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BMO and smooth truncation in Sobolev spaces

by

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Abstract. Let $L^{\alpha,p}$ be a Sobolev-potential space, $F_p^{\alpha q}$ an inhomogeneous Triebel–Lizorkin space, and BMO the space of functions of bounded mean oscillation. Let $R_1, \ldots, R_n$ be the Riesz transforms on $\mathbf{R}^n$. We show that for $1 < p < +\infty$, $\alpha > 0$, and $1 \leqslant q < +\infty$, $F_p^{\alpha q} \cap \mathrm{BMO} = F_p^{\alpha q} \cap L^\infty + \sum_{j=1}^n R_j(F_p^{\alpha q} \cap L^\infty)$. Using this, we show that if H is a smooth truncation operator, $\alpha > 0$, and $1 < p < +\infty$, then $H \circ f \in L^{\alpha,p}$ if $f \in L^{\alpha,p} \cap \mathrm{BMO}$. Examples of Dahlberg show that this is not the case for all $f \in L^{m,p}$ if $1 < p < n/m$.

1. Smooth truncation operators. For $1 \leqslant p < +\infty$ and $m \in \mathbf{Z}^+$, let $W^{m,p}(\mathbf{R}^n)$ be the usual Sobolev space of functions $f\colon \mathbf{R}^n \to \mathbf{R}$ such that

$$\|f\|_{W^{m,p}} = \sum_{|\beta| \leqslant m} \|\partial^\beta f\|_{L^p} < +\infty,$$

where β is a multi-index of order $|\beta|$ and $\partial^\beta f$ is a (distributional) partial derivative of f. If $H\colon \mathbf{R}^1 \to \mathbf{R}^1$ is Lipschitz and $H(0) = 0$, then it is well known that the composition operator $T_H f = H \circ f$, $f \in C_0^\infty(\mathbf{R}^n)$, extends as a bounded operator to $W^{1,p}$. The usual truncation operators may be obtained by taking H semibounded; for example, if $H(t) = t$ for $t \leqslant 1$ and $H(t) = 1$ for $t > 1$, then $T_H f = \min(f, 1)$. However, to obtain $T_H f \in W^{m,p}$ for $m \geqslant 2$, it is clear that additional smoothness on H must be assumed, since otherwise $H'(f) \cdot \partial f/\partial x_i$ may not be absolutely continuous on almost every line. Therefore we define a *smooth truncation operator* (*STO*) as any T_H such that

(1.1) $$H \text{ is semibounded},$$

(1.2) $$|H^{(k)}(t)| \leqslant L < +\infty \quad \text{for } k = 1, \ldots, m \text{ and all } t \in \mathbf{R},$$

(1.3) $$H(0) = 0.$$

The smoothness of H (i.e., (1.2)) insures that T_H is a densely defined and closed operator on $W^{m,p}$. However, Dahlberg has shown ([8]) that STOs are unbounded on $W^{m,p}$ if $m = 2$ and $1 < p < n/2$ or if $m \geqslant 3$ and $1 \leqslant p < n/m$.

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On the other hand, it follows from the Gagliardo–Nirenberg lemma (Lemma 3.2 below; see the discussion in § 3) that if H satisfies (1.2–3) then T_H is bounded from $W^{m,p} \cap L^\infty$ into $W^{m,p}$ whenever $1 \leqslant p < +\infty$ and $m \in Z^+$. By the Sobolev imbedding theorem, $W^{m,p} \subseteq L^\infty$ if either $p > n/m$ when $p > 1$ or $m \geqslant n$ when $p = 1$. Hence if we denote the domain of T_H in $W^{m,p}$ by

$$D(T_H)_{m,p} \equiv \{f \in W^{m,p}\colon\ T_H f \in W^{m,p}\},$$

then for these p and m and any smooth truncation operator T_H,

$$D(T_H)_{m,p} = W^{m,p}. \tag{1.4}$$

The Sobolev imbedding theorem also implies that (1.4) holds if $p = n/m \geqslant 1$. In the remaining case $m = 2$, $p = 1$, (1.4) holds under the additional assumption that $H'' \in L^1(\boldsymbol{R})$ since the identity

$$\int |H''f| \left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right) dx = \int \left(\frac{\partial^2 f}{\partial x_i\, \partial x_j}\right) \int_f^{+\infty} |H''(t)|\, dt\, dx$$

holds if $f \in C_0^\infty(\boldsymbol{R}^n)$ and $i, j \in \{1, \ldots, n\}$ (see [13]). Thus the exceptional cases noted by Dahlberg are the only cases in which $D(T_H)_{m,p}$ has not been adequately characterized.

The main results of this paper imply the inclusion

$$W^{m,p} \cap \mathrm{BMO} \subseteq D(T_H)_{m,p} \tag{1.5}$$

whenever H satisfies (1.2–3), $1 < p < +\infty$, and $m \geqslant 1$. Here BMO (bounded mean oscillation) is the set of all $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ such that

$$\|f\|_{\mathrm{BMO}} = \sup_Q |Q|^{-1} \int_Q |f - f_Q| < +\infty,$$

where $f_Q = |Q|^{-1} \int_Q f$ and the sup is taken over all cubes Q with sides parallel to the axes. In fact, our result holds more generally for the Bessel potential spaces $L^{\alpha,p} = \{G_\alpha * f\colon\ f \in L^p\}$, $\alpha > 0$, $1 < p < +\infty$, normed by $\|G_\alpha * f\|_{\alpha,p} = \|f\|_{L_p}$, where G_α is the usual Bessel potential; see e.g. [19], Ch. 5.3. By a result of Calderón ([5]), $L^{m,p} = W^{m,p}$, with equivalent norms, if $m \in Z^+$ and $1 < p < +\infty$. We prove the following:

THEOREM 1. *Suppose* $\alpha > 0$, $1 < p < +\infty$, *and* H *satisfies* (1.2–3) *for* $m \in Z^+$ *such that* $\alpha \leqslant m < \alpha + 1$. *Then*

$$L^{\alpha,p} \cap \mathrm{BMO} \subseteq D(T_H)_{\alpha,p}.$$

Furthermore, there exists $c = c(n, \alpha, p, L)$ *such that*

$$\|H \circ f\|_{\alpha,p} \leqslant c\Big[\|f\|_{\alpha,p} + \sum_{k=2}^{[\alpha]} (\|f\|_{\alpha,p} + \|f\|_{\mathrm{BMO}})^k\Big]. \tag{1.6}$$

Here $[x]$ is the greatest integer in x.

To prove Theorem 1, we first prove an analogue for $L^{\alpha,p} \cap \mathrm{BMO}$ of the Fefferman–Stein representation of BMO ([9]). The Fefferman–Stein result is that $\mathrm{BMO} = L^\infty + \sum_{j=0}^n R_j L^\infty$, where the Riesz transforms $R_1, \ldots, R_n$ are the singular integral operators satisfying $(R_j f)^\wedge(\xi) = i\xi_j |\xi|^{-1} \hat{f}(\xi)$ for $f \in \mathcal{S}$ (see e.g. [19], pp. 57–8), and "$\wedge$" is the Fourier transform. We prove that

$$L^{\alpha,p} \cap \mathrm{BMO} = L^{\alpha,p} \cap L^\infty + \sum_{j=1}^n R_j (L^{\alpha,p} \cap L^\infty)$$

if $\alpha > 0$ and $1 < p < +\infty$. The Gagliardo–Nirenberg lemma is used to obtain the result corresponding to Theorem 1 for $L^{\alpha,p} \cap L^\infty$. Then the representation of $L^{\alpha,p} \cap \mathrm{BMO}$ allows us to deduce Theorem 1.

The proof of the representation of $L^{\alpha,p} \cap \mathrm{BMO}$ closely follows Uchiyama's constructive proof ([23]) of the Fefferman–Stein result. The representation in fact holds for the larger class of Triebel–Lizorkin spaces $F_p^{\alpha q} \cap \mathrm{BMO}$ if $\alpha > 0$, $1 < p < +\infty$, and $1 \leqslant q < +\infty$ (see § 2 for the definition of $F_p^{\alpha q}$ and [22] for background). We have $L^{\alpha,p} = F_p^{\alpha 2}$ if $1 < p < +\infty$ and $\alpha \in \boldsymbol{R}$ (see e.g. [22], p. 87.) We obtain

THEOREM 2. *Suppose* $1 < p < +\infty$, $\alpha > 0$, *and* $1 \leqslant q < +\infty$. *If* $f \in F_p^{\alpha q} \cap \mathrm{BMO}$, *then there exist functions* $g_0, g_1, \ldots, g_n$ *satisfying*

$$\sum_{j=0}^n (\|g_j\|_{L^\infty} + \|g_j\|_{F_p^{\alpha q}}) \leqslant c(\|f\|_{\mathrm{BMO}} + \|f\|_{F_p^{\alpha q}})$$

such that $f = g_0 + \sum_{j=1}^n R_j g_j$. *(Here* $c = c(\alpha, p, q, n)$.*)*

The proof of Theorem 2 uses the decomposition results for $F_p^{\alpha q}$, similar to those for the homogeneous spaces $\dot{F}_p^{\alpha q}$ in [11]. We show that the conditions characterizing $F_p^{\alpha q}$ carry through Uchiyama's construction for $f \in F_p^{\alpha q} \cap \mathrm{BMO}$ and imply that the resulting bounded functions also belong to $F_p^{\alpha q}$.

