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Analytic stochastic processes

by

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Abstract. A concept of analytic stochastic process with respect to a given Brownian motion is introduced. In terms of the random Fourier transform a relationship between analytic processes and some classes of entire functions is established.

1. Preliminaries and notation. Throughout this paper $\boldsymbol{R}$ and $\boldsymbol{C}$ will denote the real and the complex field respectively. A seminorm induced by a Hermitian bilinear form on a linear space over $\boldsymbol{C}$ will be called a *Hermitian seminorm*. Let $T \in (0, \infty]$. We shall be concerned with locally convex complete topological linear spaces $\mathscr{X}$ with the topology defined by a separating family $\{p_t\colon t \in (0, T)\}$ of Hermitian seminorms fulfilling the following condition: for every pair $t, u \in (0, T)$, $t < u$, there exists a positive number $c = c(t, u)$ such that $p_t \leqslant c p_u$. It is evident that each countable system p_{t_n} with $t_n \to T$ determines the same topology in $\mathscr{X}$. Hence it follows that $\mathscr{X}$ is a B_0-space ([6], p. 59). It is convenient to have a term for such a space $\mathscr{X}$ with a fixed family $\{p_t\colon t \in (0, T)\}$. There is no standard term for this, but we shall say in this paper that $\mathscr{X}$ is a *local Hilbert space*. Two local Hilbert spaces $\mathscr{X}$ and $\mathscr{X}'$ with the families of seminorms $\{p_t\colon t \in (0, T)\}$ and $\{p'_t\colon t \in (0, T')\}$ respectively are said to be *isomorphic* if $T = T'$ and there exists a linear map l from $\mathscr{X}$ onto $\mathscr{X}'$ such that $p_t(x) = p'_t(l(x))$ for all $x \in \mathscr{X}$ and $t \in (0, T)$.

Let $\mathscr{X}_n$ $(n = 1, 2, \ldots)$ be a sequence of local Hilbert spaces with the families $\{p_{n,t}\colon t \in (0, T)\}$ of seminorms respectively. Moreover, we assume that for every pair $t, u \in (0, T)$, $t < u$, there exists a positive number $c = c(t, u)$ such that $p_{n,t} \leqslant c p_{n,u}$ for all n. The orthogonal sum $\bigoplus_{n=1}^{\infty} \mathscr{X}_n$ is defined as the set of all sequences $x = \{x_n\}$ where $x_n \in \mathscr{X}_n$ and $\sum_{n=1}^{\infty} p_{n,t}^2(x_n) < \infty$ $(t \in (0, T))$ with addition and scalar multiplication defined coordinatewise and the topology determined by the Hermitian seminorms

$$p_t(x) = \Big(\sum_{n=1}^{\infty} p_{n,t}^2(x_n)\Big)^{1/2} \qquad (t \in (0, T)).$$

It is clear that this orthogonal sum is also a local Hilbert space.

Given $n \geqslant 1$ and $t > 0$ we put

$$\Delta_n(t) = \{(t_1, \ldots, t_n)\colon 0 \leqslant t_1 \leqslant \ldots \leqslant t_n < t\}.$$

The space $L^2_{n,T}$ consists of all Borel complex-valued functions f on $\Delta_n(T)$ with finite Hermitian seminorms

(1.1) $$q_{n,t}(f) = \Big(t^{-1} \int\limits_{\Delta_n(t)} |f(t_1, \ldots, t_n)|^2 dt_1 \ldots dt_n\Big)^{1/2} \quad (t \in (0, T)).$$

It is evident that $L^2_{n,T}$ is a local Hilbert space and for $t < u$

$$q_{n,t} \leqslant (u/t)^{1/2} q_{n,u} \quad (n = 1, 2, \ldots).$$

Let (Ω, B_Ω, P) be a probability space. Throughout this paper $W = W(t, \omega)$ $(\omega \in \Omega)$ will denote the standard Brownian motion on the half-line $[0, \infty)$. Let $B^W_\Omega(s)$ be the σ-field generated by the random variables $W(t, \omega)$ $(0 \leqslant t \leqslant s)$ and B_s the σ-field of Borel subsets of the interval $[0, s]$. The space $\mathscr{M}_T$ consists of all complex-valued stochastic processes $X = X(t, \omega)$ $(t \in [0, T), \omega \in \Omega)$ such that for every $s \in [0, T)$ the function of two variables $X(t, \omega)$ is $B_s \times B^W_\Omega(s)$-measurable on $[0, s] \times \Omega$ and the Hermitian seminorms

(1.2) $$\|X\|_t = \Big(t^{-1} \int\limits_0^t \int\limits_\Omega |X(u, \omega)|^2 P(d\omega)\, du\Big)^{1/2} \quad (t \in (0, T))$$

are finite. Here we identify two stochastic processes X and Y whenever $\|X - Y\|_t = 0$ for all $t \in (0, T)$. It is evident that $\mathscr{M}_T$ is a local Hilbert space. In the sequel we shall use the notation

$$(X, Y)_t = t^{-1} \int\limits_0^t \int\limits_\Omega X(u, \omega)\, \bar{Y}(u, \omega)\, P(d\omega)\, du \quad (t \in (0, T)).$$

It is known that for every $X \in \mathscr{M}_T$ the *Itô integral*

$$(IX)(t, \omega) = \int\limits_0^t X(u, \omega)\, dW(u, \omega) \quad (t \in [0, T))$$

is well defined and the function of two variables $(IX)(t, \omega)$ is $B_s \times B^W_\Omega(s)$-measurable on $[0, s] \times \Omega$ for every $s \in [0, T)$ ([4], Ch. 4.2). Moreover, for all $X, Y \in \mathscr{M}_T$ we have the equalities

(1.3) $$\int\limits_\Omega (IX)(u, \omega)\, P(d\omega) = 0,$$

(1.4) $$\int\limits_\Omega (IX)(u, \omega)\, \overline{(IY)}(u, \omega)\, P(d\omega) = \int\limits_0^u \int\limits_\Omega X(v, \omega)\, \bar{Y}(v, \omega)\, P(d\omega)\, dv$$

([5], p. 97). Hence it follows that $\|IX\|_t \leqslant t^{1/2} \|X\|_t$ $(t \in (0, T))$ and, consequently, the Itô integral I is a continuous linear operator on $\mathscr{M}_T$. Put

$$(SX)(t, \omega) = tX(t, \omega) \quad (t \in [0, T),\ X \in \mathscr{M}_T).$$

It is evident that $SX \in \mathscr{M}_T$ and $\|SX\|_t \leqslant t\|X\|_t$ for $t \in (0, T)$. Thus S is also a continuous linear operator on $\mathscr{M}_T$.

