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Quasi-uniform convergence in compact dynamical systems

by

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Abstract. We study the notion of quasi-uniform convergence introduced by Jacobs and Keane in their work on Toeplitz sequences. We prove that the number of minimal sets, number of ergodic measures, and topological entropy of the orbit closure do not increase under a passage to the quasi-uniform limit. Moreover, the set of invariant measures varies continuously with respect to the Hausdorff distance, and so does the topological entropy in symbolic dynamics. We also present a group construction which allows us to generate all Toeplitz sequences from a single group rotation and control their quasi-uniform distance by means of an upper Riemann integral. This gives a continuous passage between any two Toeplitz sequences on the same alphabet. Finally, we observe that the quasi-uniform limit of periodic 0-1 sequences need not be a regular Toeplitz sequence.

1. Introduction. By a (*compact*) *dynamical system* we mean a pair (X, T) where X is a compact metrizable space and T is a homeomorphism of X onto itself (see also Section 8 for the more general setting of continuous maps T). A nonempty closed subset F of X is called *invariant* if $TF = F$. By compactness, for every x in X the orbit closure

$$\bar{O}(x) = \{T^j x \colon j \in \mathbf{Z}\}^-$$

contains a minimal invariant set. If $\bar{O}(x)$ carries a unique invariant (probability) measure then the system $(\bar{O}(x), T)$ is said to be *uniquely ergodic* and x is called *strictly transitive*. If $X = A^{\mathbf{Z}}$ where $2 \leqslant |A| < \infty$ and $(Tx)(j) = x(j+1)$ then we call (X, T) a *symbolic dynamical system.*

In [5], Jacobs and Keane developed a theory of quasi-uniform convergence in compact dynamics. They proved, among other things, that the quasi-uniform limit of (trajectories of) strictly transitive points is also strictly transitive and so obtained the strict transitivity of 0-1 regular Toeplitz sequences ([5, § 3], see also [8, Theorem 2.6]). A class of dynamical systems that can be gotten as quasi-uniform limits of equicontinuous (uniformly almost periodic) trajectories has been investigated in [4].

In the present paper we study further dynamical properties of the quasi-uniform convergence. The strict transitivity result of [5] is extended by proving that the set of invariant measures for $(\bar{O}(x), T)$ varies continuously in x with respect to the quasi-uniform convergence in X and the Hausdorff distance between weak-star closed subsets of probability measures. In particular, passing to the quasi-uniform limit cannot increase the number of ergodic measures on an orbit closure (Section 4). Similar assertions hold true for the number of minimal sets (Section 3) and topological entropy (Section 5). Moreover, in symbolic dynamics the topological entropy of $\bar{O}(x)$ turns out to be quasi-uniformly continuous in x (Proposition 3). Since any shift system is proved to be quasi-uniformly pathwise connected, we have a method for producing paths of continuously varying topological entropy in symbolic dynamics (Section 6). A continuous passage is also obtained within the class of Toeplitz sequences, which turn out to be all derived from a single group rotation. Finally, we show that the class of quasi-uniform limits of periodic 0-1 sequences essentially differs from the regular Toeplitz sequences of [5] (Section 7).

2. Quasi-uniform convergence. Let (X, d) be a compact metric space. The *Weyl pseudometric* in the space of all X-valued functions defined on the group of integers $\boldsymbol{Z}$ is given by the formula

$$D_{\mathrm{W}}(f, g) = \overline{\lim_{L}} \sup_{k} L^{-1} \sum_{j=k}^{k+L-1} d(f(j), g(j)).$$

An equivalent pseudometric is defined by

$$D'_{\mathrm{W}}(f, g) = \inf\{\delta\colon\ BD^*\{j\colon\ d(f(j), g(j)) > \delta\} < \delta\}$$

where $BD^*(J) = \overline{\lim}_n \sup_k |J \cap [k, k+n)|/n$ is the upper Banach density of J in $\boldsymbol{Z}$ (cf. [5, Theorem 2]). Note that any equivalent metric on X gives rise to an equivalent Weyl pseudometric on $X^{\boldsymbol{Z}}$.

The *trajectory* of a point x in the dynamical system (X, T) is the function $\bar{x}\colon \boldsymbol{Z} \to X$ defined by

$$\bar{x}(j) = T^j x.$$

We will say that x_n *converge to* x *quasi-uniformly* if the corresponding trajectories converge in D_{W}, i.e. $D_{\mathrm{W}}(\bar{x}_n, \bar{x}) \to 0$. We note that the quasi-uniform convergence of Jacobs and Keane [5] is slightly stronger since it corresponds to the metric $D_{\mathrm{W}}(\bar{x}, \bar{y}) + d(x, y)$. It will be shown that the quasi-uniform pseudometric is in general incomplete (Proposition 2).

For any function f on X and $x \in X$ we set $f^x(j) = f(T^j x)$, so that $f^x = f \circ \bar{x}$.

PROPOSITION 1. *Let (X, T) be a dynamical system and let F be a family of continuous real-valued functions on X such that the family $\{f \circ T^j\colon f \in F,\ j \in \boldsymbol{Z}\}$ separates the points of X. Then $x_n \to x$ quasi-uniformly in X iff $D_{\mathrm{W}}(f^{x_n}, f^x) \to 0$ for every $f \in F$.*

Proof. For the necessity use the equivalent pseudometric D'_{W}. To prove the sufficiency first note that the Weyl pseudometric is invariant under translations in $\boldsymbol{Z}$ so we may assume that F itself separates points. Moreover, there is no loss of generality in assuming $|f| \le 1$ for $f \in F$. Now define an equivalent metric in X by

$$d_1(x, y) = \sum a_i |f_i(x) - f_i(y)|,$$

where $f_1, f_2, \ldots$ is a separating sequence in F, $0 < a_i \le 1$, and $\sum a_i < \infty$. For any $\varepsilon > 0$ choose m large enough to ensure $\sum_{i>m} a_i < \varepsilon/4$. The distance in the Weyl pseudometric between $\bar{x}_n$ and $\bar{x}$ evaluated with respect to d_1 equals

$$\begin{aligned}\overline{\lim_{L}} \sup_{k} L^{-1} \sum_{j=k}^{k+L-1} d_1(T^j x_n, T^j x) &\le \overline{\lim_{L}} \sup_{k} \Big(\varepsilon/2 + L^{-1} \sum_{j=k}^{k+L-1} \sum_{i \le m} |f_i(T^j x_n) - f_i(T^j x)|\Big)\\ &\le \varepsilon/2 + \sum_{i \le m} D_{\mathrm{W}}(f_i^{x_n}, f_i^x) \le \varepsilon\end{aligned}$$

for all n large enough. This implies $x_n \to x$ quasi-uniformly.