Suppose H satisfies (1.1–3) and let M be the set of all $f \in W^{m,p}$ such that

$$\sup_{x \in \boldsymbol{R}^n, r > 0} r^{mp-n} \int_{B(x,r)} |\partial^\eta f|^p < +\infty,$$

for $|\eta| = m$, where $B(x, r) = \{y \in \boldsymbol{R}^n\colon\ |x - y| < r\}$. Then one consequence of (1.5) is that $M \subseteq D(T_H)_{m,p}$, since $M \subseteq W^{m,p} \cap \mathrm{BMO}$. One can also obtain this result from Theorem 3.2 of [1].

Notice that if $p = n/m$, (1.5) and the inclusion $W^{m,p} \subseteq \mathrm{BMO}$, $1 \leqslant p < +\infty$, give another proof of the fact that (1.4) holds if $p = n/m \geqslant 1$. We remark that Dahlberg's examples show that $W^{m,p} \cap L^q \nsubseteq D(T_H)_{m,p}$ if $q < +\infty$. Since $\mathrm{BMO} \subseteq L^q_{\mathrm{loc}}$ if $q < +\infty$, we consider $W^{m,p} \cap \mathrm{BMO}$ to be a natural subclass of $W^{m,p}$ for considering smooth truncation.

However, for further perspective on (1.5), we should point out that there are functions $f \in D(T_H)_{m,p}$ which are not in BMO locally or at infinity. In fact, if $L_+^p = \{\varphi \in L^p\colon \varphi \geqslant 0\}$ and we set $P = \{G_m * \varphi\colon \varphi \in L_+^p\} \subseteq W^{m,p}(\mathbf{R}^n)$, then $P \subseteq D(T_H)_{m,p}$ for $1 < p < +\infty$ if the derivatives of H satisfy the decay condition

$$\sup_{t>0} |t^{k-1} H^{(k)}(t)| < +\infty \quad \text{for } k = 0, 1, \ldots, m. \tag{1.7}$$

(This result is contained in [2]; the case $m = 2$ is due to Maz'ya [14].) Also, it follows as in [2] that $W_+^{2,p} = \{f \in W^{2,p}\colon f \geqslant 0\} \subseteq D(T_H)_{2,p}$ if (1.7) holds for $m = 2$. This can be extended to show that if $H \in C^2$, $H(0) = 0$, and $\sup_{t>0}(1+t)|H''(t)| < +\infty$, then $f \in D(T_H)_{m,p}$ whenever $f \in W^{2,p}$ and f is bounded below. A similar result holds if f is bounded above. Note that the decay condition (1.7) in a sense compensates for the growth of f, which may not be BMO.

We prove Theorem 2 in § 2. We use this in § 3 to obtain Theorem 1. We give the proof of Theorem 1 in detail for the case of the Sobolev spaces, and sketch the proof of the more technical, but similar, case of nonintegral α (which requires some interpolation-type estimates). We also make some remarks more generally about Theorem 1 in the context of the $F_p^{\alpha q}$ spaces.

Notation. The letter "c" refers to various constants depending on n and possibly other parameters, with "c" varying at each occurrence. For a cube $Q \subseteq \mathbf{R}^n$, x_Q and $l(Q)$ are the center and side length of Q, respectively. For $r > 0$, rQ is the cube concentric with Q, with side length $rl(Q)$. For $\mathbf{f}(x) = (f_0(x), \ldots, f_n(x))$, let

$$|\mathbf{f}(x)| = \Big(\sum_{j=0}^{n} |f_j(x)|^2\Big)^{1/2}, \qquad \|\mathbf{f}\| = \sum_{j=0}^{n} \|f_j\|$$

for any norm $\|\cdot\|$. Let $x_+ = \max(x, 0)$. Also, $\|\cdot\|_X \approx \|\cdot\|_Y$ means that there exist constants $c_1 > 0$ and $c_2 > 0$ such that $c_1\|\cdot\|_X \leqslant \|\cdot\|_Y \leqslant c_2\|\cdot\|_X$.

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2. Decomposition of $F_p^{\alpha q} \cap \mathrm{BMO}$. The proof of Theorem 2 is based on Uchiyama's constructive proof ([23]) of the Fefferman–Stein result that $\mathrm{BMO} = L^\infty + \sum_{j=1}^n R_j L^\infty$ ([9]). Uchiyama's proof begins with a decomposition of BMO (Lemmas 3.1 and 3.4 in [23], see also [6] and [7]) derived from Calderón's reproducing formula. This decomposition motivated the corresponding decompositions of Besov spaces in [10] and $\dot F_p^{\alpha q}$ spaces in [11]. Here the important point is that the BMO and $F_p^{\alpha q}$ decompositions hold simultaneously for $f \in F_p^{\alpha q} \cap \mathrm{BMO}$. We can then show that the $F_p^{\alpha q}$ condition carries through to the bounded functions in Uchiyama's construction, if $\alpha > 0$, $1 < p < +\infty$, and $1 \leqslant q < +\infty$.

We adopt certain technical modifications of Uchiyama's proof, introduced by Baernstein in [4]. (The key ideas are the same.) Following Baernstein's modifications allows us to avoid adapting Uchiyama's dilation techniques (Lemmas 3.3–5 in [23]) to $F_p^{\alpha q}$. Also, these techniques and our assumption that $f \in F_p^{\alpha q}$ allow us to drop the assumption in [23] that f has compact support (without appealing to duality).

To define the inhomogeneous Triebel–Lizorkin spaces $F_p^{\alpha q}(\mathbf{R})$, select functions Φ and φ belonging to $\mathscr{S}$ satisfying

$$\operatorname{supp}\hat\Phi(\xi) \subseteq \{\xi\colon |\xi| \leqslant 1\}, \qquad |\hat\Phi(\xi)| \geqslant c > 0 \text{ if } |\xi| \leqslant 5/6,$$

$$\operatorname{supp}\hat\varphi(\xi) \subseteq \{\xi\colon 1/2 \leqslant |\xi| \leqslant 2\}, \qquad |\hat\varphi(\xi)| \geqslant c > 0 \text{ if } 3/5 \leqslant |\xi| \leqslant 5/3.$$

For $\nu \in \mathbf{Z}$, $\nu \geqslant 0$, let $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$. For $\alpha \in \mathbf{R}$, $0 < p < +\infty$, and $0 < q \leqslant +\infty$, $F_p^{\alpha q}$ is the collection of all $f \in \mathscr{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{F_p^{\alpha q}} \equiv \|\Phi * f\|_{L^p} + \Big\|\Big(\sum_{\nu=0}^{\infty} (2^{\nu\alpha}|\varphi_\nu * f|)^q\Big)^{1/q}\Big\|_{L^p} < +\infty.$$

Peetre's methods show that $F_p^{\alpha q}$ is independent of the choice of Φ and φ as above; for background, see [16], [22], or [11].

We adopt the convention throughout that whenever Q appears as a summation index, the sum runs only over dyadic cubes. For $\nu \in \mathbf{Z}$, $k \in \mathbf{Z}^n$, $Q_{\nu k}$ denotes the dyadic cube $\{x \in \mathbf{R}^n\colon k_i 2^{-\nu} \leqslant x_i < (k_i+1)2^{-\nu},\ i = 1, \ldots, n\}$. We fix integers

$$K_0 = ([\alpha]+1)_+, \qquad N_0 = \max\big(\big[n[(1/\min(p, q)) - 1]_+ - \alpha\big], -1\big).$$

We also select fixed integers K, N, and M sufficiently large, so that

$$K > K_0, \qquad N > \max(N_0, K_0 + M - n + 1), \qquad M > N_0 + 10n\max(1/p, 1).$$

LEMMA 2.1. *Suppose $\alpha \in \mathbf{R}$, $0 < p < +\infty$, and $0 < q \leqslant +\infty$. If $f \in F_p^{\alpha q} \cap \mathrm{BMO}$, then there exist complex numbers $\{s_k\}_{k \in \mathbf{Z}^n}$ and $\{s_Q\}_{Q\,\text{dyadic}, l(Q) \leqslant 1}$ and complex-valued functions $\{b_k(x)\}_{k \in \mathbf{Z}^n}$ and $\{a_Q(x)\}_{Q\,\text{dyadic}, l(Q) \leqslant 1}$, such that*

$$f = \sum_{k \in \mathbf{Z}^n} s_k b_k + \sum_{l(Q) \leqslant 1} s_Q a_Q, \tag{2.1}$$

$$\operatorname{supp} b_k \subseteq 3Q_{0k}, \quad |\partial^\gamma b_k(x)| \leqslant 1 \quad \text{if } |\gamma| \leqslant K, \tag{2.2}$$

$$\operatorname{supp} a_Q \subseteq 3Q, \quad |\partial^\gamma a_Q(x)| \leqslant l(Q)^{-|\gamma|} \quad \text{if } |\gamma| \leqslant K, \tag{2.3}$$

$$\int x^\gamma a_Q(x)\,dx = 0 \quad \text{if } |\gamma| \leqslant N, \tag{2.4}$$

$$\Big(\sum_{k \in \mathbf{Z}^n} |s_k|^p\Big)^{1/p} + \Big\|\Big(\sum_{l(Q) \leqslant 1} (|Q|^{-\alpha/n}|s_Q|\chi_Q)^q\Big)^{1/q}\Big\|_{L^p} \leqslant c\|f\|_{F_p^{\alpha q}}, \tag{2.5}$$

$$\sup_{J\,\text{dyadic}, l(J) \leqslant 1} |J|^{-1} \sum_{Q \subseteq J} |s_Q|^2 |Q| \leqslant c\|f\|_{\mathrm{BMO}}^2. \tag{2.6}$$

The convergence in (2.1) is in $F_p^{\alpha q}$ (quasi-)norm if $q < +\infty$ and in $\mathscr{S}'$ if $q = +\infty$. If f is real-valued, the s_k, b_k, s_Q, and a_Q's may all be taken real.