The process identically equal to a complex number c will be briefly denoted by c. Of course, $c \in \mathscr{M}_T$. Put

$$G_1(k) = S^k 1 \quad (k = 0, 1, \ldots)$$

and for $n \geqslant 2$ and every n-tuple $(k_1, \ldots, k_n)$ of nonnegative integers

$$G_n(k_1, \ldots, k_n) = S^{k_n} I S^{k_{n-1}} I \ldots S^{k_2} I S^{k_1} 1.$$

Evidently $G_n(k_1, \ldots, k_n) \in \mathscr{M}_T$,

(1.5) $$IG(k_1, \ldots, k_n) = G_{n+1}(k_1, \ldots, k_n, 0),$$

(1.6) $$SG_n(k_1, \ldots, k_n) = G_n(k_1, \ldots, k_{n-1}, k_n + 1)$$

and, by (1.4), for $n \leqslant m$

(1.7) $$(G_n(k_1, \ldots, k_n), G_m(l_1, \ldots, l_m))_t = t^{-1} \delta^m_n \int\limits_{\Delta_n(t)} t_1^{k_1 + l_1} \ldots t_n^{k_n + l_n} dt_1 \ldots dt_n.$$

Denote by $\mathscr{M}^{(n)}_T$ the subspace of $\mathscr{M}_T$ spanned by the processes $G_n(k_1, \ldots, k_n)$ $(k_j = 0, 1, \ldots;\ j = 1, \ldots, n)$. From (1.7) it follows that $(X, Y)_t = 0$ for all $t \in (0, T)$ whenever $X \in \mathscr{M}^{(n)}_T$, $Y \in \mathscr{M}^{(m)}_T$ and $n \neq m$. The remarkable theorem of Wiener on homogeneous chaos ([11], [8], p. 407) can be formulated as follows:

(1.8) $$\mathscr{M}_T = \bigoplus_{n=1}^{\infty} \mathscr{M}^{(n)}_T.$$

Since $G_1(k)(t, \omega) = t^k$ we can deduce from (1.1), (1.2) and (1.7) that $X \in \mathscr{M}^{(1)}_T$ if and only if $X(t, \omega) = f(t)$ where $f \in L^2_{1,T}$. Moreover, in this case we have the equality $\|X\|_t = q_{1,t}(f)$ $(t \in (0, T))$. Thus $\mathscr{M}^{(1)}_T = L^2_{1,T}$. In order to describe the subspaces $\mathscr{M}^{(n)}_T$ for $n \geqslant 2$ we must define the *iterated Itô integral*

$$(I^{(n-1)} f)(t, \omega) = \int\limits_{\Delta_{n-1}(t)} f(u_1, \ldots, u_{n-1}, t)\, dW(u_1, \omega) \ldots dW(u_{n-1}, \omega)$$

for all $f \in L^2_{n,T}$. First we define it on monomials $g(t_1, \ldots, t_n) = t_1^{k_1} \ldots t_n^{k_n}$ by setting $I^{(n-1)} g = G_n(k_1, \ldots, k_n)$. Using (1.1), (1.2) and (1.7) we extend this mapping by linearity to an isomorphism from $L^2_{n,T}$ onto $\mathscr{M}^{(n)}_T$. The inverse mapping and (1.8) define the projection Π_n from $\mathscr{M}_T$ onto $L^2_{n,T}$ with the following properties:

(1.9) $$\|X\|^2_t = \sum_{n=1}^{\infty} q^2_{n,t}(\Pi_n X) \quad (t \in (0, T)),$$

$\Pi_n \mathscr{M}^{(m)}_T = \{0\}$ if $n \neq m$, Π_1 is the identity operator on $\mathscr{M}^{(1)}_T$ and

(1.10) $$\Pi_n I^{(n-1)} f = f$$

if $f \in L^2_{n,T}$ and $n \geqslant 2$. In other words, the local Hilbert spaces $\mathscr{M}_T$ and $\bigoplus_{n=1}^{\infty} L^2_{n,T}$ are isomorphic. Moreover, by (1.5) and (1.6), for $X \in \mathscr{M}_T$ we have the equalities

$$(\Pi_1 IX)(t_1) = 0 \quad (t_1 \in [0, T)) \tag{1.11}$$

and for $(t_1, \ldots, t_n) \in \Delta_n(T)$

$$(\Pi_n IX)(t_1, \ldots, t_n) = (\Pi_{n-1} X)(t_1, \ldots, t_{n-1}) \quad (n \geq 2), \tag{1.12}$$

$$(\Pi_n SX)(t_1, \ldots, t_n) = t_n(\Pi_n X)(t_1, \ldots, t_n) \quad (n \geq 1). \tag{1.13}$$

Suppose that a subspace $\mathcal{Y}$ of $\mathcal{M}_T$ contains 1 and is invariant under the operator S. Then $\mathcal{Y}$ contains all polynomials and, by the equality $\mathcal{M}_T^{(1)} = L_{1,T}^2$, we have the inclusion $\mathcal{M}_T^{(1)} \subset \mathcal{Y}$. Assuming in addition that $\mathcal{Y}$ is invariant under the operator I we prove inductively, by virtue of (1.5) and (1.6), that $\mathcal{M}_T^{(n)} \subset \mathcal{Y}$ $(n = 1, 2, \ldots)$, which, by (1.8), yields $\mathcal{Y} = \mathcal{M}_T$. Thus we have the following statement.

PROPOSITION 1.1. *A subspace of $\mathcal{M}_T$ containing* 1 *and invariant under both operators I and S coincides with $\mathcal{M}_T$.*

The Hermite polynomials $h_n(t, x)$ $(n = 0, 1, \ldots)$ of two variables are uniquely determined by the generating function

$$\sum_{n=0}^{\infty} c^n h_n(t, x) = \exp(cx - \tfrac{1}{2}c^2 t). \tag{1.14}$$

By standard calculation we get the following formulae:

$$h_n(t, x) h_m(t, x) = \sum_{k=0}^{n} \binom{n+m-2k}{n-k} \frac{t^k}{k!} h_{n+m-2k}(t, x) \quad (n \leq m), \tag{1.15}$$

$$(2\pi t)^{-1/2} \int_{-\infty}^{\infty} h_n(t, x) h_m(t, x) \exp\left(-\frac{x^2}{2t}\right) dx = \delta_m^n \frac{t^n}{n!}. \tag{1.16}$$

The stochastic processes defined by the formula

$$H_n(t, \omega) = h_n(t, W(t, \omega)) \quad (n = 0, 1, \ldots)$$

are called the *Hermite processes*. They play an important role in stochastic analysis. By (1.16) we have the formula

$$\int_{\Omega} H_n(t, \omega) H_m(t, \omega) P(d\omega) = \delta_m^n \frac{t^n}{n!}, \tag{1.17}$$

which yields

$$(H_n, H_m)_t = \delta_m^n \frac{t^n}{(n+1)!} \quad (t \in (0, \infty)). \tag{1.18}$$

Consequently, $H_n \in \mathcal{M}_T$ for every $T \in (0, \infty]$ and $n = 0, 1, \ldots$ Since $H_0 = 1$

and

$$IH_n = H_{n+1} \quad (n = 0, 1, \ldots) \tag{1.19}$$

([5], Ch. 2.7), we have $H_n = I^n 1 = G_{n+1}(0, \ldots, 0)$. Consequently,

$$H_n \in \mathcal{M}_T^{(n+1)} \quad (n = 0, 1, \ldots) \tag{1.20}$$

and, by (1.11) and (1.12),

$$(\Pi_n H_k)(t_1, \ldots, t_n) = \delta_{k+1}^n \quad (n = 1, 2, \ldots; k = 0, 1, \ldots) \tag{1.21}$$

for $(t_1, \ldots, t_n) \in \Delta_n(T)$.

2. Processes synchronically connected with the Brownian motion. Let N_T be the space of all Borel complex-valued functions f of two variables defined in the strip $[0, T) \times \boldsymbol{R}$ with finite Hermitian seminorms

$$q_t(f) = \left(t^{-1} \int_0^t \int_{-\infty}^{\infty} |f(u, x)|^2 (2\pi u)^{-1/2} \exp\left(-\frac{x^2}{2u}\right) dx\, du\right)^{1/2} \quad (t \in (0, T)). \tag{2.1}$$

It is clear that N_T is a local Hilbert space. Moreover, it is easy to check that the functions $t^k h_n(t, x)$ $(k, n = 0, 1, \ldots)$ belong to N_T.

