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Some results on symmetric subspaces of L_1

by

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Abstract. We extend the result of Dacunha-Castelle concerning subspaces of L_1 with symmetric basis to the more general setting of rearrangement invariant function spaces. Methods we use are of finite-dimensional nature, and completely different from Dacunha-Castelle's ones. We also study the embeddability of $L(E)$ into L_1, when L is a symmetric space.

Introduction. It was shown by D. Dacunha-Castelle that each infinite-dimensional subspace of L_1 with a symmetric basis is order isomorphic to a mean of Orlicz sequence spaces ([2], exposé 10, Th. 1, and exposé 11, Prop. 3).

A finite-dimensional version of this result was given by S. Kwapień and C. Schütt in [5]. The purpose of this note is to give a reduction of the infinite-dimensional result of Dacunha-Castelle to the finite one of Kwapień–Schütt, and an extension of this to the more general setting of rearrangement invariant spaces (over $[0, 1]$ or $[0, \infty]$), as well as to symmetric sublattices of $L_1(l_q)$ and $L_1(c_0)$.

Dacunha-Castelle used an ultrapower device to reduce the question to the case of exchangeable random variables; which (as known by de Finetti's theorem) is reducible to the independent identically distributed r.v. case. The random variables we consider here (in the "finite case") are discrete and not independent.

We also give some results on the embeddability of $L(E)$ into L_1, when L is a symmetric space and E a Banach space. As a consequence of the results of [5] we deduce that for $E = L$ this embeddability implies that L is equal to some L_p space. For $L = L_\varphi$, we show that L_φ order embeds into some $L_1(L_p)$ and E embeds into L_p.

The fact that $l_p(l_q)$ does not embed into L_1 if $p > q$ (Cor. 3.2 below) is a matter of "folklore", and was known for some time to several authors. See also [9] for a proof due to J. L. Krivine, or [3].

0. Definitions and notation. We say that a basis $\{x_i\}_{i\in N}$ of a Banach space is *C-symmetric* if for all $a_i \in \mathbf{R}$, $\varepsilon_i = \pm 1$, $i \in N$, and all permutations π of N,

$$\Big\|\sum_{i\in N} \varepsilon_i a_{\pi(i)} x_i\Big\| \leqslant C \Big\|\sum_{i\in N} a_i x_i\Big\|.$$

For simplicity of notation we refer to 1-symmetric sequence spaces (of finite or infinite dimension) or rearrangement invariant function spaces ("r.i. spaces") over $[0, 1]$ or $[0, \infty]$ (cf. [6] for a definition and basic properties) as *symmetric spaces.*

Let $(\Omega, \boldsymbol{P})$ be a probability space. A family $(F_\omega)_{\omega\in\Omega}$ of Orlicz functions will be said to be *measurable* if for each $\lambda \in \boldsymbol{R}_+$ the function $\omega \mapsto F_\omega(\lambda)$ is measurable. It is then clear that for each fixed vector $a \in \boldsymbol{R}^{(N)}$ (resp. for each simple measurable function f) the function $\omega \mapsto \|a\|_{l_{F_\omega}}$ (resp. $\omega \mapsto \|f\|_{L_{F_\omega}}$) is measurable.

We denote by $\| \ \|_{F_\omega}$ both the norm in l_{F_ω} (Orlicz sequence space) and in L_{F_ω} (Orlicz function space over $[0, 1]$ or $[0, \infty]$). The completion of $\boldsymbol{R}^{(N)}$ (resp. of the space of simple functions) for the norm $\int_\Omega \|a\|_{F_\omega}\, d\boldsymbol{P}(\omega)$ is called a "*mean of Orlicz spaces*".

Note that the functions F_ω may be degenerate, i.e. F_ω may be 0 for $0 < t < t_0$ and $+\infty$ for $t_1 < t < \infty$.

An Orlicz function F is said to be *normalized* if $F(1) = 1$, and *p-convex* (*q-concave*) if $\lambda \mapsto F(\lambda^{1/p})$ is convex ($\lambda \mapsto F(\lambda^{1/q})$ concave).

If N is a natural number, let $\mathfrak{S}_N$ be the group of permutations of $\{1, \ldots, N\}$ (or, equivalently, of $\boldsymbol{N}$ and fixing the numbers $n > N$) and D_N the group $\{-1, 1\}^N$ (identified with a subgroup of $D = \{-1, 1\}^{\boldsymbol{N}}$). $\operatorname{Ave}_{\pi\in\mathfrak{S}_N}$ and $\operatorname{Ave}_{\varepsilon\in D_N}$ are the natural averages on $\mathfrak{S}_N$ and D_N.