Conversely, suppose $f = \sum_{k\in Z^n} s_k m_k + \sum_{l(Q)\le 1} s_Q m_Q$, *where*

(2.7) $$|\partial^\gamma m_k(x)| \le (1+|x-k|)^{-M-|\gamma|} \quad \text{if } |\gamma| \le K_0,$$

(2.8) $$|\partial^\gamma m_Q(x)| \le l(Q)^{-|\gamma|}(1+l(Q)^{-1}|x-x_Q|)^{-M-|\gamma|} \quad \text{if } |\gamma| \le K_0,$$

(2.9) $$\int x^\gamma m_Q(x)\,dx = 0 \quad \text{if } |\gamma| \le N_0.$$

Then

(2.10) $$\|f\|_{F_p^{\alpha q}} \le c\Big(\sum_{k\in Z^n} |s_k|^p\Big)^{1/p} + c\Big\|\Big(\sum_{l(Q)\le 1} (|Q|^{-\alpha/n}|s_Q|\chi_Q)^q\Big)^{1/q}\Big\|_{L^p}.$$

If $\int m_Q(x)\,dx = 0$ (*e.g., if* $N_0 \ge 0$), *then*

(2.11) $$\|f\|_{\mathrm{BMO}} \le c\sup_{k\in Z^n}|s_k| + c\sup_{J \text{dyadic}, l(J)\le 1}\Big(|J|^{-1}\sum_{Q\subseteq J}|s_Q|^2|Q|\Big)^{1/2}.$$

Proof. We only outline this proof, since it is essentially included in [10] and [11] (see also [12]). We can select $\Theta \in \mathscr{S}$ and $\theta \in \mathscr{S}$ real-valued and radial such that $\operatorname{supp}\Theta,\ \theta \subseteq \{x\colon |x| \le 1\}$, $\hat{\Theta}(\xi) \ge c > 0$ if $|\xi| \le 1$, $\int x^\gamma \theta(x)\,dx = 0$ if $|\gamma| \le N$, and $\hat{\theta}(\xi) \ge c > 0$ if $1/2 \le |\xi| \le 2$. Then we can pick Φ and φ as in the definition of $F_p^{\alpha q}$, also real-valued and radial, so that

$$\hat{\Phi}(\xi)\hat{\Theta}(\xi) + \sum_{\nu=0}^{\infty} \hat{\varphi}(2^{-\nu}\xi)\hat{\theta}(2^{-\nu}\xi) \equiv 1.$$

Hence $f = \Phi*\Theta*f + \sum_{\nu=0}^{\infty}\varphi_\nu*\theta_\nu*f$, where $\theta_\nu(x) = 2^{\nu n}\theta(2^\nu x)$. We let

$$s_k = C\sup_{y\in Q_{0k}}|\Phi*f(y)|, \qquad b_k = s_k^{-1}\int_{Q_{0k}}\theta(x-y)\,\Phi*f(y)\,dy,$$

for C sufficiently large. Then $\Phi*\Theta*f = \sum_{k\in Z^n} s_k b_k$, and (2.2) follows. We obtain $(\sum_{k\in Z^n}|s_k|^p)^{1/p} \le c\|\Phi*f\|_{L^p}$ by the Plancherel–Pólya Theorem (see e.g. Lemma 2.4 in [10]). Similarly, let

$$s_Q = C\sup_{y\in Q}|\varphi_\nu*f(y)|, \qquad a_Q = s_Q^{-1}\int_Q \theta_\nu(x-y)\,\varphi_\nu*f(y)\,dy, \qquad \text{if } l(Q) = 2^{-\nu}.$$

If C is sufficiently large, (2.3–4) follow from the assumptions on θ. Also (2.1) follows from the identity for f above. The techniques in the proof of Theorem II A in [11] and the estimate above for $(\sum_{k\in Z^n}|s_k|^p)^{1/p}$ yield (2.5). Finally, (2.6) follows from (4.4) in [10].

For the converse, the estimates

$$|\Phi*m_k(x)| \le c(1+|x-k|)^{-M-n},$$

$$|\Phi*m_Q(x)| \le c2^{-\mu(N_0+1+n)}(1+|x-x_Q|)^{N_0+1+n-M} \quad \text{if } l(Q) = 2^{-\mu} \le 1,$$

$$|\varphi_\nu*m_k(x)| \le c2^{-\nu K_0}(1+|x-k|)^{N_0+1+n-M} \quad \text{if } \nu \ge 0,$$

are obtained as in Lemma 3.3 of [10], noting that these cases do not require vanishing moments for Φ and m_k. Then computations exactly like those in the proof of Theorem II B in [11] yield (2.10). We have the trivial estimates

$$\Big\|\sum_{k\in Z^n} s_k b_k\Big\|_{\mathrm{BMO}} \le c\Big\|\sum_{k\in Z^n} s_k b_k\Big\|_{L^\infty} \le c\sup_{k\in Z^n}|s_k|.$$

This together with Theorem 4.1.6 of [10] (or [23], Lemma 3.4) yields (2.11). ■

LEMMA 2.2. *Suppose T is a singular integral operator with smooth kernel, i.e.*

$$Tf(x) = \text{p.v.}\int \Omega(x-y)|x-y|^{-n}f(y)\,dy$$

where $\Omega \in C^\infty(S^{n-1})$, $\Omega(rx) = \Omega(x)$ *if* $r > 0$, *and* $\int_{S^{n-1}}\Omega(y)\,d\sigma(y) = 0$. *If* a_Q *satisfies* (2.3–4), *then* Ta_Q *satisfies*

$$|\partial^\gamma Ta_Q(x)| \le cl(Q)^{-|\gamma|}(1+l(Q)^{-1}|x-x_Q|)^{-M-K_0-1-|\gamma|} \quad \text{if } |\gamma| \le K,$$

$$\int x^\gamma Ta_Q(x)\,dx = 0 \quad \text{if } |\gamma| \le N.$$

Proof. The proof is as for Lemma 3.6 in [23]. To obtain the more rapid decay of $\partial^\gamma Ta_Q(x)$, use (2.4) to subtract the Taylor polynomial of degree N about $y = x_Q$ of $\partial_x^\gamma(\Omega(x-y)|x-y|^{-n})$, regarded as a function of y, in the convolution. ■

For the special case of the Riesz transforms, the results in § 2 of [23] reduce to the following lemma.

LEMMA 2.3. *Suppose* $v = (v_0, \ldots, v_n) \in \mathbf{R}^{n+1}$ *satisfies* $|v| = 1$. *If* $a_Q(x)$ *satisfies* (2.3–4) *and is real-valued, then there exists* $\mathbf{p}_Q(x) = (p_{Q,0}(x), \ldots, p_{Q,n}(x))$ *with real-valued components such that*

(2.12) $$\mathbf{R}\cdot\mathbf{p}_Q \equiv p_{Q,0} + \sum_{j=1}^n R_j p_{Q,j} = a_Q,$$

(2.13) $$v\cdot\mathbf{p}_Q \equiv \sum_{j=0}^n v_j p_{Q,j}(x) = 0 \quad \text{for all } x \in \mathbf{R}^n,$$

(2.14) $$|\partial^\gamma p_{Q,j}(x)| \le cl(Q)^{-|\gamma|}(1+l(Q)^{-1}|x-x_Q|)^{-M-K_0-1-|\gamma|}$$
$$\text{if } |\gamma| \le K \text{ and } 0 \le j \le n,$$

(2.15) $$\int x^\gamma p_{Q,j}(x)\,dx = 0 \quad \text{if } |\gamma| \le N \text{ and } 0 \le j \le n.$$

Proof (Uchiyama). Let

$$p_{Q,0} = \sum_{j=1}^n v_j^2 + v_0\sum_{j=1}^n v_j R_j a_Q,$$

and, for $j = 1, \ldots, n$,

$$p_{Q,j} = -v_j v_0 - v_0^2 R_j a_Q.$$

Then (2.12–13) follow from $(R_j f)\hat{}(\xi) = i\xi_j|\xi|^{-1}\hat{f}(\xi)$, and (2.14–15) follow from Lemma 2.2. ∎

Theorem 2 follows by an easy iteration argument, as in [23], pp. 238–9, from the following main lemma.