LEMMA 2.1. *The linear span of the functions $t^k h_n(t, x)$ $(k, n = 0, 1, \ldots)$ is dense in N_T.*

Proof. By (1.16) we have the formula

$$q_t^2(t^k h_n(t, x)) = \frac{t^{n+2k+1}}{n!(n+2k+1)} \quad (t \in (0, T)).$$

Hence it follows that for every $c \in \boldsymbol{C}$ the series $\sum_{n=0}^{\infty} c^n t^k h_n(t, x)$ is convergent in N_T. By (1.14) its sum $t^k \exp(cx - \frac{1}{2}c^2 t)$ belongs to the closure of the linear span of the functions $t^k h_n(t, x)$ $(n = 0, 1, \ldots)$.

Suppose that a continuous linear functional L on N_T vanishes on the functions $t^k h_n(t, x)$ $(k, n = 0, 1, \ldots)$. Introducing the notation $g_{k,a}(t, x) = t^k \exp(iax + \frac{1}{2}a^2 t)$ $(k = 0, 1, \ldots;\ a \in \boldsymbol{R})$ we have the equality

$$L(g_{k,a}) = 0 \quad (k = 0, 1, \ldots;\ a \in \boldsymbol{R}). \tag{2.2}$$

Since N_T is a B_0-space, we infer, by the Mazur–Orlicz Theorem ([6], p. 119), that the functional L is of the form

$$L(f) = \int_0^v \int_{-\infty}^{\infty} l(u, x) f(u, x) (2\pi u)^{-1/2} \exp\left(-\frac{x^2}{2u}\right) dx\, du,$$

where $v \in (0, T)$, l is a Borel complex-valued function in the strip $[0, v] \times \boldsymbol{R}$ and $q_v(l) < \infty$. Setting

$$g(t, x) = l(t, x) \exp\left(-\frac{x^2}{2t}\right) \quad (t \in [0, v],\ x \in \boldsymbol{R})$$

we infer that $\int_{-\infty}^{\infty}|g(t, x)|\,dx < \infty$ for almost all $t \in [0, v]$ and

$$L(g_{k,a}) = \int_0^v u^k (2\pi u)^{-1/2} \exp(\tfrac{1}{2}a^2 u) \int_{-\infty}^{\infty} e^{iax} g(u, x)\,dx\,du$$

which, by (2.2), yields $g = 0$ almost everywhere in the strip $[0, v] \times \boldsymbol{R}$. Consequently, $L = 0$ on N_T, which completes the proof.

Lemma 2.2. *If $f \in N_T$ and $g(t, x) = \int_0^x f(t, y)\,dy$, then $g \in N_T$.*

Proof. First we observe that the finiteness of the seminorms (2.1) yields the finiteness of the integral $\int_0^x |f(u, y)|\,dy$ for almost all $u \in [0, T)$ and all $x \in \boldsymbol{R}$. Thus the function g is well defined in the strip $[0, T) \times \boldsymbol{R}$. Applying the Schwarz inequality we get

$$|g(u, x)|^2 \leqslant x \int_0^x |f(u, y)|^2\,dy \qquad (u \in [0, T),\ x \in \boldsymbol{R}).$$

Consequently, for every positive number b we have the inequality

$$\int_{-b}^{b} |g(u, x)|^2 \exp\left(-\frac{x^2}{2u}\right) dx \leqslant \int_{-b}^{b} x \int_0^x |f(u, y)|^2\,dy \exp\left(-\frac{x^2}{2u}\right) dx.$$

Integrating by parts we finally obtain

$$\int_{-b}^{b} |g(u, x)|^2 \exp\left(-\frac{x^2}{2u}\right) dx \leqslant u \int_{-b}^{b} |f(u, x)|^2 \exp\left(-\frac{x^2}{2u}\right) dx.$$

The above inequality yields

$$q_t^2(g) \leqslant t q_t^2(f) \qquad (t \in (0, T)),$$

which completes the proof.

A stochastic process X from $\mathcal{M}_T$ is said to be *synchronically connected with the Brownian motion W* if it is of the form $X(t, \omega) = f(t, W(t, \omega))$, where f is a Borel complex-valued function defined in the strip $[0, T) \times \boldsymbol{R}$. The set of all processes synchronically connected with W will be denoted by $\mathcal{N}_T$.

Proposition 2.1. *$\mathcal{N}_T$ is a subspace of $\mathcal{M}_T$. The map $f \to f(t, W(t, \omega))$ is an isomorphism between local Hilbert spaces N_T and $\mathcal{N}_T$.*

Proof. Suppose that $X(t, \omega) = f(t, W(t, \omega))$. Then, by (1.2) and (2.1), $\|X\|_t = q_t(f)$. Consequently, $X \in \mathcal{N}_T$ if and only if $f \in N_T$. Since N_T is a local Hilbert space, we conclude that $\mathcal{N}_T$ is a subspace of $\mathcal{M}_T$ and the map $f \to f(t, W(t, \omega))$ is an isomorphism from N_T onto $\mathcal{N}_T$.

Theorem 2.1. *The following conditions are equivalent*:

(i) $X \in \mathcal{N}_T$.

(ii) $(\Pi_n X)(t_1, \ldots, t_n) = g_n(t_n)$ $(n = 1, 2, \ldots)$.

(iii) $X(t, \omega) = \sum_{n=0}^{\infty} f_n(t) H_n(t, \omega)$ *where*

$$\|X\|_t^2 = t^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t |f_n(u)|^2 u^n\,du < \infty \qquad (t \in (0, T)).$$

The functions f_n and g_n are connected by the relation $f_n = g_{n+1}$ $(n = 0, 1, \ldots)$.

Proof. (i) $\Rightarrow$ (ii). Put

$$X_{k,m}(t, \omega) = t^k H_m(t, \omega) = t^k h_m(t, W(t, \omega)) \qquad (k, m = 0, 1, \ldots).$$

By Lemma 2.1 and Proposition 2.1 the linear span of $X_{k,m}$ $(k, m = 0, 1, \ldots)$ is dense in $\mathcal{N}_T$. Consequently, it suffices to prove (ii) for the processes $X_{k,m}$ only. Using (1.13) and (1.21) we have

$$(\Pi_n X_{k,m})(t_1, \ldots, t_n) = \delta_{m+1}^n t_n^k \qquad (n = 1, 2, \ldots;\ k, m = 0, 1, \ldots),$$

which completes the proof.

(ii) $\Rightarrow$ (iii). Suppose that the projections $\Pi_n X$ are of the form (ii). Then, by the definition of the iterated Itô integral we have the formula $(I^{(n-1)} \Pi_n X)(t, \omega) = g_n(t)(I^{n-1} 1)(t, \omega) = g_n(t) H_{n-1}(t, \omega)$ $(n = 1, 2, \ldots)$. Consequently, $g_n(t) H_{n-1}(t, \omega)$ is the projection of X onto $\mathcal{M}_T^{(n)}$. Thus, taking into account (1.8), we get a series representation

$$X(t, \omega) = \sum_{n=1}^{\infty} g_n(t) H_{n-1}(t, \omega).$$

Finally from (1.1) and (1.9) we get the formula

$$\|X\|_t^2 = t^{-1} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_0^t |g_n(u)|^2 u^{n-1}\,du \qquad (t \in (0, T)),$$

which yields condition (iii).

(iii) $\Rightarrow$ (i). Suppose now that X has a series representation (iii). Then, by Proposition 2.1, the series $f(t, x) = \sum_{n=0}^{\infty} f_n(t) h_n(t, x)$ is convergent in N_T and $X(t, \omega) = f(t, W(t, \omega))$. The theorem is thus proved.

We conclude this section with some comments concerning pointwise multiplication of stochastic processes. A subspace $\mathcal{Y}$ of $\mathcal{M}_T$ is said to be *invariant under multiplication* if $XY \in \mathcal{Y}$ provided $X, Y \in \mathcal{Y}$ and $XY \in \mathcal{M}_T$. It is evident that the space $\mathcal{N}_T$ is invariant under multiplication. On the other hand, we have the following statement.