By taking $F = \{\pi_0\}$, where $\pi_0(x) = x(0)$ we obtain the following characterization of quasi-uniform convergence in symbolic dynamics.

COROLLARY 1. *In every symbolic dynamical system, $x_n \to x$ quasi-uniformly iff $BD^*\{j\colon x_n(j) \ne x(j)\} \to 0$.*

It follows that the pseudometric

$$D''_{\mathrm{W}}(\bar{x}, \bar{y}) = BD^*\{j\colon\ x(j) \ne y(j)\}$$

is equivalent to $D_{\mathrm{W}}(\bar{x}, \bar{y})$ in $A^{\boldsymbol{Z}}$.

The following example shows that the quasi-uniform limit of periodic points need not have a minimal orbit closure, which settles a question in [5], p. 126 (see, however, Theorem 1 below).

EXAMPLE 1. Let $x_n(j) = 1$ if $n|j$, $x_n(j) = 0$ otherwise, and let $x(0) = 1$, $x(j) = 0$ for $j \ne 0$. We have $D''_{\mathrm{W}}(\bar{x}_n, \bar{x}) \to 0$ as well as $x_n \to x$ coordinatewise in $\{0, 1\}^{\boldsymbol{Z}}$. On the other hand, the x_n are periodic, while the orbit closure of x is not a minimal invariant set.

3. Minimal sets. Define $m(x)$ to be the number of minimal invariant subsets of (X, T) (and let $m(x) = \infty$ if the number is infinite). In this section we prove that $m(x)$ is lower semicontinuous (l.s.c.) for the quasi-uniform convergence in X.

For any $E \subset X$ we let E^{ε} be the open ε-neighborhood of E. For a fixed x in X we put $J(E) = \{j \in \boldsymbol{Z}\colon T^j x \in E\}$. If $m < n$ are integers then we shall frequently write $[m, n)$ instead of $[m, n) \cap \boldsymbol{Z}$. A subset S of $\boldsymbol{Z}$ is said to be *syndetic* (or *relatively dense*) provided there exists a natural number L such that $S \cap [k, k+L) \neq \emptyset$ for every $k \in \mathbf{Z}$. It is well known that $\bar{O}(x)$ is minimal iff for every $\varepsilon > 0$ the set $J(\{x\}^\varepsilon)$ is syndetic (see e.g. [2], Ch. 1, § 4). We prove a similar criterion for the value of $m(x)$.

LEMMA 1. *Let $x \in X$ and $m \in N$. Then $m(x) \leqslant m$ iff there exists a set $K = \{z_1, \ldots, z_m\}$ such that for every $\varepsilon > 0$ the set $J(K^\varepsilon)$ is syndetic.*

Proof. We first prove the necessity. Suppose $m(x) \leqslant m$ and let $z_1, \ldots, z_m$ be representatives of all minimal subsets in $\bar{O}(x)$. Suppose $J(K^\varepsilon)$ is not syndetic for some $\varepsilon > 0$. Then for any $n \in N$ there exist intervals $T^{k_n - n}x, \ldots, T^{k_n + n}x$ of the orbit of x entirely contained in $X \setminus K^\varepsilon$. Now the orbit closure $\bar{O}(y)$ of any limit point y of $T^{k_n}x$ is also contained in $X \setminus K^\varepsilon$, and so is any minimal subset of $\bar{O}(y)$. This is a contradiction, because K intersects all minimal sets in $\bar{O}(x)$.

To prove the sufficiency suppose $m(x) > m$. We fix $m+1$ distinct minimal subsets $F_1, \ldots, F_{m+1}$ of $\bar{O}(x)$ and let $4\varepsilon = \min\{d(F_i, F_j): i \neq j\}$. The set K^ε is now disjoint from at least one F_i^ε. On the other hand, it is easy to see that $J(F_i^\varepsilon)$ contains arbitrarily long intervals so $J(K^\varepsilon)$ cannot be syndetic.

THEOREM 1. *The function $m(x)$ is l.s.c. with respect to the quasi-uniform convergence.*

Proof. Let $m(x) > m \in N$. We show that $m(y) > m$ in some quasi-uniform neighborhood of x. Let $F_1, \ldots, F_{m+1}$ be distinct minimal subsets of $\bar{O}(x)$ and choose $0 < \varepsilon \leqslant 3$ such that $\varepsilon \leqslant \min\{d(F_s, F_t): s \neq t\}$. We are going to prove that $m(y) > m$ for any y satisfying $D'_W(\bar{x}, \bar{y}) < \varepsilon/6$.

For any such y there exists an $n \in N$ such that

$$|\{j \in [k, k+n): d(T^j x, T^j y) > \varepsilon/6\}| < n\varepsilon/6 \leqslant n/2$$

for all $k \in \mathbf{Z}$. Choose $\delta > 0$ such that for $u, v \in X$

$$d(u, v) < 2\delta \Rightarrow d(T^j u, T^j v) < \varepsilon/3$$

whenever $j \in [0, n)$. By Lemma 1, there exists $K = \{z_1, \ldots, z_{m(y)}\}$ for which the set $J(K^\delta)$ (for the orbit of y) is syndetic (with some constant L). Fix a natural number $s \leqslant m+1$ and choose an interval $[k, k+L+n)$ in $J(F_s^{\varepsilon/6})$. Among the points $T^j y, j \in [k, k+L)$, there exists an element $u = T^r y$ of K^δ, i.e. $u \in \{z_{h(s)}\}^\delta$ for some $h(s) \leqslant m(y)$. Since $d(T^j x, T^j y) \leqslant \varepsilon/6$ for more than $n/2$ of the numbers j from $[r, r+n)$, we obtain $T^j u \in F_s^{\varepsilon/3}$ for more than a half of the numbers $j \in [0, n)$.

Now suppose $h(s) = h(t)$. This implies the existence of $v \in \{z_{h(s)}\}^\delta$ such that for more than a half of the numbers $j \in [0, n)$ we have $T^j v \in F_t^{\varepsilon/3}$. Consequently, there exists a j for which both $T^j u \in F_s^{\varepsilon/3}$ and $T^j v \in F_t^{\varepsilon/3}$. Since $d(u, v) < 2\delta$, this implies $s = t$. We have proved that the mapping $s \to h(s)$ is 1-1, which gives $m+1 \leqslant m(y)$ and the proof is complete.