LEMMA 2.4. *Suppose $1 < p < +\infty$, $\alpha > 0$, and $1 \leqslant q < +\infty$. There exists $A = A(n, \alpha, p, q) > 0$ such that if $f\colon R^n \to R$ satisfies $\|f\|_{F_p^{\alpha q}} + \|f\|_{\mathrm{BMO}} \leqslant 1$, then there exists $\mathbf{g} = (g_0, \ldots, g_n)$ satisfying*

$$\|\mathbf{g}\|_{F_p^{\alpha q}} + \|\mathbf{g}\|_{L^\infty} \leqslant A, \qquad \|f - \mathbf{R}\cdot\mathbf{g}\|_{F_p^{\alpha q}} + \|f - \mathbf{R}\cdot\mathbf{g}\|_{\mathrm{BMO}} < 1/2.$$

Proof. Apply Lemma 2.1 to f to obtain (2.1–6) with all quantities real. Pick $R > 0$ sufficiently large. For $k = -1, 0, 1, 2, \ldots$, we define $\mathbf{h}_k(x)$, $\mathbf{g}_k(x)$, and $\varphi_k(x)$ inductively as follows. Let $\mathbf{g}_{-1}(x) = (0, R, 0, \ldots, 0)$, $\mathbf{h}_{-1}(x) = \mathbf{0}$, and $\varphi_{-1}(x) = \mathbf{0}$. Now suppose that $\mathbf{h}_{k-1}$, $\mathbf{g}_{k-1}$, and φ_{k-1} have been defined. If Q is dyadic with $l(Q) = 2^{-k}$, obtain $\mathbf{p}_Q(x)$ by Lemma 2.3 so that (2.12–15) are satisfied for $\nu = \mathbf{g}_{k-1}(x_Q)/|\mathbf{g}_{k-1}(x_Q)|$. Define

$$\mathbf{h}_k(x) = \sum_{l(Q)=2^{-k}} s_Q \mathbf{p}_Q(x), \tag{2.16}$$

$$\mathbf{g}_k(x) = \frac{R\left(\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)\right)}{|\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)|}, \tag{2.17}$$

$$\varphi_k(x) = \mathbf{g}_{k-1}(x) + \mathbf{h}_k(x) - \mathbf{g}_k(x) = R^{-1}\left(|\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)| - R\right)\mathbf{g}_k(x). \tag{2.18}$$

Note that $|s_Q| \leqslant c$, by (2.6), so that by (2.14) and (2.16),

$$|\mathbf{h}_k(x)| \leqslant c \quad \text{for all } x \in R^n. \tag{2.19}$$

By (2.17),

$$|\mathbf{g}_k(x)| = R \quad \text{for all } x \in R^n. \tag{2.20}$$

By induction, (2.19–20) guarantee that (2.17) is defined for all $k = 1, 2, 3, \ldots$, if R is large enough.

From (2.2), (2.5), and (2.10), we obtain the simple estimate

$$\Big\|\sum_{k\in Z^n} s_k b_k\Big\|_{L^\infty} + \Big\|\sum_{k\in Z^n} s_k b_k\Big\|_{F_p^{\alpha q}} \leqslant c. \tag{2.21}$$

We will establish the following key estimates:

$$\sum_{j=0}^{k} \mathbf{h}_j \text{ converges in } F_p^{\alpha q} \text{ as } k \to +\infty, \qquad \Big\|\sum_{j=0}^{\infty} \mathbf{h}_j\Big\|_{F_p^{\alpha q}} \leqslant c, \tag{2.22}$$

$$\sum_{j=0}^{k} \varphi_j \text{ converges in } F_p^{\alpha q} \text{ as } k \to +\infty, \qquad \Big\|\sum_{j=0}^{\infty} \varphi_j\Big\|_{F_p^{\alpha q}} \leqslant c/R, \tag{2.23}$$

$$\Big\|\sum_{j=0}^{\infty} \varphi_j\Big\|_{\mathrm{BMO}} \leqslant c/R. \tag{2.24}$$

Accepting (2.22–24) temporarily, we show how these estimates imply the desired result. From (2.18), $\mathbf{g}_k - \mathbf{g}_{-1} = \sum_{j=0}^{k}(\mathbf{h}_j - \varphi_j)$; hence (2.22–23) imply that $\mathbf{g}_k - \mathbf{g}_{-1}$ converges in $F_p^{\alpha q}$ as $k \to +\infty$. Let

$$\mathbf{g} = \lim_{k\to+\infty}(\mathbf{g}_k - \mathbf{g}_{-1}) + \Big(\sum_{k\in Z^n} s_k b_k, 0, 0, \ldots, 0\Big).$$

By (2.12), $\mathbf{R}\cdot\mathbf{h}_k = \sum_{l(Q)=2^{-k}} s_Q a_Q$, so that

$$\mathbf{R}\cdot\mathbf{g} = \mathbf{R}\cdot\sum_{j=0}^{\infty}(\mathbf{h}_j - \varphi_j) + \sum_{k\in Z^n} s_k b_k = f - \mathbf{R}\cdot\sum_{j=0}^{\infty}\varphi_j.$$

Therefore by (2.23–24) and the boundedness of the Riesz transforms on BMO and $F_p^{\alpha q}$ for $1 < p < +\infty$ (see e.g. § 7 of [11]), we obtain

$$\|f - \mathbf{R}\cdot\mathbf{g}\|_{F_p^{\alpha q}} + \|f - \mathbf{R}\cdot\mathbf{g}\|_{\mathrm{BMO}} \leqslant c/R < 1/2,$$

if R is large enough. By (2.20–21), $\|\mathbf{g}\|_{L^\infty} \leqslant 2R + c$, while by (2.21–23), $\|\mathbf{g}\|_{F_p^{\alpha q}} \leqslant c$.

Hence only (2.22–24) remain. By (2.14–16) and (2.5), (2.22) follows from Lemma 2.1 ((2.10)). To prove (2.23), we require the following estimates: There exist $B > 0$ and $D > 0$ such that

$$|\partial^\gamma \mathbf{h}_k(x)| \leqslant 2^{k|\gamma|} B \quad \text{if } |\gamma| \leqslant K, \tag{2.25}$$

$$|\partial^\gamma \mathbf{g}_k(x)| \leqslant 2^{k|\gamma|} D \quad \text{if } 0 < |\gamma| \leqslant K. \tag{2.26}$$

For (2.25–26), we choose B, D, and R so that B/D and D/R are sufficiently small. By (2.14), (2.16), and the fact that $|s_Q| \leqslant c$, (2.25) holds.

We prove (2.26) by induction on k; it is obvious for $k = -1$. Assuming (2.26) for $k-1$, we obtain

$$\begin{aligned}
|\partial^\gamma \mathbf{g}_k(x)| \leqslant{} & R|\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)|^{-1}|\partial^\gamma \mathbf{g}_{k-1}(x) + \partial^\gamma \mathbf{h}_k(x)| \\
& + R|\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)|\left|\partial^\gamma\left(|\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)|^{-1}\right)\right| \\
& + R\sum_{\substack{|\beta|+|\delta|=|\gamma| \\ |\beta|\neq 0, |\delta|\neq 0}} c_{\beta,\delta}|\partial^\beta \mathbf{g}_{k-1}(x) + \partial^\beta \mathbf{h}_k(x)|\left|\partial^\delta\left(|\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)|^{-1}\right)\right| \\
\equiv{} & \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

By (2.19–20), $R|\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)|^{-1} \leqslant 1 + \varepsilon$ with ε small if R is sufficiently large. Therefore by (2.26) for $k-1$,

$$\mathrm{I} \leqslant (1+\varepsilon)(2^{(k-1)|\gamma|}D + 2^{k|\gamma|}B) \leqslant \tfrac{2}{3}\cdot 2^{k|\gamma|}D \quad \text{if } 0 < |\gamma| \leqslant K,$$

if ε and B/D are sufficiently small. To estimate II and III, let

$$F(x) = |\mathbf{g}_{k-1}(x) + \mathbf{h}_k(x)|^2 = R^2 + 2\mathbf{g}_{k-1}(x)\cdot\mathbf{h}_k(x) + |\mathbf{h}_k(x)|^2.$$

Then (2.26) for $k-1$ and (2.25) imply that $|\partial^\eta F| \leqslant c_\eta 2^{k|\eta|}RB$ if $0 < |\eta| \leqslant K$. From this one can obtain $|\partial^\delta(F^{-1/2})| \leqslant c_\delta R^{-2}2^{k|\delta|}B$ if $B < R$ and

$0<|\delta|\leqslant K$. It follows that

$$\mathrm{II}\leqslant(1+\varepsilon)R^2c_\gamma R^{-2}2^{k|\gamma|}B\leqslant\tfrac{1}{6}\cdot 2^{k|\gamma|}D\quad \text{if } 0<|\gamma|\leqslant K,$$
$$\mathrm{III}\leqslant c_\gamma R2^{k|\gamma|}DR^{-2}B\leqslant\tfrac{1}{6}\cdot 2^{k|\gamma|}D\quad \text{if } 0<|\gamma|\leqslant K,$$

if B/D and B/R are sufficiently small. Hence (2.26) holds.