Proposition 2.2. *A subspace of $\mathcal{M}_T$ containing* 1 *and invariant under multiplication and Itô integration coincides with $\mathcal{M}_T$.*

Proof. Suppose that a subspace $\mathcal{Y}$ of $\mathcal{M}_T$ fulfils the above conditions. Since $H_n = I^n 1$, we infer that $H_n \in \mathcal{Y}$ $(n = 0, 1, \ldots)$. Further, from (1.15) we get the formula $t = H_1^2(t, \omega) - 2H_2(t, \omega)$. Thus $tX(t, \omega) \in \mathcal{Y}$ whenever $X \in \mathcal{Y}$. In other words, the subspace $\mathcal{Y}$ is invariant under the operator S and our assertion is an immediate consequence of Proposition 1.1.

3. Analytic processes. We begin with a simple lemma.

LEMMA 3.1. *Suppose that* $X, Y \in \mathcal{M}_T$ *and* $X = IY + c$ *where* $c \in C$. *Then for every pair* $t, u \in (0, T)$, $t < u$,

$$|c|^2 + u^{-1} t(u-t) \|Y\|_t^2 \leqslant \|X\|_u^2.$$

Proof. From (1.3) and (1.4) integrating by parts we get the equality

$$\|X\|_u^2 = |c|^2 + u^{-1} \int_0^u (u-v) \int_\Omega |Y(v, \omega)|^2 P(d\omega)\, dv.$$

Since $\int_0^u (u-v) \int_\Omega |Y(v, \omega)|^2 P(d\omega)\, dv \geqslant t(u-t)\|Y\|_t^2$, we obtain the assertion of the lemma.

In view of the above lemma we may conclude that the representation $X = IY + c$ is unique provided it exists. If this is the case the process Y will be denoted by $D_I X$ and called the *Itô derivative* of X. The domain $\mathcal{D}_T(D_I)$ of the operator D_I in $\mathcal{M}_T$ consists of all processes of the form $IY + c$ with $Y \in \mathcal{M}_T$ and $c \in C$. By Lemma 3.1 the linear map D_I from $\mathcal{D}_T(D_I)$ into $\mathcal{M}_T$ is continuous.

The *k-th Itô derivative* D_I^k $(k = 1, 2, \ldots)$ is defined inductively: $X \in \mathcal{D}_T(D_I^k)$ whenever $D_I^{k-1} X \in \mathcal{D}_T(D_I)$ and then we put $D_I^k X = D_I(D_I^{k-1} X)$. It is evident that $H_n \in \mathcal{D}_T(D_I^k)$ $(k = 1, 2, \ldots;\ n = 0, 1, \ldots)$ and, by (1.19), $D_I^k H_n = 0$ $(n = 0, 1, \ldots, k-1)$ and $D_I^k H_n = H_{n-k}$ $(n = k, k+1, \ldots)$. Moreover, one can easily prove that the domain $\mathcal{D}_T(D_I^k)$ consists of all processes of the form $I^k Y + \sum_{j=0}^{k-1} c_j H_j$ where $Y \in \mathcal{M}_T$ and $c_0, c_1, \ldots, c_{k-1} \in C$.

A process X from $\mathcal{M}_T$ is called *analytic* if it is infinitely Itô differentiable, i.e. if $X \in \bigcap_{k=1}^\infty \mathcal{D}_T(D_I^k)$. The set of all analytic processes from $\mathcal{M}_T$ will be denoted by $\mathcal{A}_T$. For two-parameter stochastic processes defined on the positive quadrant of the plane a concept of analyticity with respect to the Brownian sheet in terms of path independent integrals has been introduced and studied by R. Cairoli and J. B. Walsh in [2], [3], [9] and [10].

Various characterizations of analytic processes are given by the following theorem.

THEOREM 3.1. *The following conditions are equivalent:*

(i) $X \in \mathcal{A}_T$.

(ii) $(\Pi_n X)(t_1, \ldots, t_n) = c_n \in C$ $(n = 1, 2, \ldots)$.

(iii) $X = \sum_{n=0}^\infty a_n H_n$ *and* $\limsup_{n\to\infty} n^{-1/2} |a_n|^{1/n} \leqslant (eT)^{-1/2}$.

(iv) $X(t, \omega) = f(t, W(t, \omega))$ *where* $f \in N_T$,

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0 \quad (t \in [0, T),\ x \in \boldsymbol{R})$$

and f *can be extended to an analytic function* $f(z_1, z_2)$ *of two complex variables in the strip* $|z_1| < T$, $z_2 \in C$.

(v) $IX \in \mathcal{N}_T$.

(vi) $X \in \mathcal{N}_T \cap \mathcal{D}_T(D_I)$.

Proof. (i) $\Rightarrow$ (ii). Given an arbitrary positive integer n we have $X \in \mathcal{D}_T(D_I^n)$. Consequently, $X = I^n Y_n + \sum_{j=0}^{n-1} c_j(n) H_j$ for some $Y_n \in \mathcal{M}_T$ and $c_0(n), c_1(n), \ldots, c_{n-1}(n) \in C$. Applying formulae (1.11), (1.12) and (1.21) we get the equality

$$(\Pi_n X)(t_1, \ldots, t_n) = c_{n-1}(n),$$

which yields condition (ii).

(ii) $\Rightarrow$ (iii). By Theorem 2.1 the process X has a series representation $X = \sum_{n=0}^\infty a_n H_n$ and

$$\|X\|_t^2 = \sum_{n=0}^\infty |a_n|^2 t^n/(n+1)! < \infty \quad (t \in (0, T)). \tag{3.1}$$

In other words, the radius of convergence of the power series

$$\sum_{n=0}^\infty |a_n|^2 z^n/(n+1)!$$

is at least T, which yields the inequality

$$\limsup_{n\to\infty} n^{-1/2} |a_n|^{1/n} \leqslant (eT)^{-1/2}.$$

(iii) $\Rightarrow$ (iv). Taking the series representation (iii) of X we conclude, by the Rosenbloom–Widder Theorems ([7], Theorems 5.3 and 5.5) that the series $f(z_1, z_2) = \sum_{n=0}^\infty a_n h_n(z_1, z_2)$ converges absolutely and uniformly on every compact subset of the strip $|z_1| < T$, $z_2 \in C$. Thus f is an analytic function in this strip. Moreover, $\partial f/\partial t + \frac{1}{2}\partial^2 f/\partial x^2 = 0$. Since $X(t, \omega) = f(t, W(t, \omega))$, we obtain condition (iv).

(iv) $\Rightarrow$ (v). Taking the representation (iv) of X we put

$$h(t) = -\frac{1}{2}\frac{\partial}{\partial x} f(t, x)\Big|_{x=0}, \qquad g(t, x) = \int_0^x f(t, y)\, dy + \int_0^t h(u)\, du.$$

Then, by Lemma 2.2, $g \in N_T$. Moreover,

$$\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} = 0, \quad \frac{\partial g}{\partial x} = f.$$

Now applying the Itô formula ([5], Ch. 2.6, [4], p. 118) and Proposition 2.1 we get the formula

$$(IX)(t, \omega) = g(t, W(t, \omega)) \in \mathcal{N}_T.$$

(v) $\Rightarrow$ (ii). If $IX \in \mathcal{N}_T$, then, by part (ii) of Theorem 2.1, we have

$$(\Pi_n IX)(t_1, \ldots, t_n) = g_n(t_n) \quad (n = 1, 2, \ldots).$$

On the other hand, by (1.12),

$$(\Pi_n IX)(t_1, \ldots, t_n) = (\Pi_{n-1} X)(t_1, \ldots, t_{n-1}) \quad (n = 2, 3, \ldots).$$

Comparing these equalities we infer that the functions $(\Pi_n X)(t_1, \ldots, t_n)$ $(n = 1, 2, \ldots)$ are constant.

