sum_{k=1}^{n-1} \Gamma_k - C_1 \geqslant \int_1^n v_1(x)\,dx - C_1 \geqslant V_1(t) - C_2$$

since $\sum 1/\big(n(\ln(n+1))^2\big) < \infty$. So

$$G(t) \leqslant W(t) := V(t+C_2) \leqslant G(t+C_2)+1 \quad \text{for large } t.$$

Fix $\varepsilon > 1$. Then we have for large n and $t \in [n, n+1]$

$$v_1(t) \geqslant \Gamma_{n+1} \geqslant 1/\big((n+1)(\ln(n+2))^2\big) \geqslant t^{-\varepsilon},$$

$$v(t) = V'(t) = \frac{1}{v_1(V_1^{-1}(t))} \leqslant V_1^{-1}(t)^\varepsilon = V(t)^\varepsilon \quad \text{for large } t.$$

So W has the above-stated properties.

So (1.4) and (1.5) do not imply a priori bounds for the growth of the weights by the remarks after 3.3. Moreover, the whole space of distributions of finite order may be filled with weighted spaces satisfying the assumptions of this paper. However, there is no solution operator for hypoelliptic equations $P(D)$ in the space of distributions of finite order. Indeed, if the kernel of $P(D)$ were complemented in $D_F(\mathbf{R}^N)'_b$, then it would be complemented in $C^\infty(\mathbf{R}^N)$, contrary to the result of Vogt ([18]).

Similarly, $(W_{\exp G,\infty})'_b = (W_{\exp W,\infty})'_b$ is the union of the spaces $W'_k := C^\infty(\exp W(k\cdot), \exp W(k\cdot))'_b$, which are general splitting spaces by 3.4. Nevertheless, $(W_{\exp G,\infty})'_b$ in general allows no linear continuous solution operator for partial differential equations ([8]).

If P is hypoelliptic, then $N_P \cap C^\infty(-W, V)^s$ may coincide (topologically) with $N_P \cap A^s$ for some continuously embedded space $A \hookrightarrow C^\infty(-W, V)$. Then $N_P \cap A^s$ is complemented in A^s if it is complemented in $C^\infty(-W, V)^s$. So Th. 3.2 may give solution operators in weighted spaces of ultradifferentiable functions assuming only (1.4) (and not (1.5)). Using this argument for $(C^\infty(W, V)'_b)^s$, one may get solution operators in spaces which are not tractable directly by the methods of this paper, since they fail to be power series spaces of infinite type (for example $\{f \in (C^\infty(\mathbf{R}^N) \mid \|f^{(k)} e^{-(W+nV)}\|_\infty < \infty$ for some $n \in N$ and any $k \in N\}$).

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Added in proof (November 1987). In [20] optimal cut off functions are constructed, which may be used to improve the results of this paper for ultradifferentiable functions: Let m_p/p he increasing and let $C_1 m_p \leqslant m_{Cp}$ for any $p \in N$ and some $C_1 > C \in N$. (These are conditions (M.1)* and (M.3) of H. Komatsu, which are stronger than (M.1) and (M.3)′). Then there are functions $0 \leqslant \mu_n \in C^\infty(\mathbf{R})$ (see [20]) such that $\int \mu_n(t)\,dt = 1$, $\operatorname{supp}\mu \subset [-n/64, n/64]$ and

$$\forall k\ \exists C_k\ \forall n:\ \sup_{j,x} |\mu_n^{(j)}(x)|\, k^j/M_j \leqslant C_k \exp(\tilde{M}(C_k/n))/n,$$

where $\tilde{M}(t) := \ln(\sup_j(t^j j!/M_j))$.

Using $\mu_{\varepsilon(t)}$ instead of $\varphi(\cdot/\varepsilon(t))/\varepsilon(t)$ in the definition of φ_y in Lemma 1.2, one can show the following improvement of Lemma 1.2: Let (M_p) satisfy (M.1)*, (M.2)′ and (M.3) and let

$$(*)\qquad M((|v|+|w|)(t)) \leqslant CV(t) \quad \text{for some } C>0 \text{ and any } t.$$

Then for any $k \in N$ there are $C_i > 0$ such that for any $f \in C^\infty_{(M_j)}(\mathbf{R})$ and any $y \neq 0$

$$\sup_{z\in \mathbf{C}} |f\varphi_y)\hat{}\,(z)| \exp(M(kz)) \leqslant C_1 (\sup_{j,x\in I_\zeta} |f^{(j)}(x)|\, k_1^j/M_j) \exp(C_2 V(\zeta)) \int_{I_\zeta} \exp(x \operatorname{Im} z)\,dx.$$

Any result of this paper for ultradifferentiable functions then holds if (*) is assumed instead of (1.5). (*) is strictly weaker than (1.5): It already holds if (M_p) also satisfies (M.2) and if $(|v|+|w|)(t) \leqslant m_{[CV(t)]}$ for some C and any t. So (*) is valid for the Gevrey sequence $M_p = (p!)^\alpha$ if $(|v|+|w|)(t) \leqslant CV(t)^\alpha$ for some C and and t. This improves Remark 1.3 and also gives new examples in 3.3. Similarly, condition (3.13) in Th. 3.4 may be substituted by a weaker one coming from (*).

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