The following example shows that $m(x)$ need not be continuous.

EXAMPLE 2. Let T be the shift on $\{0, 1\}^{\mathbf{Z}}$. Define $a_n = 10\ldots0$, $o_n = 0\ldots0$, both of length n. Now let

$$x_n = \ldots o_n o_n o_n a_n a_n o_n a_n a_n o_n o_n o_n \ldots$$

Each $\bar{O}(x_n)$ has at least two minimal orbits: that of $\ldots a_n a_n a_n \ldots$ and the fixed point $o = \ldots o_n o_n o_n \ldots$ Thus $m(x_n) \geqslant 2$. On the other hand, $D_W(x_n, o) \to 0$ and $m(o) = 1$.

As an application of Theorem 1 we show that in general neither $D_W(\bar{x}, \bar{y})$ nor any equivalent pseudometric on X is complete.

PROPOSITION 2. *If the shift on $\{0, 1\}^{\mathbf{Z}}$ occurs as a subsystem of (X, T) then no pseudometric D equivalent to $D_W(\bar{x}, \bar{y})$ is complete on X.*

Proof. Given D, we construct inductively a Cauchy sequence without limit points. Set $x_0 = \ldots000\ldots$ and suppose we have already defined x_{n-1} to be $\ldots b_{n-1} b_{n-1} b_{n-1} \ldots$ where b_{n-1} is a 0-1 block. We let $x_n = \ldots b_n b_n b_n \ldots$ with $b_n = b_{n-1} \ldots b_{n-1} s \ldots s$, where $s = 0$ or 1 according as n is odd or even, the terminal block $s \ldots s$ has length $|b_{n-1}|$, and b_{n-1} is repeated r_n times in b_n. By making r_n large enough we may have $D(x_{n+1}, x_n) < 2^{-n}$, so the sequence x_n is Cauchy in D. Suppose $D(x_n, y) \to 0$ for some $y \in X$. Then also $D'_W(\bar{x}_n, \bar{y}) \to 0$. In view of Theorem 1 we will arrive at a contradiction if we are able to show that $\bar{O}(y)$ contains the fixed points $\ldots000\ldots$ and $\ldots111\ldots$

Suppose e.g. $d(\ldots000\ldots, \bar{O}(y)) = \varepsilon > 0$ and let $L \in N$ be such that $d(\ldots000\ldots, z) < \varepsilon/2$ whenever $z(j) = 0$ for $|j| \leqslant L$ $(z \in \{0, 1\}^{\mathbf{Z}} \subset X)$. Choose n such that $b_n = b_{n-1} \ldots b_{n-1} 0 \ldots 0$ has the terminal block $b = 0\ldots0$ of length $|b| \geqslant 2L+1$. Clearly b appears in x_n with density of at least $|b_n|^{-1} = \delta$. Observe that the appearance of b in x_{n+1} is with density greater than $\delta(1 - r_{n+1}^{-1})$ since only one in every sequence of $r_{n+1}+1$ subsequent blocks b_n is altered. By the same token, b appears in x_m $(m > n)$ with density greater than

$$\eta = \delta \prod_{i > n} (1 - r_i^{-1}),$$

and $\eta > 0$ if the r_n grow rapidly enough. Now choose $m > n$ with

$$D'_W(\bar{x}_m, \bar{y}) < \min(\varepsilon/2, \eta).$$

Since $d(\ldots000\ldots, T^j y) \geqslant \varepsilon$, we have $d(T^j x_m, T^j y) > \varepsilon/2$ whenever b appears in $T^j x_m$ around zero (so that $x_m(j+i) = 0$ for $|i| \leqslant L$). This happens with density greater than η, contradicting the choice of m.

4. Invariant measures. Let (X, T) be a compact dynamical system. An *invariant measure* is a Borel probability measure μ on X such that $\mu = \mu \circ T^{-1}$. It is well known that invariant measures always exist and the extreme points of the convex set of all invariant measures are exactly the ergodic measures for T (see e.g. [1]). For any $x \in X$ we denote by $P(x)$ the (weak-star) compact convex set of invariant probability measures on $\bar{O}(x)$ and define $e(x)$ to be the number of ergodic measures (extreme points) in $P(x)$. By the Krein–Milman theorem we have $e(x) \geqslant 1$. We may assume that the metric ϱ in the space of all probability measures on X is given by

$$\varrho(\mu, \nu) = \sum_i 2^{-i} |\int f_i \, d\mu - \int f_i \, d\nu|,$$

where $\{f_i\}$, $|f_i| \leqslant 1$, is a suitably chosen sequence of continuous functions on X. Recall that the *Hausdorff distance* between nonempty closed subsets of a metric space with metric ϱ is defined by the formula

$$\varrho_{\mathrm{H}}(E, F) = \max\big(\sup_{x \in E} \varrho(x, F), \sup_{x \in F} \varrho(x, E)\big).$$

It was shown in [5] that if $e(x_n) = 1$ and $x_n \to x$ quasi-uniformly, then $e(x) = 1$.

THEOREM 2. *If $x_n \to x$ quasi-uniformly then $P(x_n) \to P(x)$ in ϱ_{H}.*

Proof. Given $\varepsilon > 0$ we find $m \in N$ and $0 < \delta < \varepsilon/8$ such that $2^{-m} \leqslant \varepsilon/4$ and $d(x, y) < \delta \Rightarrow |f_i(x) - f_i(y)| < \varepsilon/4$ for $i = 1, \ldots, m$. Now it suffices to show that $\varrho_{\mathrm{H}}(P(x), P(y)) < \varepsilon$ whenever $D'_{\mathrm{W}}(\bar{x}, \bar{y}) < \delta$. To this end fix any $\mu \in \operatorname{ex} P(x)$. Then

$$\mu = \lim_n L_n^{-1} \sum_{j=k_n}^{k_n+L_n-1} \delta_{T^j x}$$

for some sequences $\{k_n\}$ and $\{L_n\}$, $L_n \to \infty$ (see [2, Proposition 3.9]). Let

$$\nu = \lim_n L_n^{-1} \sum_{j=k_n}^{k_n+L_n-1} \delta_{T^j y}$$

(choose a subsequence if necessary). Obviously $\nu \in P(y)$ and we have

$$\varrho(\mu, P(y)) \leqslant \varrho(\mu, \nu) \leqslant \sum_i 2^{-i} \overline{\lim_n} L_n^{-1} \sum_{j=k_n}^{k_n+L_n-1} |f_i(T^j x) - f_i(T^j y)|$$
$$\leqslant \varepsilon/2 + \sum_{i \leqslant m} 2^{-i} \overline{\lim_n} L_n^{-1} \sum_{j=k_n}^{k_n+L_n-1} |f_i(T^j x) - f_i(T^j y)|$$
$$\leqslant \varepsilon/2 + \sum_{i \leqslant m} 2^{-i}(\varepsilon/4 + 2\delta) < \varepsilon.$$

If $\mu \in P(x)$ is arbitrary, the inequality $\varrho(\mu, P(y)) \leqslant \varepsilon$ now easily follows from the Krein–Milman theorem. By symmetry we also obtain $\varrho(\nu, P(x)) \leqslant \varepsilon$ for every $\nu \in P(y)$, which ends the proof of the theorem.