(iii) $\Rightarrow$ (i). Suppose that X has a series representation (iii). Put $X_k = \sum_{n=0}^{\infty} a_{n+k} H_n$ $(k = 1, 2, \ldots)$. Since

$$\limsup_{n\to\infty} n^{-1/2} |a_{n+k}|^{1/n} \leqslant (eT)^{-1/2},$$

we infer, by (3.1), that $\|X_k\|_t < \infty$ for all $t \in (0, T)$. Consequently, $X_k \in \mathcal{M}_T$. Moreover, by (1.19),

$$X = I^k X_k + \sum_{j=0}^{k-1} a_j H_j \quad (k = 1, 2, \ldots),$$

which yields $X \in \mathcal{D}_T(D_I^k)$ $(k = 1, 2, \ldots)$.

We have thus proved that conditions (i)–(v) are equivalent. It remains to consider condition (vi). If $X \in \mathcal{A}_T$, then, by (ii) and Theorem 2.1, $X \in \mathcal{N}_T$. Since $\mathcal{A}_T \subset \mathcal{D}_T(D_I)$, we get condition (vi). Conversely, suppose that $X \in \mathcal{N}_T \cap \mathcal{D}_T(D_I)$. Then $X = IY + c$, where $Y \in \mathcal{M}_T$ and $c \in C$. Hence it follows that $IY \in \mathcal{N}_T$ and, consequently, the process Y fulfils condition (ii). Using formulae (1.11) and (1.12) we conclude that the process X fulfils this condition too. The theorem is thus proved.

From part (ii) of the theorem just proved and from Theorem 2.1 it follows that $\mathcal{A}_T$ is a subspace of $\mathcal{N}_T$ and, consequently, it is also a local Hilbert space.

THEOREM 3.2. *A subset $\mathcal{B}$ of $\mathcal{A}_T$ is conditionally compact if and only if*

$$\sup\{\|X\|_t : X \in \mathcal{B}\} < \infty \tag{3.2}$$

for all $t \in (0, T)$.

Proof. The necessity of (3.2) is evident. To prove the sufficiency we introduce the notation

$$c_m(t) = \sup\Big\{\sum_{n=m}^{\infty} |a_n|^2 t^n/(n+1)!: X \in \mathcal{B}\Big\} \quad (t \in (0, T))$$

where $X = \sum_{n=0}^{\infty} a_n H_n$. It is clear that

$$c_{m+1}(t) \leqslant c_m(t) \quad (m = 0, 1, \ldots)$$

and for $t, u \in (0, T)$, $t < u$,

$$c_m(t) t^{-m} u^m \leqslant c_m(u) \quad (m = 0, 1, \ldots),$$

which yields

$$c_m(t) \leqslant u^{-m} t^m c_0(u) \quad (m = 0, 1, \ldots).$$

Since, by (3.1) and (3.2), $c_0(u) < \infty$, we have

$$\lim_{m\to\infty} c_m(t) = 0 \quad (t \in (0, T)). \tag{3.3}$$

By (3.2), for any n the coefficients a_n are uniformly bounded for $X \in \mathcal{B}$. Let $X_k \in \mathcal{B}$ and $X_k = \sum_{n=0}^{\infty} a_{n,k} H_n$ $(k = 1, 2, \ldots)$. Passing to a subsequence if necessary we may assume without loss of generality that the limits $\lim_{k\to\infty} a_{n,k} = b_n$ $(n = 0, 1, \ldots)$ exist. Evidently,

$$\sum_{n=m}^{\infty} |b_n|^2 t^n/(n+1)! \leqslant c_m(t) \quad (m = 0, 1, \ldots; t \in (0, T)).$$

Thus setting $Y = \sum_{n=0}^{\infty} b_n H_n$ we conclude, by (3.1), that $\|Y\|_t < \infty$ for all $t \in (0, T)$ and, consequently, $Y \in \mathcal{A}_T$. Given a positive integer m we have the inequality

$$\|X_k - Y\|_t^2 \leqslant \sum_{n=0}^{m-1} \frac{|a_{n,k} - b_n|^2 t^n}{(n+1)!} + 4c_m(t) \quad (t \in (0, T)),$$

which, by (3.3), yields $\|X_k - Y\|_t \to 0$ as $k \to \infty$. This proves the conditional compactness of $\mathcal{B}$.

We note that, by (3.1), the equality $\|X\|_t = 0$ for an index $t \in (0, T)$ and $X \in \mathcal{A}_T$ yields $X = 0$. Consequently, if $0 < T < U$, then the restriction of processes from $\mathcal{A}_U$ to the interval $[0, T)$ is an embedding of $\mathcal{A}_U$ into $\mathcal{A}_T$.

A representation of analytic processes appearing in part (iv) of Theorem 3.1 will be called a *continuous version* of analytic processes. Of course, the continuous version has continuous paths with probability 1.

THEOREM 3.3. *Let Δ be an infinite subset of the interval $(0, T)$ with at least one cluster point belonging to $[0, T)$. Suppose that the continuous version X from $\mathcal{A}_T$ fulfils the condition $P(\{\omega: X(t, \omega) = 0\}) > 0$ for all $t \in \Delta$. Then $X = 0$.*

Proof. Introducing the notation $X(t, \omega) = f(t, W(t, \omega))$ and $Q(t) = \{x: f(t, x) = 0\}$ where $f(z_1, z_2)$ is analytic in the strip $|z_1| < T$, $z_2 \in C$ we have the formula

$$P(\{\omega: X(t, \omega) = 0\}) = (2\pi t)^{-1} \int_{Q(t)} \exp\Big(-\frac{x^2}{2t}\Big) dx.$$

Consequently, the sets $Q(t)$ have positive Lebesgue measure for $t \in \Delta$. By the analyticity of f we conclude that $f(t, z_2) = 0$ for $t \in \Delta$ and $z_2 \in C$. Since Δ

has at least one cluster point in $[0, T)$, the function f vanishes identically in the strip $|z_1| < T$, $z_2 \in C$. Thus $X = 0$.

The following immediate consequence of formula (3.1) can be regarded as an analogue of the Liouville Theorem for entire functions.

THEOREM 3.4. *Let $X \in \mathcal{A}_\infty$. If*

$$\sup\{\|X\|_t:\ t \in (0, \infty)\} < \infty,$$

then X is a constant process.

We now proceed to the study of the multiplication of analytic processes.

THEOREM 3.5. *If $X, Y \in \mathcal{A}_T$ and $XY \in \mathcal{A}_T$, then at least one of the processes X, Y is constant.*

Proof. Taking the continuous versions

$$X(t, \omega) = f\big(t, W(t, \omega)\big), \qquad Y(t, \omega) = g\big(t, W(t, \omega)\big),$$
$$(XY)(t, \omega) = h\big(t, W(t, \omega)\big)$$

where f, g and h fulfil condition (iv) of Theorem 3.1 we infer that $h = fg$. Since

$$\frac{\partial h}{\partial t}+\frac{1}{2}\frac{\partial^2 h}{\partial x^2} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\left(\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)g+\left(\frac{\partial g}{\partial t}+\frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right)f,$$

we conclude that $(\partial f/\partial x)(\partial g/\partial x) = 0$ in the strip $[0, T)\times \mathbf{R}$. By the analyticity of f and g at least one of the functions $\partial f/\partial x$ and $\partial g/\partial x$ vanishes identically in this strip. Without loss of generality we may assume that $\partial f/\partial x = 0$ in $[0, T)\times\mathbf{R}$. Consequently, $\partial f/\partial t = -\frac{1}{2}\partial^2 f/\partial x^2 = 0$, which yields that f is constant in $[0, T)\times\mathbf{R}$. The theorem is thus proved.

