Some results on symmetric subspaces of $L_1$

by

Y. RAYNAUD (Paris) and C. SCHÜTT (Kiel)

Abstract. We extend the result of Dacunha-Castelle concerning subspaces of $L_1$ with symmetric basis to the more general setting of rearrangement invariant function spaces. Methods we use are of finite-dimensional nature, and completely different from Dacunha-Castelle’s ones. We also study the embeddability of $L(E)$ into $L_1$, when $L$ is a symmetric space.

Introduction. It was shown by D. Dacunha-Castelle that each infinite-dimensional subspace of $L_1$ with a symmetric basis is order isomorphic to a mean of Orlicz sequence spaces ([2], exposé 10, Th. 1, and exposé 11, Prop. 3). A finite-dimensional version of this result was given by S. Kwapień and C. Schütt in [5]. The purpose of this note is to give a reduction of the infinite-dimensional result of Dacunha-Castelle to the finite one of Kwapień–Schütt, and an extension of this to the more general setting of rearrangement invariant spaces (over $[0, 1]$ or $[0, \infty]$), as well as to symmetric sublattices of $L_1(l_q)$ and $L_1(c_0)$.

Dacunha-Castelle used an ultrapower device to reduce the question to the case of exchangeable random variables; which (as known by de Finetti’s theorem) is reducible to the independent identically distributed r.v. case. The random variables we consider here (in the “finite case”) are discrete and not independent.

We also give some results on the embeddability of $L(E)$ into $L_1$, when $L$ is a symmetric space and $E$ a Banach space. As a consequence of the results of [5] we deduce that for $E = L$, this embeddability implies that $L$ is equal to some $L_p$ space. For $L = L_p$, we show that $L_p$ order embeds into some $L_{1(q)}$ and $E$ embeds into $L_q$.

The fact that $L_1(l_q)$ does not embed into $L_1$ if $p > q$ (Cor. 3.2 below) is a matter of “folklore”, and was known for some time to several authors. See also [9] for a proof due to J. L. Krivine, or [3].

0. Definitions and notation. We say that a basis $\{x_i\}_{i \in N}$ of a Banach space is $C$-symmetric if for all $a_i \in \mathbb{R}$, $a_i = \pm 1$, $i \in N$, and all permutations $\pi$ of $N$,

$$\|\sum_{i \in N} a_i x_{\pi(i)}\| \leq C \|\sum_{i \in N} a_i x_i\|.$$
For simplicity of notation we refer to $1$-symmetric sequence spaces (of finite or infinite dimension) or rearrangement invariant function spaces ("i.e. spaces") over $[0, 1]$ or $[0, \infty]$ (cf. [6] for a definition and basic properties) as symmetric spaces.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $(F_{\omega})_{\omega \in \mathcal{F}}$ of Orlicz functions will be said be measurable if for each $\lambda \in \mathbb{R}_+$ the function $\omega \mapsto F_{\omega}(\lambda)$ is measurable. It is then clear that for each fixed vector $\alpha \in \mathbb{R}^N$ (resp. each fixed measurable function $f$) the function $\omega \mapsto \|\alpha\|_{F_{\omega}}$ (resp. $\omega \mapsto \|f\|_{F_{\omega}}$) is measurable.

We denote by $\|\|_{F_{\omega}}$ both the norm in $L_{F_{\omega}}$ (Orlicz sequence space) and in $L_{F_{\omega}}$ (Orlicz function space over $[0, 1]$ or $[0, \infty]$). The completion of $\mathcal{F}^N$ (resp. of the space of simple functions) for the norm $\int_{\Omega} \|d\|_{F_{\omega}} d\mathbb{P}(\omega)$ is called a "mean of Orlicz spaces".

Note that the functions $F_{\omega}$ may be degenerate, i.e. $F_{\omega}$ may be $0$ for $0 < t < t_0$ and $+\infty$ for $t_0 < t < \infty$.