If a and b are positive real numbers, we write $a \underset{c}{\sim} b$ if $(1/\sqrt{c})\, b \leqslant a \leqslant \sqrt{c}\, b$.

1. The Representation Theorem.

THEOREM 1.1. *There is a universal constant K such that every symmetric space X which is C-isomorphic to a subspace Y of L_1 is KC-isomorphic to a mean of Orlicz spaces with 2-concave Orlicz functions.*

More precisely,

$$\|f\|_X \underset{KC}{\sim} m \int_{\mathcal{T}} \|f\|_{F_\tau}\, d\pi(\tau)$$

where $(\mathcal{T}, \pi)$ is a separable probability space, $m \underset{KC}{\sim} 1$, and the F_τ, $\tau \in \mathcal{T}$, are 2-concave normalized Orlicz functions.

In this representation, the Orlicz spaces are sequence spaces (finite or infinite) or function spaces (over $[0, 1]$ or $[0, \infty]$) according to the nature of the symmetric space X.

Note that by the 2-concavity of the Orlicz functions, each of the spaces appearing in the average is isomorphic (with a universal isomorphism constant) to a subspace of L_1, by [1].

We give the proof of Theorem 1.1 after recalling the treatment of the finite case [5].

(A) *The finite case.* Let $(x_i)_{i=1}^N$ be a C-symmetric sequence in L_1. We have $\|\sum a_i x_i\|_1 \underset{C}{\sim} \|\sum a_i x_i\|_X$ where X is a (finite) sequence space with 1-symmetric basis $(x_i)_{i=1}^N$. We have

$$\Big\|\sum_{i=1}^N a_i x_i\Big\|_X = \operatorname*{Ave}_{\substack{\pi\in\mathfrak{S}_N\\ \varepsilon\in D_N}} \Big\|\sum_{i=1}^N \varepsilon_i a_{\pi(i)} x_i\Big\|_X \underset{C}{\sim} \operatorname*{Ave}_{\substack{\pi\in\mathfrak{S}_N\\ \varepsilon\in D_N}} \Big\|\sum_{i=1}^N \varepsilon_i a_{\pi(i)} x_i\Big\|_1,$$

$$\operatorname*{Ave}_{\substack{\pi\in\mathfrak{S}_N\\ \varepsilon\in D_N}} \Big\|\sum_{i=1}^N \varepsilon_i a_{\pi(i)} x_i\Big\|_1 \underset{\mathrm{Kh}}{\sim} \int \operatorname*{Ave}_{\pi\in\mathfrak{S}_N}\Big(\sum_{i=1}^N |a_{\pi(i)} x_i(\omega)|^2\Big)^{1/2} d\boldsymbol{P}(\omega)$$

where $\mathrm{Kh} = \sqrt{2}$ (Khintchine equivalence constant).

By Kwapień–Schütt's results we have

$$\operatorname*{Ave}_{\pi\in\mathfrak{S}_N}\Big(\sum_{i=1}^N |a_{\pi(i)} x_i(\omega)|^2\Big)^{1/2} \underset{K_1}{\sim} \|a\|_{F_{N,\omega}} \Big(N^{-1}\sum_{i=1}^N |x_i(\omega)|\Big)$$

where K_1 is a universal constant and $F_{N,\omega}$ is a 2-concave normalized Orlicz function depending measurably on ω (cf. [5], Lemma 2.10).

In fact, if we set, as in [5], formula (2.3),

$$M_N(t) = \begin{cases} \dfrac{N^2}{2N-1}\, t^2 & \text{for } 0 \leqslant t \leqslant \dfrac{1}{N}, \\[2ex] \dfrac{N}{2N-1}\Big(2t - \dfrac{1}{N}\Big) & \text{for } t \geqslant \dfrac{1}{N}, \end{cases}$$

$$F_{N,\omega}(t) = \sum_{i=1}^N M_N\Big(t \frac{|x_i(\omega)|}{\|(x_i(\omega))_{i=1}^N\|_{M_N}}\Big)$$

it is clear that $F_{N,\omega}$ is 2-concave, normalized and depends measurably on ω.