THEOREM 3.6. *$\mathcal{N}_T$ is the least subspace of $\mathcal{M}_T$ containing $\mathcal{A}_T$ and invariant under multiplication.*

Proof. Let $\mathcal{Y}$ be the least subspace of $\mathcal{M}_T$ containing $\mathcal{A}_T$ and invariant under multiplication. We already know that $\mathcal{A}_T \subset \mathcal{N}_T$ and $\mathcal{N}_T$ is invariant under multiplication. Consequently, $\mathcal{Y} \subset \mathcal{N}_T$. Using formula (1.15) we have $t = H_1^2(t, \omega)-2H_2(t, \omega) \in \mathcal{Y}$, which yields $t^k H_m(t, \omega) \in \mathcal{Y}$ $(k, m = 0, 1, \ldots)$. By Lemma 2.1 and Proposition 2.1 the linear span of the processes $t^k H_m(t, \omega)$ $(k, m = 0, 1, \ldots)$ is dense in $\mathcal{N}_T$. Thus $\mathcal{N}_T \subset \mathcal{Y}$, which completes the proof.

We define an auxiliary function φ for $a, b \in (0, \infty)$ by setting

$$\varphi(a, b) = \frac{2ab}{a+b+\big((a+b)^2+12ab\big)^{1/2}}.$$

It is evident that φ is monotone nondecreasing,

$$\varphi(a, b) < \min(a, b), \qquad 3\varphi^2(a, b)+(a+b)\varphi(a, b)-ab = 0,$$

which yields

$$3t^2+(a+b)t-ab < 0 \quad \text{if } t \in \big(0, \varphi(a, b)\big). \tag{3.4}$$

Given $c \in (0, \infty)$ we put

$$V_c(t, \omega) = \sum_{n=0}^{\infty}(n+1)!\,c^{-n}H_n^2(t, \omega). \tag{3.5}$$

LEMMA 3.2. *For every pair $a, b \in (0, \infty)$ and $t \in \big(0, \varphi(a, b)\big)$ the inequality $(V_a, V_b)_t < \infty$ holds.*

Proof. Applying formula (1.15) we get, by a standard calculation,

$$V_c(u, \omega) = (1-c^{-1}u)^{-2}\sum_{k=0}^{\infty}\binom{2k}{k}(k+1)!\,(c-u)^{-k}H_{2k}(u, \omega).$$

Consequently, by (1.17), for $u \in \big(0, \min(a, b)\big)$

$$\int_\Omega V_a(u, \omega)V_b(u, \omega)P(d\omega)$$
$$= (1-a^{-1}u)^{-2}(1-b^{-1}u)^{-2}\sum_{k=0}^{\infty}\binom{2k}{k}(k+1)^2\left(\frac{u^2}{(a-u)(b-u)}\right)^k.$$

Taking $t \in \big(0, \varphi(a, b)\big)$ and setting $y = t^2/\big((a-t)(b-t)\big)$ we have, by (3.4), $0 < y < \frac{1}{4}$ and for $u \in [0, t]$

$$(V_a, V_b)_t \leqslant (1-a^{-1}t)^{-2}(1-b^{-1}t)^{-2}\sum_{k=0}^{\infty}\binom{2k}{k}(k+1)^2 y^k.$$

Since the radius of convergence of the power series

$$\sum_{k=0}^{\infty}\binom{2k}{k}(k+1)^2 z^k$$

is equal to $\frac{1}{4}$ we get the assertion of the lemma.

We extend the definition of the function φ by setting $\varphi(a, \infty) = \varphi(\infty, a) = \lim_{b\to\infty}\varphi(a, b) = a$ and $\varphi(\infty, \infty) = \infty$.

THEOREM 3.7. *Let $T, U \in (0, \infty]$. The mapping $(X, Y) \to XY$ from $\mathcal{A}_T \times \mathcal{A}_U$ into $\mathcal{N}_{\varphi(T,U)}$ is continuous.*

Proof. Suppose that $X = \sum_{n=0}^{\infty} a_n H_n \in \mathcal{A}_T$ and $Y = \sum_{n=0}^{\infty} b_n H_n \in \mathcal{A}_U$. Given $t \in \big(0, \varphi(T, U)\big)$ we can find a pair of positive numbers a and b satisfying the conditions $a < T$, $b < U$ and $t < \varphi(a, b)$. Using the Schwarz inequality and formulae (3.1) and (3.5) we have for $u\in[0, t]$

$$|X(u, \omega)|^2 = \left|\sum_{n=0}^{\infty}\frac{a_n a^{n/2}}{((n+1)!)^{1/2}}((n+1)!)^{1/2} a^{-n/2} H_n(u, \omega)\right|^2 \leqslant \|X\|_a^2 V_a(u, \omega).$$

The same calculation leads to the inequality

$$|Y(u, \omega)|^2 \leqslant \|Y\|_b^2 V_b(u, \omega).$$

Hence we get the inequality

$$\|XY\|_t^2 \leqslant \|X\|_a^2 \|Y\|_b^2 (V_a, V_b)_t,$$

which, by Lemma 3.2, yields the assertion of the theorem.

Remark. The function φ appearing in the above theorem is the best possible. This can be shown by the following example. For any $T\in(0, \infty)$, $|z_1| < T$ and $z_2\in C$ we put

$$f_T(z_1, z_2) = \left(\frac{T}{T+z_1}\right)^{1/2} \exp\left(\frac{1}{2}\frac{z_2^2}{T+z_1}\right)$$

where $z^{1/2}$ denotes the principal branch of the square root. One can easily check that f_T is analytic in the strip $|z_1| < T$, $z_2\in C$, belongs to N_T and fulfils the equation

$$\frac{\partial}{\partial t} f_T + \frac{1}{2}\frac{\partial^2}{\partial x^2} f_T = 0.$$

Consequently, by part (iv) of Theorem 3.1 the process $X_T(t, \omega) = f_T(t, W(t, \omega))$ belongs to $\mathscr{A}_T$. Given $t > \varphi(T, U)$ we get, by a standard calculation, the inequality

$$f_T(v, x) f_U(v, x) \geqslant (1+T^{-1}t)^{-1/2}(1+U^{-1}t)^{-1/2}\exp\frac{x^2}{4v}$$

whenever $\varphi(T, U) \leqslant v \leqslant t$. Hence and from (2.1) it follows immediately that $q_t(f_T f_U) = \infty$, which shows that $f_T f_U \notin N_V$ for any $V > \varphi(T, U)$. Applying Proposition 2.1 we conclude that $X_T X_U \notin \mathscr{N}_V$ for any $V > \varphi(T, U)$.

4. Random Fourier transform. We define a family of Borel probability measures λ_t $(t\in(0, \infty))$ on the complex plane by setting

$$\lambda_t(B) = -(\pi t)^{-1}\int_0^{2\pi}\int_0^{\infty} 1_B(re^{i\theta})\,\mathrm{Ei}\left(-\frac{r^2}{t}\right) r\,dr\,d\theta$$

where 1_B denotes the indicator of the set B and Ei is the *integral exponential*

function

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-y}}{y}dy \qquad (x < 0).$$

Given $T\in(0, \infty]$ by A_T we shall denote the set of all entire functions f with finite Hermitian seminorms

$$s_t(f) = \left(\int_C |f(z)|^2 \lambda_t(dz)\right)^{1/2} \qquad (t\in(0, T)).$$

If $f(z) = \sum_{n=0}^{\infty} b_n z^n$, then, by a standard calculation, we get the formula

$$s_t^2(f) = \sum_{n=0}^{\infty} n!\,|b_n|^2 (n+1)^{-1} t^n. \tag{4.1}$$

Hence it follows that $f\in A_T$ if and only if the radius of convergence of the power series $\sum_{n=0}^{\infty} n!\,|b_n|^2(n+1)^{-1} z^n$ is at least T. Using Stirling's formula we can calculate this radius to obtain the following statement.