COROLLARY 2. *$e(x)$ is l.s.c. with respect to the quasi-uniform convergence.*

Proof. In view of Theorem 2 it suffices to show that the number of extreme points of a compact convex set is an l.s.c. function with respect to ϱ_{H} provided all the sets in question are contained in a common compact subset (with metric ϱ) of a locally convex space. A standard proof of this general fact is omitted.

Note that Example 2 in Section 3 shows that $e(x)$ need not be continuous.

5. Topological entropy. For every open cover $\mathscr{U}$ of X we denote by $N(\mathscr{U})$ the minimal cardinality of a subcover of $\mathscr{U}$. By

$$\bigvee_{i=0}^{n} T^{-i} \mathscr{U}$$

we denote the open cover by the sets $\bigcap_{i=0}^{n} T^{-i} U_i$, where $U_i \in \mathscr{U}$. Recall that the *topological entropy* $h(T)$ of the dynamical system (X, T) is defined as the supremum over all open covers $\mathscr{U}$ of the numbers

$$h(\mathscr{U}, T) = \lim_n n^{-1} \log N\Big(\bigvee_{i=0}^{n-1} T^{-i} \mathscr{U}\Big).$$

A subset E of X is called *(n, ε)-separated* if for any two distinct elements x, y of E we have $d(T^j x, T^j y) > \varepsilon$ for some $j \in [0, n)$. A subset F of X is said to be *(n, ε)-spanning* if for every $x \in X$ there exists $y \in F$ such that $d(T^j x, T^j y) \leqslant \varepsilon$ for all $j \in [0, n)$. If $s_n(\varepsilon)$ is the maximal cardinality of an (n, ε)-separated set and $r_n(\varepsilon)$ the minimal cardinality of an (n, ε)-spanning set then

$$h(T) = \lim_{\varepsilon \to 0} \bar{s}(\varepsilon) = \lim_{\varepsilon \to 0} \bar{r}(\varepsilon)$$

where $\bar{s}(\varepsilon) = \overline{\lim}_n n^{-1} \log s_n(\varepsilon)$, $\bar{r}(\varepsilon) = \overline{\lim}_n n^{-1} \log r_n(\varepsilon)$ (see [1] or [7]).

In this section we study a function $h(x)$ defined as the topological entropy of the system $(\bar{O}(x), T)$. First we extend the notions of separated and spanning sets.

DEFINITION. A subset E of X is *(n, ε, δ)-separated* if for any $x \neq y$ in E

$$|\{j \in [0, n) \colon d(T^j x, T^j y) > \varepsilon\}|/n > \delta.$$

A subset F of X is *(n, ε, δ)-spanning* if for every $x \in X$ there exists $y \in F$ such that

$$|\{j \in [0, n) \colon d(T^j x, T^j y) \leqslant \varepsilon\}|/n > 1 - \delta.$$

Denote by $s_n(\varepsilon, \delta)$ and $r_n(\varepsilon, \delta)$ the maximal cardinality of an (n, ε, δ)-separated set and the minimal cardinality of an (n, ε, δ)-spanning set, respectively. We define

$$\bar{s}(\varepsilon, \delta) = \overline{\lim_n}\, n^{-1} \log s_n(\varepsilon, \delta), \qquad \bar{r}(\varepsilon, \delta) = \overline{\lim_n}\, n^{-1} \log r_n(\varepsilon, \delta).$$

Note that both $\bar{s}(\varepsilon, \delta)$ and $\bar{r}(\varepsilon, \delta)$ are nonincreasing in both variables. The following lemma is straightforward (cf. [1] or [7]).

LEMMA 2. $r_n(\varepsilon, \delta) \leqslant s_n(\varepsilon, \delta) \leqslant r_n(\varepsilon/2, \delta/2)$ *for all* $n \in N$, $\varepsilon > 0$, $\delta > 0$.

LEMMA 3. $h(T) = \sup_{\varepsilon,\delta>0} \bar{s}(\varepsilon, \delta)$.

Proof. By Lemma 2, it suffices to prove

$$h(T) = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \bar{r}(\varepsilon, \delta).$$

First we show that if $\mathscr{U}$ is an open cover with Lebesgue number $\varepsilon = \varepsilon(\mathscr{U})$ (i.e. every ε-ball in X is contained in some $U \in \mathscr{U}$) then for every $0 < \delta < 1/2$

$$h(\mathscr{U}, T) \leqslant \bar{r}(\varepsilon, \delta) + \lambda(\delta) + \delta \log N(\mathscr{U}),$$

where

$$\lambda(\delta) = -\delta \log \delta - (1-\delta) \log (1-\delta) \qquad (\lambda(\delta) \to 0 \text{ as } \delta \to 0).$$

Let F be an (n, ε, δ)-spanning set of cardinality $r_n(\varepsilon, \delta)$. For any $J \subset [0, n)$ define $U(y, J) = \{x \in X\colon d(T^j x, T^j y) < \varepsilon \text{ for all } j \in J\}$. The sets $U(y, J)$ where $y \in F$ and $|J|/n > 1-\delta$ form an open cover of X and clearly each $U(y, J)$ is contained in a member of $\bigvee_{j\in J} T^{-j} \mathscr{U}$. It follows that the family

$$\mathscr{V} = \bigcup_J \Big(\{U(y, J)\colon y \in F\} \vee \bigvee_{j\notin J} T^{-j} \mathscr{U}\Big)$$

is an open cover which is a refinement of $\bigvee_{j=0}^{n-1} T^{-j} \mathscr{U}$; here $\bigcup_J$ is taken over all sets $J \subset [0, n)$ with $|J|/n > 1-\delta$, and $\bigvee_{j\notin J}$ is taken over all j in $[0, n) \setminus J$. Without loss of generality assume $|\mathscr{U}| = N(\mathscr{U})$ and $n\delta = k \in N$. Now

$$N\Big(\bigvee_{j=0}^{n-1} T^{-j} \mathscr{U}\Big) \leqslant N(\mathscr{V}) \leqslant |\{J\colon |J|/n > 1-\delta\}| \cdot |F|\, N(\mathscr{U})^{n\delta}$$
$$\leqslant \binom{n}{k} k r_n(\varepsilon, \delta)\, N(\mathscr{U})^k.$$