An Orlicz function $F$ is said to be normalized if $F(1) = 1$, and $p$-convex ($q$-concave) if $\lambda \mapsto F(\lambda^p)$ is convex ($\lambda \mapsto F(\lambda^q)$) concave. If $N$ is a natural number, let $\mathcal{S}_N$ be the group of permutations of $\{1, \ldots, N\}$ (or, equivalently, of $N$ and fixing the numbers $n > N$) and $D_N$ the group $\{-1, 1\}^N$ (identified with a subgroup of $D = \{-1, 1\}^N$). $\text{Ave}_{\mathcal{S}_N}$ and $\text{Ave}_{D_N}$ are the natural averages on $\mathcal{S}_N$ and $D_N$.

If $a$ and $b$ are positive real numbers, we write $a \asymp b$ if $(1/\sqrt{c})b \leq a \leq cb$.

1. The Representation Theorem.

**Theorem 1.1.** There is a universal constant $K$ such that every symmetric space $X$ which is $C$-isomorphic to a subspace $Y$ of $L_1$ is $KC$-isomorphic to a mean of Orlicz spaces with 2-concave Orlicz functions.

More precisely,

$$\|f\|_{F_{\omega}} \leq M \int_{\mathcal{F}} \|f\|_{F_{\omega}} d\mathbb{P}(\tau)$$

where $(\mathcal{F}, \pi)$ is a separable probability space, $M < \infty$, and the $F_{\omega}, \tau \in \mathcal{F}$, are 2-concave normalized Orlicz functions.

In this representation, the Orlicz spaces are sequence spaces (finite or infinite) or function spaces (over $[0, 1]$ or $[0, \infty]$) according to the nature of the symmetric space $X$.

Note that by the 2-concavity of the Orlicz functions, each of the spaces appearing in the average is isomorphic (with a universal isomorphism constant) to a subspace of $L_1$, by [1].

We give the proof of Theorem 1.1 after recalling the definition of the finite case [5].

(A) The finite case. Let $(\alpha_i)_{i=1}^N$ be a C-symmetric sequence in $L_1$, we have

$$\left\| \sum_{i=1}^N \alpha_i x_i \right\| \geq \left\| \sum_{i=1}^N \alpha_i x_i \right\|_X$$

where $X$ is a (finite) sequence space with 1-symmetric basis $(x_i)_{i=1}^N$. We have

$$\left\| \sum_{i=1}^N \alpha_i x_i \right\|_X \leq \text{Ave}_{\mathcal{S}_N} \left\| \sum_{i=1}^N \alpha_i x_i \right\|_X \leq \text{Ave}_{D_N} \left\| \sum_{i=1}^N \alpha_i x_i \right\|_X$$

$$\text{Ave}_{\mathcal{S}_N} \left\| \sum_{i=1}^N \alpha_i x_i \right\|_X \leq \text{Ave}_{\mathcal{S}_N} \left\| \sum_{i=1}^N \left( \alpha_i x_i(\omega) \right)^2 \right\|_X^{1/2} \text{d}\mathbb{P}(\omega)$$

where $K_\text{H} = \sqrt{2}$ (Khintchine equivalence constant).

By Kwapien–Schutt's results we have

$$\left\| \sum_{i=1}^N \alpha_i x_i(\omega) \right\|_X^{1/2} \leq K_\text{H} \left\| \sum_{i=1}^N \left( \alpha_i x_i(\omega) \right)^2 \right\|_X^{1/2} \text{d}\mathbb{P}(\omega)$$

where $K_\text{H}$ is a universal constant and $F_{X_n}$ is a 2-concave normalized Orlicz function depending measurable on $\omega$ (cf. [5], Lemma 2.10).

In fact, if we set, as in [5], formula (2.3),

$$M_n(t) = \left\{ \left( \frac{N-1}{2N-1} \right)^{t-1} \right\}$$

it is clear that $F_{X_n}$ is a 2-concave, normalized and depends measurably on $\omega$.