We assert that we can write

$$\varphi_k(x)=R^{-1}\sum_{l(Q)=2^{-k}}s_Q\,\mathbf{m}_Q(x),\tag{2.27}$$

where

$$|\partial^\gamma\mathbf{m}_Q(x)|\leqslant cl(Q)^{-|\gamma|}(1+l(Q)^{-1}|x-x_Q|)^{-M-|\gamma|}\quad \text{if } |\gamma|\leqslant K_0.\tag{2.28}$$

To see this, note that by (2.18),

$$\varphi_k(x)=\mathbf{g}_k(x)[(1+t)^{1/2}-1]\quad \text{for } t=R^{-2}(2\mathbf{g}_{k-1}(x)+\mathbf{h}_k(x))\cdot\mathbf{h}_k(x).$$

By (2.19–20), $|t|<1$; expanding $(1+t)^{1/2}$ in a Taylor series gives

$$\varphi_k(x)=\mathbf{g}_k(x)\sum_{j=1}^{\infty}c_j a^j(x)R^{-2j},$$

where $a(x)=(2\mathbf{g}_{k-1}(x)+\mathbf{h}_k(x))\cdot\mathbf{h}_k(x)$ and $|c_j|\leqslant 1$ for all $j\in Z^+$. Since $\mathbf{g}_{k-1}(x_Q)\cdot\mathbf{p}_Q(x)\equiv 0$ by (2.13), we can write

$$a^j(x)=\sum_{l(Q)=2^{-k}}s_Q a^{j-1}(x)\left[2(\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q))+\mathbf{h}_k(x)\right]\cdot\mathbf{p}_Q(x).$$

Then (2.27) holds if we define

$$\mathbf{m}_Q(x)=\sum_{j=1}^{\infty}c_jR^{-j}\mathbf{g}_k(x)(a(x)/R)^{j-1}\left[2(\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q))+\mathbf{h}_k(x)\right]\cdot\mathbf{p}_Q(x)$$

for $l(Q)=2^{-k}$. Then if $|\beta|\leqslant K$

$$\begin{aligned}&\left|\partial^\beta\left(\left[2(\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q))+\mathbf{h}_k(x)\right]\cdot\mathbf{p}_Q(x)\right)\right|\\
&\leqslant|2(\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q))+\mathbf{h}_k(x)|\,|\partial^\beta\mathbf{p}_Q(x)|\\
&\quad+\sum_{\substack{|\delta|+|\eta|=|\beta|\\|\delta|\neq 0}}c_{\delta,\eta}|\partial^\delta[2(\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q))+\mathbf{h}_k(x)]|\,|\partial^\eta\mathbf{p}_Q(x)|\\
&\leqslant c(2^kD|x-x_Q|+B)2^{k|\beta|}(1+2^k|x-x_Q|)^{-M-K_0-1-|\beta|}\\
&\quad+\sum_{|\eta|\leqslant|\beta|}c_\beta 2^{k|\beta|}D(1+2^k|x-x_Q|)^{-M-K_0-1-|\eta|}\\
&\leqslant c_\beta D2^{k|\beta|}(1+2^k|x-x_Q|)^{-M-K_0},\end{aligned}$$

by (2.14) and (2.25–26). Noting that $|\partial^\beta a|\leqslant c_\beta 2^{k|\beta|}BR$ if $|\beta|\leqslant K$, we obtain

$$\begin{aligned}|\partial^\gamma\mathbf{m}_Q(x)|&\leqslant c_\gamma\sum_{j=1}^{\infty}2^{k|\gamma|}R^{1-j}(c_\gamma B)^{j-1}D(1+2^k|x-x_Q|)^{-M-K_0}\\
&\leqslant c_\gamma D2^{k|\gamma|}(1+2^k|x-x_Q|)^{-M-|\gamma|}\end{aligned}$$

if $|\gamma|\leqslant K_0$ and $c_\gamma B<R$. Hence (2.28) holds.

Note that for $\alpha>0$ and $p,q\geqslant 1$, $N_0=-1$ and hence (2.9) is vacuous in this case. Therefore (2.27–28) imply (2.23) by (2.5) and Lemma 2.1. We also note that $K_0\geqslant 1$ for $\alpha>0$, so that (2.27–28) imply in particular

$$|\nabla\varphi_k(x)|\leqslant c2^k/R.\tag{2.29}$$

It remains only to prove (2.24). If $\int\mathbf{m}_Q(x)\,dx=0$ for each Q, then (2.24) would follow from (2.11) in Lemma 2.1. Unfortunately, this is not the case, and sharper estimates involving $\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q)$ are required. The remainder of the proof follows Baernstein's notes ([4]) on Uchiyama's work.

The key estimate (2.24) follows easily, as in [23], p. 238, from (2.29) and the following estimate: if I is dyadic and $l(I)=2^{-l}\leqslant 1$, then

$$\int_I\sum_{k=l}^{\infty}|\varphi_k|\leqslant c|I|/R.\tag{2.30}$$

To establish (2.30), define

$$\eta_k(x)=\sum_{l(Q)=2^{-k}}|s_Q|(1+2^k|x-x_Q|)^{-n-1},\qquad k=0,1,2,\ldots$$

Uchiyama ([23]), p. 226) proves that if I is as in (2.30), then (2.6) and the assumption $\|f\|_{\mathrm{BMO}}\leqslant 1$ imply

$$\int_I\sum_{k=l}^{\infty}\eta_k^2\leqslant c|I|.\tag{2.31}$$

Set $t=R^{-2}(2\mathbf{g}_{k-1}\cdot\mathbf{h}_k+|\mathbf{h}_k|^2)$ as above; then since $|t|$ is small, $|(1+t)^{1/2}-1-\frac{1}{2}t|\leqslant t^2$. Hence by (2.18) and (2.20),

$$|\varphi_k|=R|(1+t)^{1/2}-1|\leqslant R^{-1}(|\mathbf{g}_{k-1}\cdot\mathbf{h}_k|+\tfrac{1}{2}|\mathbf{h}_k|^2)+E_k,\tag{2.32}$$

where $E_k\leqslant c|\mathbf{h}_k|^2/R$, by (2.19–20). By (2.14) and (2.16), $|\mathbf{h}_k|\leqslant c\eta_k$, so that by (2.31), (2.30) will follow from

$$\int_I\sum_{k=l}^{\infty}|\mathbf{g}_{k-1}\cdot\mathbf{h}_k|\leqslant c|I|,\tag{2.33}$$

for I as in (2.30). Note that we have obtained, from (2.32) and (2.19–20),

$$|\varphi_k|\leqslant c|\mathbf{h}_k|\leqslant c\eta_k.\tag{2.34}$$

To prove (2.33), we require the following estimates:

(2.35) if $1 \leqslant m \leqslant k$, then $|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)| \leqslant c\sum_{i=k-m}^{k-1}\eta_i(x)+c\sum_{i=k-m}^{k-1}\eta_i(y)$
$$+|\mathbf{g}_{k-m-1}(x)-\mathbf{g}_{k-m-1}(y)|,$$

(2.36) if $k \geqslant 0$, then $|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)| \leqslant c\log(1+2^k|x-y|)$,

and, if $x\in Q$, $y \in 3Q$, and $l(Q)=2^{-k}\leqslant 2^{-l}\leqslant 1$, then

(2.37) $|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)| \leqslant c(3/4)^{k-l}+c\sum_{i=1}^{k-l}(3/4)^i \inf_{z\in 3Q}\eta_{k-i}(z)$.

Writing
$$|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)| \leqslant \sum_{i=k-m}^{k-1}|\mathbf{g}_i(x)-\mathbf{g}_{i-1}(x)|+\sum_{i=k-m}^{k-1}|\mathbf{g}_i(y)-\mathbf{g}_{i-1}(y)|$$
$$+|\mathbf{g}_{k-m-1}(x)-\mathbf{g}_{k-m-1}(y)|,$$
(2.35) follows from $\mathbf{g}_i-\mathbf{g}_{i-1}=\mathbf{h}_i-\varphi_i$ (i.e. (2.18)) and (2.34). If $|x-y|\leqslant 2^{-k}$, (2.36) follows from (2.26); while if $2^{r-k-1}<|x-y|\leqslant 2^{r-k}$, $r\in Z^+$, then applying (2.35) with $m=r$ if $r\leqslant k$ and with $m=k$ if $r>k$, and using $|\eta_i|\leqslant c$ (since $|s_Q|\leqslant c$), we obtain
$$|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)| \leqslant cm+c\leqslant cr+c\leqslant c\log(1+2^k|x-y|).$$

To prove (2.37), note the elementary estimate that $|\mathbf{a}-\mathbf{b}|\leqslant\frac{3}{2}|\mathbf{a}+\mathbf{c}-(\mathbf{b}+\mathbf{d})|$ if $|\mathbf{a}|=|\mathbf{b}|=R$, $|\mathbf{c}|\ll R$, $|\mathbf{d}|\ll R$, $\mathbf{c}\,||\,\mathbf{a}$, and $\mathbf{d}\,||\,\mathbf{b}$. By (2.18–20) and (2.34), we can apply this to $\mathbf{a}=\mathbf{g}_{k-1}(x)$, $\mathbf{b}=\mathbf{g}_{k-1}(y)$, $\mathbf{c}=\varphi_{k-1}(x)$, and $\mathbf{d}=\varphi_{k-1}(y)$. Using $\mathbf{g}_{k-1}+\varphi_{k-1}=\mathbf{g}_{k-2}+\mathbf{h}_k$ ((2.18)), we obtain
$$|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)|\leqslant\tfrac{3}{2}|\mathbf{g}_{k-2}(x)-\mathbf{g}_{k-2}(y)|+\tfrac{3}{2}|\mathbf{h}_{k-1}(x)-\mathbf{h}_{k-1}(y)|.$$
Iteration of this result yields
$$|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)|\leqslant(\tfrac{3}{2})^{k-l}|\mathbf{g}_{l-1}(x)-\mathbf{g}_{l-1}(y)|+\sum_{i=1}^{k-l}(\tfrac{3}{2})^i|\mathbf{h}_{k-i}(x)-\mathbf{h}_{k-i}(y)|.$$
Our assumptions on x and y give $|\mathbf{g}_{l-1}(x)-\mathbf{g}_{l-1}(y)|\leqslant c2^l|x-y|\leqslant c2^{l-k}$, by (2.26). If $1\leqslant i\leqslant k-l$ and $x\in Q$, $y\in 3Q$, and $l(Q)=2^{-k}$, then by (2.16) and (2.14),
$$|\mathbf{h}_{k-i}(x)-\mathbf{h}_{k-i}(y)\leqslant\sum_{l(J)=2^{i-k}}|s_J|\,|\mathbf{p}_J(x)-\mathbf{p}_J(y)|$$
$$\leqslant c|x-y|\sum_{l(J)=2^{i-k}}|s_J|\,2^{k-i}\sup_{w\in 3Q}(1+2^{k-i}|w-x_J|)^{-n-1}$$
$$\leqslant c2^{-i}\sum_{l(J)=2^{i-k}}|s_J|\inf_{z\in 3Q}(1+2^{k-i}|z-x_J|)^{-n-1}$$
$$\leqslant c2^{-i}\inf_{z\in 3Q}\eta_{k-i}(z).$$

These estimates establish (2.37).