This implies

$$h(\mathscr{U}, T) \leqslant \bar{r}(\varepsilon, \delta) + \overline{\lim}\, n^{-1} \log \left(\binom{n}{k} k\right) + \delta \log N(\mathscr{U}).$$

By Stirling's formula, for n large enough,

$$\binom{n}{k} k \leqslant \frac{k n^{n+1/2}}{k^{k+1/2} (n-k)^{n-k+1/2}} = n^{1/2} \delta^{-\delta n + 1/2} (1-\delta)^{-(1-\delta)n - 1/2},$$

so $h(\mathscr{U}, T) \leqslant \bar{r}(\varepsilon, \delta) + \lambda(\delta) + \delta \log N(\mathscr{U})$. Consequently,

$$h(\mathscr{U}, T) \leqslant \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \bar{r}(\varepsilon, \delta) \leqslant \lim_{\varepsilon\to 0} \bar{r}(\varepsilon) = h(T).$$

For every $x \in X$ we define $s_n^x(\varepsilon, \delta)$, $\bar{s}^x(\varepsilon, \delta)$ as the numbers $s_n(\varepsilon, \delta)$, $\bar{s}(\varepsilon, \delta)$ for the dynamical system $(\bar{O}(x), T)$.

LEMMA 4. *If* $BD^* \{j\colon d(T^j x, T^j y) > \varepsilon\} < \delta$ *then* $s_n^y(\varepsilon, \delta) \geqslant s_n^x(3\varepsilon, 3\delta)$ *for all sufficiently large* n.

Proof. First choose n such that for every $k \in \boldsymbol{Z}$

$$|\{j \in [k, k+n)\colon d(T^j x, T^j y) > \varepsilon\}|/n < \delta.$$

Next let $s = s_n^x(3\varepsilon, 3\delta)$ and let $\{x_1, \ldots, x_s\}$ be an $(n, 3\varepsilon, 3\delta)$-separated subset of $\bar{O}(x)$. If η is sufficiently small and $d(x_i, z_i) < \eta$ $(i = 1, \ldots, s)$ then $\{z_1, \ldots, z_s\}$ is also $(n, 3\varepsilon, 3\delta)$-separated, so we may assume that $x_i = T^{n_i} x$ $(n_i \in \boldsymbol{Z})$. Now set $y_i = T^{n_i} y$. If $k \neq m$ then, by an elementary application of the triangle inequality,

$$|\{j \in [0, n)\colon d(T^j y_k, T^j y_m) > \varepsilon\}|/n > 3\delta - \delta - \delta = \delta.$$

The set $\{y_1, \ldots, y_s\}$ is (n, ε, δ)-separated, which implies $s_n^y(\varepsilon, \delta) \geqslant s$.

THEOREM 3. $h(x)$ *is l.s.c. with respect to the quasi-uniform convergence.*

Proof. If $h(x) > a$ then by Lemma 3 there exists $\delta > 0$ with $\bar{s}^x(3\delta, 3\delta) > a$. By Lemma 4, $D'_W(\bar{x}, \bar{y}) < \delta$ implies $\bar{s}^y(\delta, \delta) > a$ and consequently $h(y) > a$ by Lemma 3.

The following example shows that $h(x)$ need not be continuous.

EXAMPLE 3. Let T be the shift in $X = Y^{\boldsymbol{Z}}$, where Y is an infinite compact metric space. Let $y_n \neq y$ and $y_n \to y$ in Y. For every $n \in N$ there exists $x_n \in \{y, y_n\}^{\boldsymbol{Z}}$ such that $\bar{O}(x_n) = \{y, y_n\}^{\boldsymbol{Z}}$. Clearly $h(x_n) = \log 2$, the x_n converge quasi-uniformly to $x = \ldots yyy \ldots$ and $h(x) = 0$.

For symbolic dynamical systems we have a stronger result.

PROPOSITION 3. *Let* $X = A^{\boldsymbol{Z}}$ $(2 \leqslant |A| < \infty)$ *and let* T *be the shift transformation. Then* $h(x)$ *is continuous with respect to the quasi-uniform convergence.*

Proof. Let $\theta_n(x)$ denote the number of n-blocks occurring in x. If $x, y \in A^{\boldsymbol{Z}}$ and $BD^* \{j\colon x(j) \neq y(j)\} < \delta$ then, for n sufficiently large, any two corresponding n-blocks in x and y differ at $k \leqslant n\delta$ positions. This implies

$$\theta_n(y) \leqslant \theta_n(x)\binom{n}{k}|A|^k.$$

By using Stirling's formula as in the proof of Lemma 3 we obtain for n large enough

$$n^{-1}\log\theta_n(y) < n^{-1}\log\theta_n(x)+\lambda(\delta)+\delta\log|A|.$$

Parry's formula for the topological entropy of subshifts (see e.g. [1, Proposition 16.11]) gives $n^{-1}\log\theta_n(x) \to h(x)$ so $h(y) \leqslant h(x)+\lambda(\delta)+\delta\log|A|$. By symmetry, $|h(x)-h(y)| \leqslant \lambda(\delta)+\delta\log|A|$, which implies the uniform continuity of $h(x)$ with respect to the pseudometric $D''_W(\bar{x}, \bar{y})$ of Section 2.

6. Sequences generated by group rotations. Let G be a compact monothetic group with a dense cyclic subgroup with generator θ and normalized Haar measure μ. Consider the rotation $Tz = z+\theta$ of G. If (Y, d) is a compact metric space then for every function $f\colon G \to Y$ we obtain a sequence $f^0(j) = f(T^j 0)$, $j \in \boldsymbol{Z}$, which can be viewed as a point in the shift system $(Y^{\boldsymbol{Z}}, S)$. (This idea has been exploited e.g. in [6].) For any $f, g\colon G \to Y$ define

$$R(f, g) = \overline{\int} d(f(z), g(z))\,d\mu(z)$$

where $\overline{\int}$ denotes the upper Riemann integral, i.e. $\overline{\int}\varphi\,d\mu = \inf\int\psi\,d\mu$, where the infimum is taken over all continuous functions $\psi \geqslant \varphi$.