Putting $\mathcal{P}_n = m_N^{-1} \left( \sum_{i=1}^N |x_i(\cdot)| \right) P$ with $m_N = \sum_{i=1}^N |x_i(\omega)| \text{d}\mathbb{P}(\omega)$ we see that $\mathcal{P}_n$ is a probability measure and

$$\left\| \sum_{i=1}^N \alpha_i x_i \right\|_{\mathcal{F}} \leq m_N \int_{\mathcal{F}} \|d\|_{F_{\omega}} \text{d}\mathbb{P}(\omega).$$

Note that

$$m_N = N^{-1} \sum_{i=1}^N \|x_i\| \geq \sum_{i=1}^N \|x_i\|_X \geq 1.$$

(B) The infinite case. Let $(\alpha_i)_{i=1}^\infty$ be an infinite C-symmetric sequence in $L_1$ and $X$ the associated 1-symmetric space C-isomorphic to $\text{span}(x_i)_{i=1}^\infty$. Let $(F_{X_{n,\omega}})$ be the measurable family of Orlicz functions associated to the vectors $x_1, \ldots, x_N$ as in (A) before. It can be viewed as a random variable whose values are Orlicz functions.

More precisely, consider the space $\ell_2$ of normalized 2-concave Orlicz
functions equipped with the topology of pointwise convergence. It is a compact metrizable space, for it is closed in $\mathbb{R}^+$ and the topology of pointwise convergence coincides on $C_c$ with the topology of compact convergence. This follows from Ascoli's theorem and the fact that $F(t) \leq t^2$ for $f \in C_c$ and $t \geq 1$.

Let $\pi_N$ be the image of $P_N$ by the map $\Omega \to C_{c1}$, $\omega \mapsto F_{N,\omega}$. We can extract a subsequence $(\pi_{N_k})_k$ which $\omega^*$-converges to a probability $\pi$ on $C_c$.

Now for each $\alpha \in \mathbb{R}^+$, the map $C_{c1} \to C$, $F \mapsto F(\alpha) = \sum_{n=0}^{\infty} F(\alpha n)$ is clearly bounded and continuous. But

$$||\alpha||_{\mathbb{R}^+} = \inf \{ \alpha : F(\alpha) \leq 1 \} = \sup \{ \alpha : F(\alpha) \geq 1 \}.$$ 

The first equality shows easily that $F \mapsto ||\alpha||_{\mathbb{R}^+}$ is l.s.c., and the second that it is u.s.c.; so this map is bounded continuous, and consequently

$$\forall \alpha \in \mathbb{R}^+: \int_{C_c} ||\alpha||_{\mathbb{R}^+} d\pi_{N_k}(F) \to \int ||\alpha||_{\mathbb{R}^+} d\pi(F) \quad \text{as} \quad k \to \infty.$$ 

If we moreover $m_{N_k} \to m$ as $k \to \infty$, it follows that

$$\int_{C_c} \left( \sum_{n=0}^{\infty} \alpha_n \right) d\pi_{N_k} \to \int_{C_c} \left( \sum_{n=0}^{\infty} \alpha_n \right) d\pi.$$ 

(C) The rearrangement invariant case. Let $X$ be a r.i. space over $[0,1]$ (resp. $[0,\infty]$) and suppose that $T$ is an embedding of $X$ into $L_1$, with $||T|| = ||T^{-1}|| = C$. We apply the same procedure as for finite sequence spaces but normalize differently the Orlicz functions in order to get Orlicz function space norms.

Let $(A_{n,k})$ be the dyadic partition of order $N$ of $[0,1]$ (resp. of $[0,\infty]$), which generates a ring $\mathcal{Q}_N$, and have

$$\sum_{n=0}^{N} \lambda_i \chi_{A_{n,k}}(x) \geq \text{Ave} \sum_{n=0}^{N} \lambda_i \chi_{A_{n,k}}(x) \text{Ave} \sum_{n=0}^{N} \lambda_i \chi_{A_{n,k}}(x),$$

where $\theta_N = 2^{-N} \sum_{n=0}^{N} \chi_{A_{n,k}}(x)$, and $1_{A_{n,k}}$ is the characteristic function of the set $A_{n,k}$.