Returning to (2.33), use $\mathbf{g}_{k-1}(x_Q)\cdot\mathbf{p}_Q=0$ to write
$$\int_I\sum_{k=l}^{\infty}|\mathbf{g}_{k-1}\cdot\mathbf{h}_k|\leqslant\int_I\sum_{k=l}^{\infty}\sum_{l(Q)=2^{-k}}|s_Q|\,|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q)|\,|\mathbf{p}_Q|$$
$$=\int_I\sum_{k=l}^{\infty}\sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}}(\ldots)+\int_I\sum_{k=l}^{\infty}\sum_{\substack{l(Q)=2^{-k}\\ Q\not\subseteq 3I}}(\ldots)\equiv A+B.$$
By (2.36), (2.14), and $|s_Q|\leqslant c$,
$$A\leqslant c\int_I\sum_{k=l}^{\infty}\sum_{\substack{j\in Z^n\\ |j|\geqslant 2^{k-l}}}(1+|j|)^{-n-1}\log(1+|j|)\leqslant c|I|.$$
By (2.14) and (2.6),
$$B\leqslant\sum_{k=l}^{\infty}\sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}}\sum_{j\in Z^n:Q+jl(Q)\subseteq I}\int_{Q+jl(Q)}|s_Q|\,|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q)|\,|\mathbf{p}_Q|$$
$$\leqslant c\sum_{j\in Z^n}(1+|j|)^{-n-1}\sum_{k=l}^{\infty}\sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}}\int_{Q+jl(Q)}|s_Q|\,|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q)|$$
$$\leqslant c\sum_{j\in Z^n}(1+|j|)^{-n-1}\Big(\sum_{k=l}^{\infty}\sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}}\int_{Q+jl(Q)}|s_Q|^2\Big)^{1/2}$$
$$\times\Big(\sum_{k=l}^{\infty}\sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}}\int_{Q+jl(Q)}|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q)|^2\Big)^{1/2}$$
$$\leqslant c\sum_{j\in Z^n}(1+|j|)^{-n-1}|I|^{1/2}\Big(\sum_{k=l}^{\infty}\sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}}\int_{Q+jl(Q)}|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q)|^2\Big)^{1/2}.$$

To estimate this last term, fix $j\in Z^n$ and define $r\in Z$, $r\geqslant 0$, by $r=0$ if $|j|\leqslant 1$, while if $|j|>1$, r is such that $2^{r-1}<|j|\leqslant 2^r$. For $r\geqslant 1$, (2.36) gives
$$\sum_{k=l}^{l+r-1}\sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}}\int_{Q+jl(Q)}|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_Q)|^2$$
$$\leqslant c\sum_{k=l}^{l+r-1}\sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}}|Q|\log^2(1+|j|)\leqslant c|I|\log^3(1+|j|).$$

Now consider $k\geqslant l+r$. With $r\geqslant 1$, still, let $\tilde{Q}$ be the unique dyadic cube of side length 2^{r-k} containing Q. Note that since $|j|\leqslant 2^r$, $Q+jl(Q)\subseteq 3\tilde{Q}$. Applying (2.35) with $m = r$ and $x \in Q + jl(Q)$, we obtain

$$|g_{k-1}(x) - g_{k-1}(x_Q)|^2 \leqslant cr \sum_{i=k-r}^{k-1} \eta_i^2(x) + cr \sum_{i=k-r}^{k-1} \eta_i^2(x_Q) + c|g_{k-r-1}(x) - g_{k-r-1}(x_Q)|^2.$$

Applying (2.31) and noting that $|jl(Q)| \leqslant 2^{r-k} \leqslant 2^{-l}$ if $l(Q) = 2^{-k}$, $k \geqslant l+r$, we obtain

$$cr \sum_{k=l+r}^{\infty} \sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}} \int_{Q+jl(Q)} \sum_{i=k-r}^{k-1} \eta_i^2(x) \leqslant cr \sum_{i=1}^{r} \int_{5I} \sum_{k=l+r}^{\infty} \eta_{k-i}^2(x)$$

$$\leqslant cr^2 \int_{5I} \sum_{k=l}^{\infty} \eta_k^2 \leqslant cr^2 |5I| \leqslant c|I| \log^2(1+|j|).$$

Similarly, noting that $\eta_{k-i}(x_Q) \leqslant c\eta_{k-i}(x)$ for all $x \in Q$ if $l(Q) = 2^{-k}$ and $i > 0$, we have

$$cr \sum_{k=l+r}^{\infty} \sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}} \int_{Q+jl(Q)} \sum_{i=k-r}^{k-1} \eta_i^2(x_Q)$$

$$= cr \sum_{k=l+r}^{\infty} \sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}} \sum_{i=1}^{r} \eta_{k-i}^2(x_Q)|Q| \leqslant cr \sum_{i=1}^{r} \sum_{k=l+r}^{\infty} \sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}} \int_Q \eta_{k-i}^2(x)$$

$$\leqslant cr \sum_{i=1}^{r} \int_{3I} \sum_{k=l+r}^{\infty} \eta_{k-i}^2(x) \leqslant c|I| \log^2(1+|j|).$$

It remains to consider $|g_{k-r-1}(x) - g_{k-r-1}(x_Q)|^2$, for $k \geqslant l+r$. If $r = 0$, this is the only term that appears. We apply (2.37) with k replaced by $k-r$, x replaced by $x_Q \in \tilde{Q}$, and y replaced by $x \in Q + jl(Q) \subseteq 3\tilde{Q}$. (For $r = 0$, set $\tilde{Q} = Q$.) Noting again that $|jl(Q)| \leqslant 2^{r-k} \leqslant 2^{-l} = l(I)$ for $l(Q) = 2^{-k}$, $k \geqslant r+l$, we obtain

$$\sum_{k=l+r}^{\infty} \sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}} \int_{Q+jl(Q)} |g_{k-r-1}(x) - g_{k-r-1}(x_Q)|^2$$

$$\leqslant \sum_{k=l+r}^{\infty} \sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}} \int_{Q+jl(Q)} \Big[c\left(\tfrac{3}{4}\right)^{2(k-r-l)} + c\sum_{i=1}^{k-r-l} \left(\tfrac{3}{4}\right)^i \inf_{z\in 3Q} \eta_{k-r-1}^2(z)\Big]$$

$$\leqslant c \sum_{k=l+r}^{\infty} \left(\tfrac{3}{4}\right)^{2(k-r-l)} |3I| + \sum_{k=l+r}^{\infty} \sum_{\substack{l(Q)=2^{-k}\\ Q\subseteq 3I}} \sum_{i=1}^{k-r-l} \left(\tfrac{3}{4}\right)^i \int_{Q+jl(Q)} \eta_{k-r-1}^2(x)$$

$$\leqslant c|I| + \sum_{i=1}^{\infty} \left(\tfrac{3}{4}\right)^i \sum_{k=l+r+i}^{\infty} \int_{5I} \eta_{k-r-1}^2$$

$$\leqslant c|I| + \sum_{i=1}^{\infty} \left(\tfrac{3}{4}\right)^i |5I| \leqslant c|I|.$$

Combining these estimates in our estimate for B above yields

$$B \leqslant c|I| \sum_{j\in Z^n} (1+|j|)^{-n-1} \log^3(1+|j|) \leqslant c|I|,$$

which completes the proof of (2.33) and hence of the desired result. ∎

We make a few remarks about the result. First, the result holds with the Riesz transforms replaced by any set of singular integral operators satisfying Uchiyama's condition (1.1) in [23], since that is all that is required for the inversion problem in Lemma 2.3. (See § 2 of [23] for the proof of this.) Second, the result holds with the Besov spaces $B_p^{\alpha q}$ in place of $F_p^{\alpha q}$ everywhere (for the same α, p, and q), by the same proof, if, in (2.5) and (2.10), one replaces

$$\Big\|\Big(\sum_{l(Q)\leqslant 1} (|Q|^{-\alpha/n}|s_Q|\chi_Q)^q\Big)^{1/q}\Big\|_{L^p}$$

by

$$\Big(\sum_{\nu=0}^{\infty}\Big(\sum_{l(Q)=2^{-\nu}} (|s_Q|\,|Q|^{1/p-\alpha/n})^p\Big)^{q/p}\Big)^{1/q}$$

and uses the corresponding results from § 7 in [10] in place of Lemma 2.1.