Putting $\boldsymbol{P}_N = m_N^{-1}(N^{-1}\sum |x_i(\cdot)|)\boldsymbol{P}$ with $m_N = \int N^{-1} \sum |x_i(\omega)|\, d\boldsymbol{P}(\omega)$ we see that $\boldsymbol{P}_N$ is a probability measure and

$$\Big\|\sum_{i=1}^N a_i x_i\Big\|_X \underset{KC}{\sim} m_N \int_\Omega \|a\|_{F_{N,\omega}}\, d\boldsymbol{P}_N(\omega).$$

Note that

$$m_N = N^{-1}\sum_{i=1}^N \|x_i\|_{L^1} \underset{C}{\sim} N^{-1}\sum_{i=1}^N \|x_i\|_X = 1.$$

(B) *The infinite sequence case.* Let $(x_i)_{i=1}^\infty$ be an infinite C-symmetric sequence in L_1 and X the associated 1-symmetric space C-isomorphic to $\operatorname{span}[x_i]_{i=1}^\infty$. Let $(F_{N,\omega})_{\omega\in\Omega}$ be the measurable family of Orlicz functions associated to the vectors $x_1, \ldots, x_N$ as in (A) before. It can be viewed as a random variable whose values are Orlicz functions.

More precisely, consider the space $\mathcal{O}_2$ of normalized 2-concave Orlicz functions equipped with the topology of pointwise convergence. It is a compact metrizable space, for it is closed in $\bar{\boldsymbol{R}}_+^{R^+}$ and the topology of pointwise convergence coincides on $\mathcal{O}_2$ with the topology of compact convergence. This follows from Ascoli's theorem and the fact that $F(t) \leqslant t^2$ for $f \in \mathcal{O}_2$ and $t \geqslant 1$.

Let π_N be the image of $\boldsymbol{P}_N$ by the map $\Omega \to \mathcal{O}_2$, $\omega \mapsto F_{N,\omega}$. We can extract a subsequence $(\pi_{N_k})_{k=1}^{\infty}$ which w^*-converges to a probability π on $\mathcal{O}_2$.

Now for each $a \in \boldsymbol{R}^{(\boldsymbol{N})}$, the map $\mathcal{O}_2 \to \boldsymbol{R}_+$, $F \mapsto F(a) = \sum_{i \in \boldsymbol{N}} F(|a_i|)$ is clearly bounded and continuous. But

$$\|a\|_F = \inf\{\lambda\colon F(a/\lambda) \leqslant 1\} = \sup\{\lambda\colon F(a/\lambda) \geqslant 1\}.$$

The first equality shows easily that $F \mapsto \|a\|_F$ is l.s.c., and the second that it is u.s.c.; so this map is bounded continuous, and consequently

$$\forall a \in \boldsymbol{R}^{(\boldsymbol{N})}\colon \quad \int_{\mathcal{O}_2} \|a\|_F \, d\pi_{N_k}(F) \to \int_{\mathcal{O}_2} \|a\|_F \, d\pi(F) \quad \text{as } k \to \infty.$$

If we suppose $m_{N_k} \to m$ as $k \to \infty$ it follows that

$$\Big\|\sum_{i=1}^{\infty} a_i x_i\Big\|_X \underset{KC}{\sim} m \int_{\mathcal{O}_2} \|a\|_F \, d\pi(F).$$

(C) *The rearrangement invariant case.* Let X be a r.i. space over $[0, 1]$ (resp. $[0, \infty]$) and suppose that T is an embedding of X into L_1, with $\|T\| \cdot \|T^{-1}\| \approx C$. We apply the same procedure as for finite sequence spaces but normalize differently the Orlicz functions in order to get Orlicz function space norms.

Let $(A_{i,N})_i$ be the dyadic partition of order N of $[0, 1]$ (resp. of $[0, N]$), which generates a ring Q_N. We have

$$\Big\|\sum_{i=1}^{2^N} \lambda_i 1_{A_{i,N}}\Big\|_X \underset{C}{\sim} \operatorname*{Ave}_{\substack{\pi \in \mathfrak{S}_{2^N} \\ \varepsilon \in D_{2^N}}} \Big\|\sum_{i=1}^{2^N} \varepsilon_i \lambda_{\pi(i)} T 1_{A_{i,N}}\Big\|_{L_1} \quad \Big(\text{resp. } \operatorname*{Ave}_{\substack{\pi \in \mathfrak{S}_{N 2^N} \\ \varepsilon \in D_{N 2^N}}}\Big)$$

$$\underset{K}{\sim} \int \Big\|\sum_{i=1}^{2^N} \lambda_i e_i\Big\|_{l_{F_{N,\omega}}} h_N(\omega) \, d\boldsymbol{P}(\omega)$$

where $h_N = 2^{-N} \sum_{i=1}^{2^N} |T 1_{A_{i,N}}|$, and $1_{A_{i,N}}$ is the characteristic function of the set $A_{i,N}$.