Set $\theta_N(\omega) = F_{N,\omega}(2^{-N})$ and $\tilde{F}_{N,\omega}(t) = 2^{-N} F_{N,\omega}(\theta_N(\omega))$. Then

$$\tilde{F}_{N,\omega}(t) = 1, \quad \sum_{n=0}^{N} \lambda_i \chi_{A_{n,k}}(x) = \frac{1}{\theta_N(\omega)} \sum_{n=0}^{N} \lambda_i \chi_{A_{n,k}},$$

and the family $(\tilde{F}_{N,\omega})_{\omega}$ remains measurable, with values in $C_c$. Set $v_N = (\theta_N(\omega), P$ and $m_N = ||v_N||$; let $n_{\omega}$ be the probability on $C_c$ corresponding to $v_N/m_N$. We have

$$\int_{C_c} ||v_N||_{\mathbb{R}^+} d\pi_{N_k}(F) \to \int_{C_c} ||v_N||_{\mathbb{R}^+} d\pi(F)$$

for $f = \sum \lambda_i I_{A_{n,k}}$. The same holds for each $N \geq N_0$ when $f$ is $Q_{N_0}$ measurable.

With $f = 1_{[0,1]}$, we get

$$m_N = ||v_N||_{\mathbb{R}^+} 1,$$

so we can pass to the limit ($w^*$) along a subsequence and get

$$||f||_{L_0^0} \to \int_{C_c} ||v_N||_{\mathbb{R}^+} d\pi(F).$$

Note that the simple functions on dyadic intervals form a dense subspace of $X$. Indeed, since $X$ is isomorphic to a subspace of $L_1$, $X$ does not contain a subspace isomorphic to $c_0$. Therefore $X$ is an order continuous lattice, thus the dyadic functions are dense in $X$.

2. Symmetric sublattices of $L_1$.

Analogously to Theorem 1.1, the following theorem holds for symmetric sublattices of $L_1(q)$ or $L_1(c_0)$.

**Theorem 2.1.** A symmetric space $X$ which is C-order isomorphic to a sublattice of $L_1(c_0)$ (resp. $L_1(q)$) is KC- (resp. KC)-isomorphic to a mean of Orlicz spaces (resp. of q-concave Orlicz spaces) where $X$ is a universal constant (resp. $K_q$ depends only on $q$).

Note that a q-concave Orlicz space is in turn order isomorphic to a sublattice of $L_1(q_0)$ by [8], Th. 2.

**Proof.** (A) The finite case is treated with a suitable modification of the function $M_q$. We set in the case of $L_1(q_0)$, $q < \infty$,

$$M_q^p(t) = \left\{ \begin{array}{ll} N^q \frac{t^{q-1}}{N^q - (q-1)^{q-1}} & \text{if } 0 \leq t \leq 1 \frac{N}{N} \\
qN - (q-1)^t & \text{if } t > 1 \frac{N}{N} \end{array} \right.$$ 

and in the case of $L_1(c_0)$,

$$M_q^{0}(t) = \left\{ \begin{array}{ll} 0 & \text{if } 0 \leq t \leq 1 \frac{N}{N} \\
1-t/N & \text{if } t > 1 \frac{N}{N} \end{array} \right.$$ 

We then have the following analogue of Lemma (2.9) of [5]:

**Lemma 2.2.** Let $a_1 \geq a_2 \geq \ldots \geq a_{N+1} > 0$. Then

$$\frac{N}{3N} \sum_{i=1}^{N} \left( a_i \right)^{q_0} \leq \left( \sum_{i=1}^{N} \left( a_i \right)^{q_0} \right)^{1/q_0} \leq \frac{1}{N} \sum_{i=1}^{N} \left( a_i \right)^{q_0} \leq \frac{1}{N} \sum_{i=1}^{N} \left( a_i \right)^{q_0} \leq \frac{3}{2} \left( \sum_{i=1}^{N} \left( a_i \right)^{q_0} \right)^{1/q_0} \text{ if } q < \infty.$$
\[
\frac{1}{N} \left\| \left( a_i \right) \right\|_{\ell^q} \leq \frac{1}{N} \sum_{i=1}^{N} a_i \left\| \left( a_i \right) \right\|_{\ell^q} \quad \text{if } q = \infty.
\]

It is straightforward to verify this lemma (by adapting the proof of Lemma (2.9) of [5]).