PROPOSITION 4.1. *An entire function $\sum_{n=0}^{\infty} b_n z^n$ belongs to A_T if and only if*

$$\limsup_{n\to\infty} n^{1/2}|b_n|^{1/n} \leqslant (T^{-1}e)^{1/2}.$$

Put $m_r(f) = \max\{|f(z)|\colon |z| \leqslant r\}$. The *order* $\varrho(f)$ of an entire function f is defined by the formula

$$\varrho(f) = \limsup_{r\to\infty}\frac{\log\log m_r(f)}{\log r}.$$

A constant function has order 0, by convention. If $0 < \varrho(f) < \infty$, then the *type* $\tau(f)$ is defined by the formula

$$\tau(f) = \limsup_{r\to\infty} r^{-\varrho(f)}\log m_r(f).$$

The order and type of an entire function can be expressed in terms of the coefficients of its power series representation $f(z) = \sum_{n=0}^{\infty} b_n z^n$. Namely,

$$\varrho(f) = \limsup_{n\to\infty}\frac{n\log n}{\log(1/|b_n|)},$$

$$\tau(f) = e^{-1}\varrho(f)^{-1}\limsup_{n\to\infty} n|b_n|^{\varrho(f)/n}$$

([1], Theorems 2.2.2 and 2.2.10). Using the above formulae and Proposition 4.1 we get, by a simple calculation, the following criterion.

PROPOSITION 4.2. *An entire function f belongs to A_T if and only if either $\varrho(f) < 2$, or $\varrho(f) = 2$ and $\tau(f) \leqslant (2T)^{-1}$. Equivalently, $f\in A_T$ if and only if*

$$\limsup_{r\to\infty} r^{-2}\log m_r(f) \leqslant (2T)^{-1}.$$

Given $f(z)=\sum_{n=0}^{\infty} b_n z^n$, for any pair r, t of positive numbers we have, by (4.1) and the Schwarz inequality,

$$m_r^2(f) \leqslant \Big(\sum_{n=0}^{\infty} |b_n| r^n\Big)^2$$
$$= \Big(\sum_{n=0}^{\infty} |b_n| (n!)^{1/2} (n+1)^{-1/2} t^{n/2} \cdot (n+1)^{1/2} (n!)^{-1/2} (r^2 t^{-1})^{n/2}\Big)^2$$
$$\leqslant s_t^2(f)(1+t^{-1}r^2)\exp(t^{-1}r^2).$$

Consequently, the convergence in A_T implies the uniform convergence on every compact subset of the complex plane. Hence, in particular, it follows that the space A_T is complete. Since, by (4.1), the family s_t $(t\in(0, T))$ is monotone nondecreasing, A_T is a local Hilbert space.

From Proposition 4.2 it follows immediately that the exponential functions $e_c(z)=\exp(cz)$ $(c\in C)$ belong to every space A_T.

PROPOSITION 4.3. *Let $\{c_k\}$ be a sequence of distinct nonzero complex numbers and $\sum_{k=1}^{\infty}|c_k|^{-p}=\infty$ for an exponent $p>2$. Then the linear span of the exponential functions e_{c_k} $(k=1, 2, \ldots)$ is dense in A_T.*

Proof. Let l be a continuous linear functional on A_T vanishing on all functions e_{c_k} $(k=1, 2, \ldots)$. Since A_T is a B_0-space, we infer, by the Mazur–Orlicz Theorem ([6], p. 119), that the functional l is of the form

$$l(f)=\int_C f(z)\overline{h(z)}\,\lambda_v(dz) \tag{4.2}$$

where $v\in(0, T)$, h is an entire function and $s_v(h)<\infty$. Of course, $h\in A_v$ and, by Proposition 4.2, $\varrho(h)\leqslant 2$. Put

$$h(z)=\sum_{n=0}^{\infty} a_n z^n,\quad g(z)=\sum_{n=0}^{\infty} \bar{a}_n (n+1)^{-1} z^n.$$

Evidently, $\varrho(g)\leqslant 2$ and, by (4.2), $l(e_c)=g(cv)$ $(c\in C)$. Hence it follows that $c_k v$ $(k=1, 2, \ldots)$ are zeros of the entire function g.

Suppose that g does not vanish identically. Then taking into account the relation between the order $\varrho(g)$ and the convergence exponent of the zeros of g ([1], Theorem 2.5.18) we have the inequality $\sum_{k=1}^{\infty}|c_k|^{-q}<\infty$ for every $q>2$. But this contradicts the assumption. Consequently, $g=0$, which yields $h=0$ on C. Thus, by (4.2), the functional l vanishes on A_T, which completes the proof.

Remark. The assumption $p>2$ in the above proposition is essential. In fact, taking $T>2\pi$, $v\in(2\pi, T)$ and setting

$$h(z)=\sin(\pi v^{-2}z^2)+2\pi v^{-2}z^2\cos(\pi v^{-2}z^2)$$

we have $s_v(h)<\infty$. Evidently, the functional l determined by h in (4.2) does not vanish identically on A_T. On the other hand, setting $c_k=k^{1/2}$ $(k=1, 2, \ldots)$ we have the equality

$$l(e_{c_k})=\sum_{n=1}^{\infty}(-1)^n\big((2n+1)!\big)^{-1}(\pi k)^{2n+1}=\sin\pi k=0 \quad (k=1, 2, \ldots),$$

which shows that the linear span of e_{c_k} $(k=1, 2, \ldots)$ is not dense in A_T.

Given T, $U\in(0, \infty]$ we denote by $\psi(T, U)$ the harmonic mean of T and U, i.e.

$$\psi(T, U)^{-1}=T^{-1}+U^{-1}.$$

PROPOSITION 4.4. *Let T, $U\in(0, \infty]$. The mapping $(f, g)\to fg$ from $A_T\times A_U$ into $A_{\psi(T,U)}$ is continuous.*

Proof. Suppose that $f(z)=\sum_{n=0}^{\infty} a_n z^n\in A_T$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n\in A_U$. Given $t\in(0, \psi(T, U))$ we can find a pair a, b of positive numbers fulfilling the conditions $a<T$, $b<U$ and $t<\psi(a, b)$. From (4.1) we get the inequalities

$$|b_n|^2\leqslant(n!)(n+1)b^{-n}s_b^2(g)\quad (n=0, 1, \ldots),$$

which yield

$$\sum_{k=0}^{n}(k!)^{-1}(k+1)|b_{n-k}|^2 a^{-k}\leqslant s_b^2(g)(n!)^{-1}(n+1)^2\psi(a, b)^{-n} \quad (n=0, 1, \ldots). \tag{4.3}$$

Setting $(fg)(z)=\sum_{n=0}^{\infty} c_n z^n$, we have, by (4.1), (4.3) and the Schwarz inequality,

$$|c_n|^2=\Big|\sum_{k=0}^{n} a_k (k+1)^{-1/2}(k!)^{1/2}a^{k/2}\cdot b_{n-k}(k+1)^{1/2}(k!)^{-1/2}a^{-k/2}\Big|^2$$
$$\leqslant s_a^2(f)s_b^2(g)(n!)^{-1}(n+1)^2\psi(a, b)^{-n}\quad (n=0, 1, \ldots).$$

Applying now formula (4.1) we get the inequality

$$s_t(fg)\leqslant c s_a(f)s_b(g)$$

where $c^2=\sum_{n=0}^{\infty}(n+1)\big(\psi(a, b)^{-1}t\big)^n<\infty$, which completes the proof.