Set $\theta_N(\omega) = F_{N,\omega}^{-1}(2^{-N})$ and $\hat{F}_{N,\omega}(t) = 2^N F_{N,\omega}(\theta_N(\omega) t)$. Then

$$\hat{F}_{N,\omega}(1) = 1, \qquad \Big\|\sum_{i=1}^{2^N} \lambda_i e_i\Big\|_{l_{F_{N,\omega}}} = \frac{1}{\theta_N(\omega)} \Big\|\sum_{i=1}^{2^N} \lambda_i 1_{A_{i,N}}\Big\|_{L_{\hat{F}_{N,\omega}}}$$

and the family $(\hat{F}_{N,\omega})_N$ remains measurable, with values in $\mathcal{O}_2$. Set $\nu_N = (h_N/\theta_N) \boldsymbol{P}$ and $m_N = \|\nu_N\|$; let π_N be the probability on $\mathcal{O}_2$ corresponding to ν_N/m_N. We have

$$\|f\|_X \underset{CK}{\sim} \int \|f\|_{L_{\hat{F}_{N,\omega}}} \, d\nu_N(\omega) = m_N \int \|f\|_{L_F} \, d\pi_N(F)$$

for $f = \sum_i \lambda_i 1_{A_{i,N}}$. The same holds for each $N \geqslant N_0$ when f is Q_{N_0}-measurable.

With $f = 1_{[0,1]}$, we get

$$m_N = \|\nu_N\| \underset{CK}{\sim} 1,$$

so we can pass to the limit (w^*) along a subsequence and get

$$\|f\|_X \underset{CK}{\sim} m \int \|f\|_{L_F} \, d\pi.$$

Note that the simple functions on dyadic intervals form a dense subspace of X. Indeed, since X is isomorphic to a subspace of L_1, X does not contain a subspace isomorphic to c_0. Therefore X is an order continuous lattice, thus the dyadic functions are dense in X.

2. Symmetric sublattices of $L_1(l_q)$, $L_1(c_0)$. Analogously to Theorem 1.1, the following theorem holds for symmetric sublattices of $L_1(l_q)$ or $L_1(c_0)$.

THEOREM 2.1. *A symmetric space X which is C-order isomorphic to a sublattice of $L_1(c_0)$ (resp. $L_1(l_q)$) is KC- (resp. $K_q C$-)isomorphic to a mean of Orlicz spaces (resp. of q-concave Orlicz spaces) where K is a universal constant (resp. K_q depends only on q).*

Note that a q-concave Orlicz space is in turn order isomorphic to a sublattice of $L_1(L_q)$, by [8], Th. 2.

Proof. (A) The finite case is treated with a suitable modification of the function M_N. We set in the case of $L_1(l_q)$, $q < \infty$,

$$M_N^{(q)}(t) = \begin{cases} \dfrac{N^q}{qN-(q-1)} t^q & \text{if } 0 \leqslant t \leqslant \dfrac{1}{N}, \\[2ex] \dfrac{N}{qN-(q-1)} \Big(qt - \dfrac{q-1}{N}\Big) & \text{if } t \geqslant \dfrac{1}{N}, \end{cases}$$

and in the case of $L_1(c_0)$,

$$M_N^{(\infty)}(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \dfrac{1}{N}, \\[2ex] \dfrac{t-1/N}{1-1/N} & \text{if } t \geqslant \dfrac{1}{N}. \end{cases}$$

We then have the following analogue of Lemma (2.9) of [5]:

LEMMA 2.2. *Let $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{N^2} \geqslant 0$. Then*

$$\frac{1}{3N} \|(a_i)_{i=1}^{N^2}\|_{M_N^{(q)}} \leqslant \frac{1}{N} \sum_{i=1}^{N} a_i + \Big(\frac{1}{N} \sum_{i=N+1}^{N^2} a_i^{(q)}\Big)^{1/q} \leqslant \frac{3q}{N} \|(a_i)_{i=1}^{N^2}\|_{M_N^{(q)}} \quad \textit{if } q < \infty,$$

$$\frac{1}{N}\|(a_i)_{i=1}^{N^2}\|_{M_N^{(\infty)}} \leqslant \frac{1}{N}\sum_{i=1}^{N} a_i \leqslant \frac{3}{N}\|(a_i)_{i=1}^{N^2}\|_{M_N^{(\infty)}} \quad \text{if } q = \infty.$$

It is straightforward to verify this lemma (by adapting the proof of Lemma (2.9) of [5]).

Then (by using Ths. 1.1 and 1.2 of [5]) it is clear that

$$\|\sum_{i=1}^{N} a_i x_i\|_{L_1(l_q)} \underset{C}{\sim} \int \operatorname{Ave}_{\pi}\Big(\sum_{i=1}^{N} |a_{\pi(i)}|\,\|x_i(\omega)\|_{l_q}^q\Big)^{1/q} dP(\omega)$$
$$\underset{K_q C}{\sim} \int \|(a_i)\|_{l_{F_{N,\omega}}} dP(\omega)$$

with $F_{N,\omega}$ deduced from $M_N^{(q)}$ as in the L_1 case.

(B) *The infinite case* is obtained, in the case of $L_1(l_q)$ sublattices, by convergence of probability measures on $\mathcal{O}_q$, the compact metrizable space of q-concave normalized Orlicz functions.

In the case of $L_1(c_0)$ one has to consider the space $\mathcal{O}$ of normalized (possibly degenerate) Orlicz functions, which are left-continuous at irrational points, endowed with the topology of simple convergence on $\boldsymbol{Q}_+$. It is easy to see that $\mathcal{O}$ is again a compact metrizable space. Moreover, it is clear that, for example, the functions $F \mapsto \|a\|_F$ are continuous on $\mathcal{O}$ when $a \in \boldsymbol{Q}^{(N)}$. So by the convergence procedure we identify the norm of $\|\sum a_i x_i\|$ for $a \in \boldsymbol{Q}^{(N)}$ and conclude by approximation. In the r.i. case one has to notice, moreover, that the simple dyadic functions are dense in r.i. spaces order isomorphic to sublattices of $L_1(c_0)$: in fact, sublattices of $L_1(c_0)$ are order continuous (cf. [6], Def. 1.a.6). ∎

3. Embeddings of $L(E)$ into L_1 where L is a symmetric space. We recall the following result, given as a remark (without proof) in [11] (the proof will appear in another paper of Kwapień and Schütt).

PROPOSITION 3.1. *Let L and F be two spaces with 1-symmetric bases of the same dimension (finite or infinite). If $L(F)$ is embeddable into L_1 then the identity map from L to F is bounded, and more precisely*

$$\|\mathrm{id}\colon L \to F\| \leqslant 5\sqrt{2}\, d_1(L(F))$$

where $d_1(L(F)) = \inf\{d(L(F), G)\colon G \text{ is a subspace of } L_1\}$.

COROLLARY 3.2. *If the space $l_p(l_q)$ embeds into L_1 then $p \leqslant q$.*

PROPOSITION 3.3. *An infinite-dimensional symmetric space L such that $L(L)$ embeds (isomorphically) into L_1 is order isomorphic (by the identity map) to l_p or L_p for some p, $1 \leqslant p \leqslant 2$.*

Proof. By Krivine's results [4] the space L contains l_p^n uniformly as sublattices. So $l_p^n(L)$ and $L(l_p^n)$ uniformly embed into L_1 and the same is true for every symmetric sublattice Λ of L. By Proposition 3.1 we see that every finite-dimensional symmetric sublattice Λ of L is order isomorphic to an l_p^n with isomorphism constants uniformly bounded. So L itself is order isomorphic to an L_p (or l_p) space. ∎

COROLLARY 3.4. *If $l_\varphi(l_\varphi)$ (resp. $L_\varphi(L_\varphi)$) embeds into L_1 then $l_\varphi \approx l_p$ (resp. $L_\varphi \approx L_p$) for some p, $1 \leqslant p \leqslant 2$.*

More precisely, the function $\varphi(t)$ is equivalent to t^p (at 0 in the case of l_φ, at $+\infty$ in the case of $L_\varphi([0,1])$, at 0 and $+\infty$ in the case of $L_\varphi(\boldsymbol{R})$).