Then (by using Ths. 1.1 and 1.2 of [5]) it is clear that
\[
\left\| \sum_{i=1}^{n} a_i x_i \right\|_{\ell^q} \leq \int \sum_{i=1}^{n} \left| a_i \right| \left\| x_i \right\|_{\ell^q} \frac{dP}{\ell^q}(\omega)
\]

\[
\sum_{i=1}^{n} \left\| a_i \right\|_{\ell^q} \frac{dP}{\ell^q}(\omega)
\]

with \(F_{N,\omega} \) deduced from \(M_{N\omega}^q\) as in the \(L_1\) case.

(B) The infinite case is obtained, in the case of \(L_1(L_p)\) sublattices, by convergence of probability measures on \(\ell^q\), the compact metrizable space of \(q\)-concave normalized Orlicz functions.

In the case of \(L_1(c_0)\) one has to consider the space \(\ell^q\) of normalized (possibly degenerate) Orlicz functions, which are left-continuous at irrational points, endowed with the topology of simple convergence on \(\ell^q\). It is easy to see that \(\ell^q\) is again a compact metrizable space. Moreover, it is clear that, for example, the functions \(F \mapsto \left\| a \right\|_{\ell^p}\) are continuous on \(\ell^q\) when \(a \in \ell^q\). So by the convergence procedure we identify the norm of \(\sum_{i=1}^{n} a_i x_i\) for \(a \in \ell^q\) and conclude by approximation. In the r.i. case one has to notice, moreover, that the simple dyadic functions are dense in r.i. spaces order isomorphic to sublattices of \(L_1(c_0)\); in fact, sublattices of \(L_1(c_0)\) are order continuous (cf. [6], Def. 1.9.6).

3. Embeddings of \(L(E)\) into \(L_1\), where \(L\) is a symmetric space. We recall the following result, given as a remark (without proof) in [11] (the proof will appear in another paper of Kwapień and Schlöti.

**Proposition 3.1.** Let \(L_1\) be the space of \(1\)-symmetric bases of the same dimension (finite or infinite). If \(L_1(F)\) is embeddable into \(L_1\) then the identity map from \(L_1\) to \(L_1(F)\) is bounded, and more precisely
\[
\left\| \text{id} : L_1 \rightarrow L_1(F) \right\| \leq \sqrt{2} \left\| d_2 \left( L_1(F) \right) \right\|
\]

where \(d_1(L_1(F)) = \inf \left\{ \left\| d(L_1(F), G) \right\| : G \text{ is a subspace of } L_1 \left( \right) \right\}\).

**Corollary 3.2.** If the space \(L_1(L_p)\) embeds into \(L_1\) then \(p \leq q\).

**Proposition 3.3.** An infinite-dimensional symmetric space \(L\) such that \(L_1(L)\) embeds (isomorphically) into \(L_1\) is order isomorphic (by the identity map) to \(L_1\) or \(L_p\) for some \(p\), \(1 \leq p \leq 2\).

**Proof.** By Krivine’s results [4] the space \(L\) contains \(L_p\) uniformly as sublattices. So \(L_p(L)\) and \(L(L)\) uniformly embed into \(L_1\) and the same is true for every symmetric sublattice \(A\) of \(L_1\). By Proposition 3.1 we see that every finite-dimensional symmetric sublattice \(A\) of \(L_1\) is order isomorphic to \(L_p\) with isomorphism constants uniformly bounded. So \(L_1\) itself is order isomorphic to an \(L_p\) (or \(L_2\) ) space.

**Corollary 3.4.** If \(L_1(L_p)\) (resp. \(L_1(L_2)\) ) embeds into \(L_1\) then \(L_1 \approx L_p\) (resp. \(L_2 \approx L_2\) ) for some \(p\), \(1 \leq p \leq 2\).

More precisely, the function \(q(t)\) is equal to \(p(t)\) at \(0\) in the case of \(L_p\), at \(+\infty\) in the case of \(L_2\) at \(0\) and \(+\infty\) in the case of \(L_1(R)\).