Lemma 5. $D_W(f^0, g^0) \leqslant R(f, g)$.

Proof. Let $h\colon G \to \boldsymbol{R}$ be a continuous function with $d(f(z), g(z)) \leqslant h(z)$. We have

$$D_W(f^0, g^0) = \overline{\lim_L}\sup_k L^{-1}\sum_{j=k}^{k+L-1} d(f(T^j 0), g(T^j 0)) \leqslant \lim_L\sup_k L^{-1}\sum_{j=k}^{k+L-1} h(T^j 0).$$

Since the system (G, T) is equicontinuous, the last limit exists and equals $\int h\,d\mu$, which clearly ends the proof.

Now we let $K = \{z \in \boldsymbol{C}\colon |z| = 1\}$, the unit circle, and $\theta = \exp 2\pi i\alpha$ with α irrational. Note that every element of the shift space $Y^{\boldsymbol{Z}}$ is derived in the above manner from the irrational rotation (K, T). In fact, given $x \in Y^{\boldsymbol{Z}}$, we can define f on the orbit $\{T^j 1\colon j \in \boldsymbol{Z}\}$ by $f(T^j 1) = x(j)$. We use this simple observation to prove our next result.

Proposition 4. *The shift space $Y^{\boldsymbol{Z}}$ is pathwise connected with respect to D_W.*

Proof. Without loss of generality assume diam $Y \leqslant 1$. Given $f, g\colon K \to Y$ we will construct functions f_t, $0 \leqslant t \leqslant 1$, such that $f_0 = f$, $f_1 = g$, and the mapping $t \to f_t$ is R-continuous. The assertion will then follow from Lemma 5.

Let $I(0) = \emptyset$ and for $0 < t \leqslant 1$ denote by $I(t)$ the closed arc centered at 1 with $\mu(I(t)) = t$. Now define

$$f_t(z) = \begin{cases} g(z) & \text{if } z \in I(t), \\ f(z) & \text{otherwise.} \end{cases}$$

Clearly $f_0 = f$, $f_1 = g$ and if $s, t \in [0, 1]$ then $f_s(z) = f_t(z)$ except for $z \in J = I(s) \triangle I(t)$. This implies

$$R(f_s, f_t) = \overline{\int}\chi_J(z)\,d(f_s(z), f_t(z))\,d\mu(z) \leqslant \mu(J) = |t-s|,$$

so $t \to f_t$ is continuous, which ends the proof.

The following in particular gives us a method of obtaining 0-1 sequences of any topological entropy varying continuously between 0 and log 2.

Corollary 3. *For any x, y in the symbolic dynamical system $(A^{\boldsymbol{Z}}, S)$ $(2 \leqslant |A| < \infty)$ there exists a "path" $x(t)$, $0 \leqslant t \leqslant 1$, in $A^{\boldsymbol{Z}}$ such that:*

(1) $x(0) = x$, $x(1) = y$.

(2) *$t \to h(x(t))$ is a continuous function.*

(3) *$t \to P(x(t))$ is a continuous mapping into the space of nonempty weak* closed subsets of the probability measures endowed with the Hausdorff distance.*

Proof. Combine Proposition 4, Proposition 3 and Theorem 2.

7. Toeplitz sequences. Let Y be a compact metric space. We recall some basic definitions and facts (see [5] and [8]). A sequence $\eta \in Y^{\boldsymbol{Z}}$ is called a *Toeplitz sequence* if for every $k \in \boldsymbol{Z}$ there exists $p \in \boldsymbol{N}$ such that $\eta(k) = \eta(k+jp)$, for all $j \in \boldsymbol{Z}$. In other words,

$$\bigcup_{p \in \boldsymbol{N}} \mathrm{Per}_p(\eta) = \boldsymbol{Z}$$

where $\mathrm{Per}_p(\eta) = \{j \in \boldsymbol{Z}\colon \eta(k) = \eta(j) \text{ whenever } k \equiv j \bmod p\}$. The orbit closure of a Toeplitz sequence is always minimal in the shift system $(Y^{\boldsymbol{Z}}, S)$. We set

$$d(\eta) = \sup_{p \in \boldsymbol{N}} BD^*(\mathrm{Per}_p(\eta)).$$

The Toeplitz sequence η is said to be *regular* if $d(\eta) = 1$. The orbit closure of η is then uniquely ergodic.

If p is the smallest period of the restriction $\eta|\mathrm{Per}_p(\eta)$ then p is called an *essential period* of η. For every Toeplitz sequence η there exists a sequence (p_i) of essential periods such that $p_i | p_{i+1}$ and $\bigcup_i \mathrm{Per}_{p_i}(\eta) = \boldsymbol{Z}$. Every such sequence is called a *period structure* for η ([8], p. 97). It is easy to see that $d(\eta) = \lim_i p_i^{-1} |\mathrm{Per}_{p_i}(\eta) \cap [0, p_i)|$.

Our aim is to describe Toeplitz sequences in terms of a certain 0-dimensional group rotation (cf. [6]). Next, as an application we prove that the space of Toeplitz sequences is pathwise connected with respect to D_W. In particular, within 0-1 Toeplitz sequences, we can produce "paths" of continuously varying topological entropy in the interval $[0, \log 2)$ (see [8, Cor. 5.2] for the upper bound of $\log 2$).

First consider the group of $\boldsymbol{a}$-adic integers $\Delta_{\boldsymbol{a}}$ where $\boldsymbol{a} = (q_1, q_2, \ldots)$ is a sequence of primes containing every prime infinitely many times. It is known that $\Delta_{\boldsymbol{a}}$ is a 0-dimensional compact monothetic group, and it is maximal in the sense that every 0-dimensional compact monothetic group is a continuous homomorphic image of $\Delta_{\boldsymbol{a}}$ ([3, § 25]). Clearly $\theta = (1, 0, 0, \ldots)$ is a topological generator of $\Delta_{\boldsymbol{a}}$. Let $Tz = z + \theta$ on $\Delta_{\boldsymbol{a}}$.