When $q < 1$ and $\alpha > n(1/q-1)_+$, the proof works with minor modifications, coming from the fact that $\|\cdot\|_{F_p^{\alpha q}}$ is only a quasi-norm for $q < 1$. However, the restriction $1 < p < +\infty$ may be essential because the Riesz transforms are not bounded on $F_p^{\alpha q}$ if $p \leqslant 1$. Our proof requires $\alpha > 0$ so that (2.9) is vacuous for our $\mathbf{m}_Q$'s. However, we do not have any reason to suspect that the result is false for $\alpha \leqslant 0$. Of particular interest would be the case of $L^p = F_p^{02}$, $1 < p < +\infty$. Perhaps sharper estimates like those leading to (2.24) could also yield (2.23), and hence the result, if $\alpha = 0$.

3. Smooth truncation in $L^{\alpha,p} \cap \mathrm{BMO}$. For simplicity, we first give the proof of Theorem 1 for the case where α is an integer, i.e. in the case where $L^{\alpha,p}$ is a Sobolev space. We require the following lemmas.

LEMMA 3.1 (Calderón, [5]). *If $1 < p < +\infty$, $m \in Z^+$, and $\varphi \in C_0^\infty(R^n)$, then*

$$\|\varphi\|_{m,p} \approx \|\varphi\|_{L^p} + \sum_{|\xi|=m} \|\partial^\xi \varphi\|_{L^p}.$$

LEMMA 3.2 (Gagliardo–Nirenberg, see [15]). *Suppose $1 < p < +\infty$, $j, m \in Z^+$, and $j < m$. Then there exists $c = c(j, m, p)$ such that for all $\varphi \in C_0^\infty(R^n)$,*

$$\|\partial^\eta \varphi\|_{L^{mp/j}} \leqslant c\|\varphi\|_{m,p}^{j/m} \|\varphi\|_{L^\infty}^{1-j/m} \tag{3.1}$$

holds for all multi-indices η with $|\eta| = j$.

Proof of Theorem 1 for $\alpha \in Z^+$. By density, we may assume $f \in C_0^\infty(R^n)$. Our assumptions (1.2–3) clearly imply the estimate $\|H \circ f\|_{L^p} \leqslant \|H'\|_{L^\infty} \|f\|_{L^p}$.

By Lemma 3.1, then, it suffices to obtain

$$\|\partial^\eta H\circ f\|_{L^p}\leqslant c\big[\|f\|_{\alpha,p}+\sum_{k=2}^{\alpha}(\|f\|_{\alpha,p}+\|f\|_{\mathrm{BMO}})^k\big], \tag{3.2}$$

for $|\eta|=\alpha$. But if $|\eta|=\alpha$, then

$$\partial^\eta H\circ f=\sum_{k=1}^{\alpha}H^{(k)}(f)\sum c_{\eta^1\ldots\eta^k}\,\partial^{\eta^1}f\ldots\partial^{\eta^k}f, \tag{3.3}$$

where the second sum in (3.3) is over all sets of multi-indices $\{\eta^1,\ldots,\eta^k\}$ such that $|\eta^1|+\ldots+|\eta^k|=|\eta|=\alpha$. For such a set of multi-indices, letting $p_j=\alpha/|\eta^j|$ and applying Hölder's inequality yields

$$\Big\|\prod_{j=1}^{k}\partial^{\eta^j}f\Big\|_{L^p}\leqslant\prod_{j=1}^{k}\|\partial^{\eta^j}f\|_{L^{pp_j}}. \tag{3.4}$$

By Theorem 2, we can write $f=g_0+\sum_{i=1}^{n}R_i g_i$ with

$$\sum_{i=0}^{n}(\|g_i\|_{L^\infty}+\|g_i\|_{\alpha,p})\leqslant c(\|f\|_{\mathrm{BMO}}+\|f\|_{\alpha,p}).$$

Using the boundedness of the Riesz transforms on L^{pp_j} and Lemma 3.2, and setting $\theta_j=|\eta^j|/\alpha$, we obtain

$$\|\partial^{\eta^j}f\|_{L^{pp_j}}\leqslant c\sum_{i=0}^{n}\|\partial^{\eta^j}g_i\|_{L^{pp_j}}\leqslant c\sum_{i=0}^{n}\|g_i\|_{\alpha,p}^{\theta_j}\|g_i\|_{L^\infty}^{1-\theta_j}\leqslant c(\|f\|_{\alpha,p}+\|f\|_{\mathrm{BMO}}).$$

Therefore

$$\prod_{j=1}^{k}\|\partial^{\eta^j}f\|_{L^{pp_j}}\leqslant c(\|f\|_{\alpha,p}+\|f\|_{\mathrm{BMO}})^k$$

if $k\geqslant 2$. If $k=1$, then $|\eta^1|=|\eta|=\alpha$, so that for this case

$$\prod_{j=1}^{k}\|\partial^{\eta^j}f\|_{L^{pp_j}}=\|\partial^\eta f\|_{L^p}\leqslant c\|f\|_{\alpha,p}.$$

Thus (3.2) follows by using these estimates and (1.2) in (3.3). ∎

Remark 3.3. From (3.3), it is easy to see that the domain of any STO on a Sobolev space can be characterized by

$$D(T_H)_{m,p}=\{f\in W^{m,p}:$$
$$\Big\|\sum_{k=2}^{m}H^{(k)}(f)\sum_{|\xi^1|+\ldots+|\xi^k|=m}c_{\xi^1\ldots\xi^k}\,\partial^{\xi^1}f\ldots\partial^{\xi^k}f\Big\|_{L^p}<+\infty\}.$$

For $m=2$, we have the particularly simple form

$$D(T_H)_{m,p}=\{f\in W^{2,p}:\ \|H''(f)|\nabla f|^2\|_{L^p}<+\infty\},$$

where ∇f denotes the gradient of f. Thus, any $f\in W^{m,p}$ for which the right side of (3.4) is finite belongs to $D(T_H)_{m,p}$. By the Sobolev imbedding theorem, this is always the case if $p=n/m\geqslant 1$.

Proof of Theorem 1 for $\alpha\notin Z^+$ (sketch). Following the techniques of Adams–Polking ([3]), we can adapt the proof above for $\alpha\in Z^+$ to the nonintegral case. We first require a fractional version of Lemma 3.2.

Lemma 3.4. *Suppose* $1<p<+\infty$, $0<\theta<1$, $\alpha>0$, *and* $\varphi\in C_0^\infty(R^n)$. *Then there exists* $c=c(\alpha,\sigma,\theta)$ *such that*

$$\|\varphi\|_{\alpha\theta,p/\theta}\leqslant c\|\varphi\|_{\alpha,p}^{\theta}\|\varphi\|_{L^\infty}^{1-\theta}. \tag{3.5}$$

Proof. We use the fact that the complex intermediate spaces between $L^{p'}$ and the Hardy space H^1 are more Lebesgue spaces. More precisely, $[L^{p'},H^1]_\theta=L^{r'}$ if $r=p/(1-\theta)$, $0<\theta<1$; see [9]. Hence for $h\in L^{r'}$ given, there exists $H\colon S\to L^{p'}+H^1$, where S is the strip $\{z\in C\colon 0\leqslant \operatorname{Re}z\leqslant 1\}$, such that H is continuous and bounded on S, H is a B-valued analytic function on the interior of S, and $H(z)=H_z\in L^{p'}+H^1$ satisfies $H_\theta=h$. Also,

$$\|h\|_{L^{r'}}\approx\inf_{H:H_\theta=h}\max\big(\sup_{\zeta\in R}\|H_{i\zeta}\|_{L^{p'}},\ \sup_{\zeta\in R}\|H_{1+i\zeta}\|_{H^1}\big).$$

For $f\in C_0^\infty(R^n)$, let

$$g(z)=e^{z^2}\int J^{\alpha z}f\cdot H_z\,dx,$$

where $(J^{\alpha z}f)\hat{}\,(\xi)=(1+|\xi|^2)^{-\alpha z/2}\hat f(\xi)$.

We need the following Fourier multiplier estimates which follow from the Mikhlin multiplier theorem (see [19], p. 232 for the H^1 case): for $\zeta\in R$,

$$\|J^{i\zeta}f\|_{L^p}\leqslant c_{n,p}(1+|\zeta|^n)\|f\|_{L^p},\qquad 1<p<+\infty, \tag{3.6}$$

$$\|J^{i\zeta}f\|_{H^1}\leqslant c_n(1+|\zeta|^n)\|f\|_{H^1}. \tag{3.7}$$

Note that (3.7) implies that $\|J^{i\zeta}f\|_{\mathrm{BMO}}\leqslant c_n(1+|\zeta|^n)\|f\|_{L^\infty}$, by the H^1-BMO duality ([9]). Thus

$$\sup_{\zeta\in R}|g(i\zeta)|\leqslant c\|f\|_{L^p}\|H_{i\zeta}\|_{L^{p'}},$$

$$\sup_{\zeta\in R}|g(1+i\zeta)|\leqslant c\|J^\alpha f\|_{L^\infty}\|H_{1+i\zeta}\|_{H^1}.$$

The three lines theorem ([20]), p. 180) implies

$$|g(\theta)|\leqslant c\|f\|_{L^p}^{1-\theta}\|J^\alpha f\|_{L^\infty}^{\theta}\|h\|_{L^{r'}},$$

which yields (3.5). ∎

Remark 3.5. If $1 \leqslant p, q < +\infty$, $0 < \alpha < +\infty$, $0 < \theta < 1$, and $\varphi \in C_0^\infty(\boldsymbol{R}^n)$, a similar argument, using results from [22] on complex interpolation and Fourier multipliers, yields

$$\|\varphi\|_{F^{\alpha\theta, q}_{p/\theta}} \leqslant c \|\varphi\|^{\theta}_{F^{\alpha q}_p} \|\varphi\|^{1-\theta}_{L^\infty}. \tag{3.8}$$

We recover (3.5) when $q = 2$.