Now if L is a lattice and E a Banach space (of infinite dimensions) and if $L(E)$ embeds into L_1, then let

$$q(L) = \inf\{q\colon L \text{ is } q\text{-concave}\} = \inf\{q\colon L \text{ has a } q \text{ lower estimate}\},$$

$$p(E) = \sup\{p\colon E \text{ is of type } p\}.$$

Then $l_{q(L)}$ is latticially finitely representable in L (this is implicitly contained in Krivine's paper [4]) and, by Maurey–Pisier [7], $l_{p(E)}$ is finitely representable in E.

So $l_{q(L)}(l_{p(E)})$ is finitely representable in L_1, which (by Corollary 3.2) implies $q(L) \leqslant p(E)$.

Next, $l_{q(L)}(E)$ is finitely representable (hence embeddable by standard ultraproduct arguments) in L_1. So by a result of N. Kalton [3], E embeds into $L_{q(L)}$.

In the case where L is an Orlicz space, we can make the situation more precise.

PROPOSITION 3.5. *$L_\varphi(E)$ is embeddable into L_1 iff there exists p such that L_φ is order isomorphic to a sublattice of $L_1(L_p)$ and E is embeddable into L_p.*

Proof (of the "only if" part). If $q(L_\varphi) < p(E)$, choose $q(L_\varphi) < p < p(E)$. L_φ being p-concave, by [8], Th. 2, it is order isomorphic to a sublattice of $L_1(L_p)$ (note that there exists a p-concave Orlicz function ψ such that $L_\psi \approx L_\varphi$). On the other hand, E is embeddable into L_p by a well-known result of H. P. Rosenthal [10].

If $q(L_\varphi) = p(E)$ let p be the common value. Note that E is embeddable into L_p and that $L_\varphi(l_p)$ is finitely representable, hence embeddable, into L_1.

The same is true if we replace L_φ by one of its symmetric finite-dimensional sublattices. It is a consequence of Proposition 3.1 that for each choice of n functions $f_1, \ldots, f_n$, L_φ-normalized, disjointly supported and having the same distribution,

$$\|\sum_{i=1}^{n} f_i\| \geqslant K n^{1/p}$$

where K does not depend on the f_i's and on n.

It is then an easy exercise to show that L_φ is in fact p-concave (cf. [6], proof of Prop. 2.b.5) and therefore is a sublattice of $L_1(L_p)$. ■

4. Open questions. It seems not clear whether convexity or concavity conditions can be translated on the representation of symmetric subspaces as means of Orlicz spaces given in Section 1. More precisely:

QUESTION 1. If L is a q-concave symmetric subspace of L_1, is it representable as a mean of q-concave Orlicz spaces?

QUESTION 2. If L is a p-convex symmetric subspace of L_1, is it representable as a p-mean of p-convex Orlicz functions, i.e.

$$\|f\|_L^p \sim \int \|f\|_F^p \, d\pi(F) \ ?$$

These questions are equivalent to the following, which have affirmative answers in the case of Orlicz spaces:

QUESTION 1′. If a q-concave symmetric subspace of L_1 order embeddable into $L_1(L_q)$?

QUESTION 2′. Is a p-convex symmetric subspace of L_1 embeddable into L_p?

It is well known (see [10]) that a weaker form of Question 2′ has an affirmative answer: a subspace of L_1 with p-convex lattice structure is embeddable into L_{p_1} for $1 \leqslant p_1 < p \leqslant 2$. We do not know if, analogously, the following weakening of Question 1′ has an affirmative answer:

QUESTION 1″. Is a q-concave ($q < 2$) symmetric subspace of L_1 order embeddable into $L_1(L_{q_1})$, for each q_1, $q < q_1 \leqslant 2$?

One can also ask for the equivalent of Proposition 3.5, for symmetric subspaces of L_1 in place of Orlicz spaces.

To reproduce the proof of Proposition 3.5, it would be sufficient to answer Question 1″ in the affirmative, because, on the other hand, we are able to prove (by "stability" methods, in the sense of Krivine and Maurey, which notably differ from the methods we use here) that a lattice L such that $L(l_q)$ embeds into L_1 is order embeddable into $L_1(L_q)$.

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