It should be noted that any individual Toeplitz sequence η can be easily obtained from the group $\Delta_{\boldsymbol{b}}$ where $\boldsymbol{b} = (p_1, p_2/p_1, p_3/p_2, \ldots)$ for some period structure $(p_1, p_2, \ldots)$ of η (the system $(\Delta_{\boldsymbol{b}}, T)$ is then the maximal equicontinuous factor of $\bar{O}(\eta)$; see [8, Theorem 2.2]). We are going to derive all possible Toeplitz sequences in $Y^{\boldsymbol{Z}}$ from the same group $\Delta_{\boldsymbol{a}}$, which will enable us to control the quasi-uniform distance by means of the R-distance between functions on $\Delta_{\boldsymbol{a}}$. The functions generating Toeplitz sequences will turn out to have a nowhere dense set of discontinuity points (cf. [6]).

For every $k \in \boldsymbol{N}$ denote by G_k the subgroup of $\Delta_{\boldsymbol{a}}$ consisting of all elements beginning with k zeros ($n\theta \in G_k$ iff $q_1 q_2 \ldots q_k \mid n$). Then $\mu(G_k) = (q_1 q_2 \ldots q_k)^{-1}$, where μ is the normalized Haar measure on $\Delta_{\boldsymbol{a}}$. Since G_k is open, the cosets of G_k form a partition $\mathscr{P}_k$ into $q_1 \ldots q_k$ clopen sets. For any $f \colon \Delta_{\boldsymbol{a}} \to Y$ denote by $U(f)$ the union of all sets in $\bigcup_{k \in \boldsymbol{N}} \mathscr{P}_k$ on which f is constant. The following proposition is similar to a characterization in Markley [6].

PROPOSITION 5. *Let $f \colon \Delta_{\boldsymbol{a}} \to Y$. If $U(f) \supset \{j\theta \colon j \in \boldsymbol{Z}\}$ then $\eta(j) = f(j\theta)$ is a Toeplitz sequence. Conversely, if η is a Toeplitz sequence in $Y^{\boldsymbol{Z}}$ then there exists $f \colon \Delta_{\boldsymbol{a}} \to Y$ with $U(f) \supset \{j\theta \colon j \in \boldsymbol{Z}\}$ and $\eta(j) = f(j\theta)$.*

Proof. If $U(f) \supset \{j\theta \colon j \in \boldsymbol{Z}\}$ then for given $m \in \boldsymbol{Z}$, $f(x) = f(m\theta)$ on some coset $m\theta + G_k$. Let $p = q_1 \ldots q_k$ and $j \in \boldsymbol{Z}$. Since $jp\theta \in G_k$, we have $\eta(m+jp) = f\big((m+jp)\theta\big) = f(m\theta) = \eta(m)$, whence η is Toeplitz.

Now, let η be a Toeplitz sequence and let (p_k) be its period structure. First find k_1 such that $q_1 \ldots q_{k_1} = L_1 p_1$. If $y_1, \ldots, y_{r_1}$ are the symbols that occur p_1-periodically in η at positions $j_1, \ldots, j_{r_1} \pmod{p_1}$, respectively, then put $f(x) = y_i$ whenever $x \in (Lp_1 + j_i)\theta + G_{k_1}$, $i = 1, \ldots, r_1$, $L = 0, \ldots, L_1 - 1$. By continuing this process for $p_2, p_3, \ldots$ we will have defined f on a union of cosets containing $\{j\theta \colon j \in \boldsymbol{Z}\}$. Clearly $f(j\theta) = \eta(j)$ $(j \in \boldsymbol{Z})$. On the remaining part of $\Delta_{\boldsymbol{a}}$ the function is defined arbitrarily.

Remark 1. It is not hard to see that if f is as in the second part of the proof then $\mu(U(f)) = d(\eta)$, so η is regular iff $\mu(U(f)) = 1$. Moreover, if $\mu(U(g)) = 1$ then $\eta(j) = g(j\theta)$ is always a regular Toeplitz sequence.

PROPOSITION 6. *Let Y be a compact metric space. The space of all Toeplitz sequences in the shift system $Y^{\boldsymbol{Z}}$ is pathwise connected with respect to D_W.*

Proof. First embed $\Delta_{\boldsymbol{a}}$ topologically into the unit interval in the following manner. Divide $[0, 1]$ into $2q_1 - 1$ intervals of equal length and choose every other (closed) interval to form H_1. Next divide every component of H_1 into $2q_2 - 1$ equal intervals and choose every other one to form H_2. By continuing this construction we obtain a nested sequence of compact sets H_n and define $H = \bigcap H_n$. It is easy to see (as in the standard construction of the Cantor set) that H is a homeomorphic copy of $\Delta_{\boldsymbol{a}}$, with the elements $j\theta$ $(j \in \boldsymbol{Z})$ mapped into the endpoints of the components of H_k $(k \in \boldsymbol{N})$. Moreover, every G_k-coset of $\Delta_{\boldsymbol{a}}$ is now mapped into an interval in H_k. In the sequel we shall identify $\Delta_{\boldsymbol{a}}$ with its image H.

Now let $f, g \colon \Delta_{\boldsymbol{a}} \to Y$ satisfy $U(f), U(g) \supset \{j\theta \colon j \in \boldsymbol{Z}\}$. We construct a family $\{f_t \colon 0 \leqslant t \leqslant 1\}$ such that $f_t \colon \Delta_{\boldsymbol{a}} \to Y$, $U(f_t) \supset \{j\theta \colon j \in \boldsymbol{Z}\}$, $f_0 = f$, $f_1 = g$ and $R(f_s, f_t) \to 0$ as $s - t \to 0$. To this end we let $f_t(x) = g(x)$ if $x > t$ and $f_t(x) = f(x)$ if $x < t$. We put $f_t(t) = f(t)$ or $g(t)$ according as t is the left or right endpoint of an interval in some H_k; if $t \notin \{j\theta \colon j \in \boldsymbol{Z}\}$ we let $f_t(t) = f(t)$. Clearly $\{j\theta \colon j \in \boldsymbol{Z}\} \setminus \{t\} \subset U(f_t)$. If $t = j\theta$ for some j and e.g. t is the left endpoint in some H_k then $f_t(x) = f(x)$ on the G_k-coset containing t. Since $t \in U(f)$, we have $t \in U(f_t)$. We have proved that f_t generates a Toeplitz sequence (Proposition 5). Finally, if $0 \leqslant s < t \leqslant 1$ then

$$R(f_s, f_t) = \overline{\int} \chi_{[s,t] \cap \Delta_{\boldsymbol{a}}}(x)\, d\big(f_s(x), f_t(x)\big)\, d\mu(x) \leqslant (\operatorname{diam} Y) \overline{\int} \chi_{[s,t] \cap \Delta_{\boldsymbol{a}}}\, d\mu.$$

One easily verifies that $\overline{\int} \chi_{[s,t] \cap \Delta_{\boldsymbol{a}}}\, d\mu = \mu([s, t] \cap \Delta_{\boldsymbol{a}})$. Since μ is nonatomic, we obtain $R(f_s, f_t) \to 0$ as $t - s \to 0$.