Now if \(L\) is a lattice and 

\[ q(L) = \inf \left\{ q : \text{ L is } q\text{-concave } = \inf \left\{ q : \text{ L has a q lower estimate} \right\} \right\} \]

\[ p(E) = \sup \left\{ p : E \text{ is of type } p \right\} \]

Then \(L_1(L_p)\) is lattically finitely representable in \(L\) (this is implicitly contained in Krivine’s paper [4]) and, by Maurey–Pisier [7], \(L_{1/2}\) is finitely representable in \(L_1\).

So \(L_1(L_1)\) is finitely representable in \(L_1\), which (by Corollary 3.2) implies \(q(L) \leq p(E)\).

Next, \(L_1(E)\) is finitely representable (hence embeddable by standard ultraproduct arguments) in \(L_1\). So by a result of N. Kallion [3], \(E\) embeds into \(L_1(E)\).

In the case where \(L\) is an Orlicz space, we can make the situation more precise.

**Proposition 3.5.** \(L_1(E)\) is embeddable into \(L_1\) iff there exists \(p\) such that \(L_1\) is order isomorphic to a sublattice of \(L_1(L_p)\) and \(E\) is embeddable into \(L_p\).

**Proof.** (of the “only if” part). If \(q(L) \leq p(E)\), choose \(q(L) \leq p < p(E)\). \(L_p\) being \(p\)-concave, by [3], Th. 2, it is order isomorphic to a sublattice of \(L_1(L_p)\), there exists a \(p\)-concave Orlicz function \(\psi\) such that \(L_p \neq L_1\). On the other hand, \(E\) is embeddable into \(L_p\) by a well-known result of H. P. Rosenthal [10].

If \(q(L) = p(E)\) let \(p\) be the common value. Note that \(E\) is embeddable into \(L_p\) and that \(L_1(E)\) is finitely representable, hence embeddable, into \(L_1\).

The same is true if we replace \(L_p\) by one of its symmetric finite-dimensional sublattices. It is a consequence of Proposition 3.1 that for each choice of \(n\) functions \(f_1, \ldots, f_n\) \(L_p\)-normalized, disjointly supported and having the same distribution,
\[
\left\| \sum_{i=1}^{n} f_i \right\| \geq K n^{1/p}
\]

where \(K\) does not depend on the \(f_i\)’s and on \(n\).
It is then an easy exercise to show that $L_0$ is in fact $p$-concave (cf. [6], proof of Prop. 2.b.5) and therefore is a sublattice of $L_1(L_0)$.

4. Open questions. It seems not clear whether convexity or concavity conditions can be translated on the representation of symmetric subspaces as means of Orlicz spaces given in Section 1. More precisely:

**Question 1.** If $L$ is a $q$-concave symmetric subspace of $L_2$, is it representable as a mean of $q$-concave Orlicz spaces?

**Question 2.** If $L$ is a $p$-convex symmetric subspace of $L_1$, is it representable as a $p$-mean of $p$-convex Orlicz functions, i.e.

$$\|\|F\|\|_p \sim \|\|F\|\|_p^\infty \text{ d} \alpha(F)?$$

These questions are equivalent to the following, which have affirmative answers in the case of Orlicz spaces:

**Question 1’.** If a $q$-concave symmetric subspace of $L_1$ order embeddable into $L_1(L_q)$?

**Question 2’.** Is a $p$-convex symmetric subspace of $L_1$ embeddable into $L_p$?

It is well known (see [10]) that a weaker form of Question 2’ has an affirmative answer: a subspace of $L_1$ with $p$-convex lattice structure is embeddable into $L_p$, for $1 < p < 2$. We do not know if, analogously, the following weakening of Question 1’ has an affirmative answer:

**Question 1”.** Is a $q$-concave ($q < 2$) symmetric subspace of $L_1$ order embeddable into $L_1(L_{q_1})$ for each $q_1$, $q < q_1 < 2$?

One can also ask for the equivalent of Proposition 3.5, for symmetric subspaces of $L_1$ in place of Orlicz spaces.

To reproduce the proof of Proposition 3.5, it would be sufficient to answer Question 1” in the affirmative, because, on the other hand, we are able to prove (by "stability" methods, in the sense of Krivine and Maurey, which notably differ from the methods we use here) that a lattice $L$ such that $L(L_q)$ embeds into $L_1$ is order embeddable into $L_1(L_q)$.

References