Remark. Let $T\in(0, \infty)$. Put $f_T(z)=\exp(z^2/2T)$. It is easy to check using Proposition 4.2 that $f_T\in A_T$ and $f_T\notin A_V$ for any $V>T$. Since $f_T f_U=f_{\psi(T,U)}$ we conclude that the function ψ appearing in Proposition 4.4 is the best possible.

Given $c\in C$ the *exponential process* $E(c)$ is defined as the unique solution

X in $\mathscr{A}_\infty$ of the equation $X = cIX+1$. For a continuous version of X this equation can be written in the differential form $D_I X = cX$ with the initial condition $X(0, \omega) = 1$. It is well known that

$$E(c)(t, \omega) = \exp\big(cW(t, \omega) - \tfrac{1}{2}c^2 t\big) = \sum_{n=0}^{\infty} c^n H_n(t, \omega)$$

([5], Ch. 2.7). Moreover, the mapping $c \to E(c)$ from C into $\mathscr{A}_\infty$ is continuous.

We define a Borel probability measure λ on C by setting

$$\lambda(B) = \pi^{-1} \int_0^{2\pi} \int_0^{\infty} 1_B(re^{i\theta}) e^{-r^2} r\, dr\, d\theta.$$

In what follows K_r will denote the closed disk $\{z\colon |z| \leqslant r\}$. It is clear that for every complex-valued continuous function f on K_r the Bochner integral $\int_{K_r} \overline{E(z)} f(z) \lambda(dz)$ exists in $\mathscr{A}_\infty$. Moreover, by a simple calculation, we get the formula

$$\int_{K_r} \overline{E(z)} f(z) \lambda(dz) = \sum_{n=0}^{\infty} c_n(f, r) H_n \tag{4.4}$$

where $c_n(f, r) = \int_{K_r} \bar{z}^n f(z) \lambda(dz)$ $(n = 0, 1, \ldots)$. In particular,

$$c_n(z^m, r) = \delta_n^m \int_0^{r^2} e^{-y} y^n\, dy \qquad (m, n = 0, 1, \ldots). \tag{4.5}$$

Lemma 4.1. *For every entire function f from A_T the limit*

$$\hat{f} = \lim_{r\to\infty} \int_{K_r} \overline{E(z)} f(z) \lambda(dz)$$

exists in $\mathscr{A}_T$. If $f(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\hat{f} = \sum_{n=0}^{\infty} n!\, b_n H_n.$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} b_n z^n \in A_T$. From (4.4) and (4.5) we get the formula

$$Z_r = \int_{K_r} \overline{E(z)} f(z) \lambda(dz) = \sum_{n=0}^{\infty} b_n \int_0^{r^2} e^{-y} y^n\, dy\, H_n. \tag{4.6}$$

Given $t \in (0, T)$ we have, by (3.1) and (4.1), the inequality

$$\|Z_r\|_t^2 = \sum_{n=0}^{\infty} \big((n+1)!\big)^{-1} |b_n|^2 \Big(\int_0^{r^2} e^{-y} y^n\, dy\Big)^2 t^n \leqslant s_t^2(f) \qquad (r \in (0, \infty)).$$

Consequently, by Theorem 3.2 the family Z_r $(r \in (0, \infty))$ is conditionally compact in $\mathscr{A}_T$. Moreover, by formula (4.6) we conclude that the family Z_r has exactly one cluster point $\sum_{n=0}^{\infty} n!\, b_n H_n$ as $r \to \infty$. This completes the proof.

The mapping $f \to \hat{f}$ from A_T into $\mathscr{A}_T$ is called the *random Fourier transform.*

Theorem 4.1. *The random Fourier transform is an isomorphism from A_T onto $\mathscr{A}_T$.*

Proof. By (3.1), (4.1) and the second part of Lemma 4.1 for every $f \in A_T$ we have the formula $s_t(f) = \|\hat{f}\|_t$ $(t \in (0, T))$. Consequently, to prove that the random Fourier transform is an isomorphism it suffices to show that $\mathscr{A}_T$ is its range. By Theorem 3.1 each process from $\mathscr{A}_T$ has a series representation $X = \sum_{n=0}^{\infty} a_n H_n$ where

$$\limsup_{n\to\infty} n^{-1/2} |a_n|^{1/n} \leqslant (eT)^{-1/2}.$$

Setting $b_n = (n!)^{-1} a_n$ $(n = 0, 1, \ldots)$ we have

$$\limsup_{n\to\infty} n^{1/2} |b_n|^{1/n} \leqslant (T^{-1} e)^{-1/2}.$$

Consequently, by Proposition 4.1, the function $f(z) = \sum_{n=0}^{\infty} b_n z^n$ belongs to A_T and, by Lemma 4.1, $\hat{f} = X$, which completes the proof.

Using the second part of Lemma 4.1 we get the following simple formulae: $\hat{z} = W$, $(z^n)\hat{} = n!\, H_n$, $(e_c)\hat{} = E(c)$ and

$$\Big(\frac{d}{dz} f\Big)\hat{} = D_I \hat{f}. \tag{4.7}$$

Further, as an immediate consequence of Proposition 4.3 and Theorem 4.1 we get the following statement.

Theorem 4.2. *Let $\{c_k\}$ be a sequence of distinct nonzero complex numbers and $\sum_{k=1}^{\infty} |c_k|^{-p} = \infty$ for an exponent $p > 2$. Then the linear span of the exponential processes $E(c_k)$ $(k = 1, 2, \ldots)$ is dense in $\mathscr{A}_T$.*

We note that using the random Fourier transform one can define a convolution of stochastic processes. Namely, if $X \in \mathscr{A}_T$, $Y \in \mathscr{A}_U$, $X = \hat{f}$ and $Y = \hat{g}$ where $f \in A_T$ and $g \in A_U$ then we put $X * Y = (fg)\hat{} \in \mathscr{A}_V$ provided $fg \in A_V$. As an immediate consequence of Proposition 4.4 and Theorem 4.1 we get the following result.

Theorem 4.3. *Let $T, U \in (0, \infty]$. The mapping $(X, Y) \to X * Y$ from $\mathscr{A}_T \times \mathscr{A}_U$ into $\mathscr{A}_{\psi(T,U)}$ is continuous.*

Hence in particular it follows that $\mathscr{A}_\infty$ is an algebra under convolution. Moreover, if $X \in \mathscr{A}_\infty$ and $Y \in \mathscr{A}_T$, then $X * Y \in \mathscr{A}_T$. By (4.7) we have the formulae

$$D_I(X*Y) = (D_I X)*Y + X*(D_I Y)$$

and $H_n * H_m = \binom{n+m}{n} H_{n+m}$ $(m, n = 0, 1, \ldots)$. The last formula yields $H_n = (n!)^{-1} W^{*n}$ where W^{*n} is the nth power of the Brownian motion W under $*$. Thus $E(c) = \sum_{n=0}^{\infty} (c^n/n!) W^{*n}$ and, consequently, $E(a)*E(b) = E(a+b)$. Criterion (iii) from Theorem 3.1 can be rewritten in the following form.

THEOREM 4.4. *$X \in \mathscr{A}_T$ if and only if $X = \sum_{n=0}^{\infty} c_n W^{*n}$ where* $\limsup_{n\to\infty} n^{1/2} |c_n|^{1/n} \leqslant (T^{-1} e)^{1/2}$.

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