Remark 2. If f and g generate regular Toeplitz sequences then the same is true of f_t, $0 < t < 1$. In fact, by construction, $\mu(U(f_t)) = 1$ if $\mu(U(f)) = \mu(U(g)) = 1$ and the statement follows by Remark 1. Therefore the space of regular Toeplitz sequences in $Y^{\boldsymbol{Z}}$ is pathwise connected with respect to D_W.

Remark 3. Proposition 6 can also be proved directly, by constructing a family $\{Z_t\}_{t \in [0,1]}$ of subsets of $\boldsymbol{Z}$ such that $Z_0 = \emptyset$, $Z_1 = \boldsymbol{Z}$, $s \leqslant t \Rightarrow Z_s \subset Z_t$, $BD^* Z_t = t$ and each Z_t is equal to the union of its periodic subsets (the construction runs inductively through dyadic rationals). If we have this family and η_0, η_1 (two Toeplitz sequences) then we let $\eta_t(n) = \eta_1(n)$ or $\eta_0(n)$ according as n belongs to Z_t or not.

Every regular Toeplitz sequence is a quasi-uniform limit of periodic sequences. In the rest of this section we show that the converse is not true even in the class of 0-1 Toeplitz sequences. First we give an example of a D_W limit of periodic sequences that has positive D_W distance from every sequence having a periodically occurring symbol.

EXAMPLE 4. Define $a_{00} = 00$, $a_{01} = 01$, $a_{02} = 10$. Let $k_0 = 3$, $k_{n+1} = \binom{k_n}{2} + 1$. Next we define two sequences of blocks a_{ni} and b_{ni}, $i = 0, \ldots, k_n - 1$, by induction on n. If $a_{n0}, \ldots, a_{n,k_n-1}$ $(n \geq 0)$ are already defined, put

$$b_{ni} = a_{ni} \ldots a_{ni} \quad (n+2 \text{ times}), \qquad i = 0, \ldots, k_n - 1,$$

$$a_{n+1,i} = b_{n,\tau_i(0)} \ldots b_{n,\tau_i(k_n-1)}, \qquad i = 0, \ldots, k_{n+1} - 1,$$

where $\tau_0 = \mathrm{id}$ and $\tau_1, \ldots, \tau_{k_{n+1}-1}$ are the transpositions of the set $\{0, \ldots, k_n - 1\}$. Write $L_n = |a_{n0}|$; note that $|b_{ni}| = (n+2) L_n$. Now let a_n $(n \geq 1)$ be the periodic sequence $\ldots a_{n0} a_{n0} a_{n0} \ldots$ with a_{n0} occurring at positions $0, \pm L_n, \ldots$, and let a_∞ be a coordinatewise limit point of the sequence (a_n) in $\{0,1\}^{\boldsymbol{Z}}$. Note that for $m > n$ the a_m (including a_∞) are made of the blocks $a_{n0}, \ldots, a_{n,k_n-1}$ occurring at positions $0, \pm L_n, \ldots$ Clearly a_{n0} and a_{ni} differ at $\leq 2|b_{n-1}|$ places, which implies $D''_W(\bar{a}_n, \bar{a}_m) \leq 2/k_{n-1}$, $D''_W(\bar{a}_n, \bar{a}_\infty) \to 0$.

Now consider any sequence b having a symbol $s \in \{0, 1\}$ repeated periodically with a period $p = n+2$ for some $n \geq 1$ (every Toeplitz sequence is such). Note that a_∞ is made of the blocks $b_{n0}, \ldots, b_{n,k_n-1}$, each b_{ni} occurring with Banach density $1/(k_n |b_{n0}|)$. Moreover, $|b_{ni}| = pL_n$, so s occurs in b at each of positions $k + j|b_{n0}|$ for some $0 \leq k < |b_{n0}|$ and every $j \in \boldsymbol{Z}$. On the other hand, it is easily seen from the construction of a_∞ that both 0 and 1 appear as the kth symbol in some b_{ni}. This implies $D''_W(\bar{a}_\infty, \bar{b}) \geq 1/(pL_n k_n)$.

Remark 4. By modifying the above construction it is possible to obtain a 0-1 Toeplitz sequence which is the quasi-uniform limit of periodic sequences without being regular. In fact, it suffices to adjoin b_{n0} to each $a_{n+1,i}$ (for all n large enough, say $n \geq n_0$) as a new initial and terminal block, so now

$$a'_{n+1,i} = b_{n0} b_{n,\tau_i(0)} \ldots b_{n,\tau_i(k_n-1)} b_{n0}.$$

Since $|b_{n0}| \to \infty$, we see that each symbol in the limit sequence η occurs periodically, so η is a Toeplitz sequence. Moreover, it is not hard to see that for $n \geq n_0$ the L_n-periodic part of η has density less than $\sum_{n \geq n_0} 2/k_n$. Since, as before, for every p there exists n with $p|L_n$, we obtain $d(\eta) \leq \sum_{n \geq n_0} 2/k_n < 1$ for n_0 sufficiently large.

8. Continuous mappings. If T: $X \to X$ is a continuous mapping (not necessarily homeomorphic) then the closed orbit $\bar{O}_+(x)$ of $x \in X$ is defined as the closure in X of the set $\{x, Tx, T^2 x, \ldots\}$. If the upper Banach density is only considered on the subsets of the nonnegative integers rather than $\boldsymbol{Z}$, then we can define a corresponding pseudometric D_W^+ on X as in Section 2. Now, just as in Sections 3–5 we define $m_+(x)$, $P_+(x)$, $e_+(x)$, and $h_+(x)$ to be the number of minimal sets, the set of invariant measures, the number of ergodic measures, and the topological entropy of $\bar{O}_+(x)$, respectively. Obviously, if T is a homeomorphism we have $D_W^+ \leq D_W$, $m_+ \leq m$, $P_+ \subset P$, $e_+ \leq e$ and $h_+ \leq h$.

All the results of this paper remain valid for continuous mappings with D_W, m, P, e, h replaced by D_W^+, m_+, P_+, e_+ and h_+, respectively. The proofs are essentially the same (for a general reference on topological entropy of continuous mappings see e.g. [7, Ch. 5]). Proposition 3 and all the examples are valid for the unilateral shift on $A^{\boldsymbol{N}}$.

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