Another result needed for the fractional case is a simpler version of a result of Polking ([8]). We state this for the $F^{\alpha q}_p$ spaces, since that is the context in which the proof is the most clear. We require the following results which follow from the characterization of the $F^{\alpha q}_p$ spaces by "ball means of differences"; see Triebel [22], Ch. 2.5.11. These results go back to Strichartz ([21]) in the Bessel potential case. Let

$$D^{\alpha}_{r,s}(\varphi)(x) = \Big(\int_0^\infty \Big[\int_{|y| \leqslant 1} |\varphi(x+\varrho y) - \varphi(x)|^r \, dy\Big]^{s/r} \varrho^{-1-\alpha s} \, d\varrho\Big)^{1/s}$$

for $\varphi \in C_0^\infty(\boldsymbol{R}^n)$.

Lemma 3.6. (a) *If $0 < \alpha < 1$, $1 < p, q < +\infty$, and $\varphi \in C_0^\infty(\boldsymbol{R}^n)$, then*

$$\|\varphi\|_{F^{\alpha q}_p} \approx \|\varphi\|_{L^p} + \|D^{\alpha}_{1,q}(\varphi)\|_{L^p}.$$

(b) *If $1 \leqslant r < p$, $q = s/r \geqslant 1$, $0 < \alpha < 1$, and $\varphi \in C_0^\infty(\boldsymbol{R}^n)$, then there exists $c = c(\alpha, p, r, s, n)$ such that*

$$\|D^{\alpha}_{r,s}(\varphi)\|_{L^p} \leqslant c \|\varphi\|_{F^{\alpha q}_p}.$$

Finally, we need the two following, more elementary facts ([22], [18]): if $1 < p < +\infty$ and $\varphi \in C_0^\infty(\boldsymbol{R}^n)$, then

$$\|\varphi\|_{F^{\alpha q}_p} \approx \sum_{|\xi| = [\alpha]} \|\partial^\xi \varphi\|_{F^{\alpha - [\alpha], q}_p} + \|\varphi\|_{F^{\alpha - [\alpha], q}_p}, \tag{3.9}$$

and, if $0 < \lambda, \mu < 1$ and $\varphi, \psi \in C_0^\infty(\boldsymbol{R}^n)$, then

$$D^{\alpha}_{r,s}(\psi\varphi) \leqslant |\varphi| \, D^{\alpha}_{r,s}(\psi) + |\psi| \, D^{\alpha}_{r,s}(\varphi) + D^{\lambda\alpha}_{r/\mu, s/\mu}(\varphi) \, D^{(1-\lambda)\alpha}_{r/(1-\mu), s/(1-\mu)}(\psi). \tag{3.10}$$

With these facts, the proof follows as in [3]. We use Theorem 2 above to obtain the following estimate: for $0 < \sigma < \alpha - [\alpha]$, $|\xi| \leqslant [\alpha]$, $r = \alpha/(\sigma + |\xi|)$, and $t = \alpha p/(\sigma + |\xi|)$, we have

$$\|D^{\sigma}_{r,2r}(\partial^\xi f)\|_{L^t} \leqslant c \|\partial^\xi f\|_{\sigma, t} \leqslant c \sum_{j=0}^{n} \|\partial^\xi g_j\|_{\sigma, t}.$$

Then by Lemma 3.4,

$$\|D^{\sigma}_{r,2r}(\partial^\xi f)\|_{L^t} \leqslant c \sum_{j=0}^{n} \|g_j\|^{\theta}_{\alpha, p} \|g_j\|^{1-\theta}_{L^\infty} \leqslant c (\|f\|_{\alpha, p} + \|f\|_{\mathrm{BMO}}),$$

where now $\theta = (\sigma + |\xi|)/\alpha$. ■

Remark 3.7. We have written the lemmas for the case $\alpha \notin \boldsymbol{Z}$ in the context of the $F^{\alpha q}_p$ spaces to make a point about the possible validity of the result in general for $f \in F^{\alpha q}_p$. In fact, the above arguments can be carried through to obtain

$$\|H \circ f\|_{F^{\alpha q}_p} \leqslant c \Big[\|f\|_{F^{\alpha q}_p} + \sum_{k=2}^{[\alpha]} (\|f\|_{F^{\alpha q}_p} + \|f\|_{\mathrm{BMO}})^k\Big] \tag{3.11}$$

if $0 < \alpha \notin \boldsymbol{Z}$, $1 < p < +\infty$, and $1 < q \leqslant 2$. It is not known if (3.11) is valid for other values of q. However, the difficulty involved for the case $\alpha \in \boldsymbol{Z}$ can be illustrated by considering the case $\alpha = 1$. We wish to show that if $f \in F^{1q}_p$, then $H \circ f \in F^{0q}_p$ and $H'(f) \cdot \partial f/\partial x_i \in F^{0q}_p$. Certainly $\partial f/\partial x_i \in F^{0q}_p$, but it is not clear that $H'(f)$ is a pointwise multiplier for F^{0q}_p. If $q = 2$ this is obvious since $F^{02}_p = L^p$ and $H'(f) \in L^\infty$, but L^∞ does not in general multiply F^{0q}_p if $q \neq 2$ (see [12]). However, if $H'(f)$ is Hölder continuous of any order, then it is a pointwise multiplier on F^{0q}_p (see [21], p. 1043). Thus the result for $\alpha = 1$ holds if $p > n$ since $H'(f)$ is Hölder continuous if f is (when H'' is bounded), and $F^{\alpha q}_p$ functions are Hölder continuous when $\alpha > n/p$.

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Analytic stochastic processes

by

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Abstract. A concept of analytic stochastic process with respect to a given Brownian motion is introduced. In terms of the random Fourier transform a relationship between analytic processes and some classes of entire functions is established.

1. Preliminaries and notation. Throughout this paper $\boldsymbol{R}$ and $\boldsymbol{C}$ will denote the real and the complex field respectively. A seminorm induced by a Hermitian bilinear form on a linear space over $\boldsymbol{C}$ will be called a *Hermitian seminorm*. Let $T \in (0, \infty]$. We shall be concerned with locally convex complete topological linear spaces $\mathscr{X}$ with the topology defined by a separating family $\{p_t: t \in (0, T)\}$ of Hermitian seminorms fulfilling the following condition: for every pair $t, u \in (0, T)$, $t < u$, there exists a positive number $c = c(t, u)$ such that $p_t \leqslant c p_u$. It is evident that each countable system p_{t_n} with $t_n \to T$ determines the same topology in $\mathscr{X}$. Hence it follows that $\mathscr{X}$ is a B_0-space ([6], p. 59). It is convenient to have a term for such a space $\mathscr{X}$ with a fixed family $\{p_t: t \in (0, T)\}$. There is no standard term for this, but we shall say in this paper that $\mathscr{X}$ is a *local Hilbert space*. Two local Hilbert spaces $\mathscr{X}$ and $\mathscr{X}'$ with the families of seminorms $\{p_t: t \in (0, T)\}$ and $\{p'_t: t \in (0, T')\}$ respectively are said to be *isomorphic* if $T = T'$ and there exists a linear map l from $\mathscr{X}$ onto $\mathscr{X}'$ such that $p_t(x) = p'_t(l(x))$ for all $x \in \mathscr{X}$ and $t \in (0, T)$.

Let $\mathscr{X}_n$ $(n = 1, 2, \ldots)$ be a sequence of local Hilbert spaces with the families $\{p_{n,t}: t \in (0, T)\}$ of seminorms respectively. Moreover, we assume that for every pair $t, u \in (0, T)$, $t < u$, there exists a positive number $c = c(t, u)$ such that $p_{n,t} \leqslant c p_{n,u}$ for all n. The orthogonal sum $\bigoplus_{n=1}^{\infty} \mathscr{X}_n$ is defined as the set of all sequences $x = \{x_n\}$ where $x_n \in \mathscr{X}_n$ and $\sum_{n=1}^{\infty} p_{n,t}^2(x_n) < \infty$ $(t \in (0, T))$ with addition and scalar multiplication defined coordinatewise and the topology determined by the Hermitian seminorms

$$p_t(x) = \Big(\sum_{n=1}^{\infty} p_{n,t}^2(x_n)\Big)^{1/2} \quad (t \in (0, T)).$$

It is clear that this orthogonal sum is also a local Hilbert space.

Given $n \geqslant 1$ and $t > 0$ we put

$$\Delta_n(t) = \{(t_1, \ldots, t_n): 0 \leqslant t_1 \leqslant \ldots \leqslant t_n < t\}.$$