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Integral extension procedures in weakly σ-complete lattice-ordered groups, II

by

M. WILHELM (Wrocław)

Abstract. Interrelations between generalized MacNeille–Mikusiński, Stone and Daniell integral extension procedures are studied.

1. Introduction. Let G be a commutative l-group, L an l-subgroup of G and ν a finite σ-subadditive l-seminorm on L. (For basic notation and terminology see Section 2.) In part I of this work ([7]) we defined and studied the generalized MacNeille–Mikusiński extension (L_M, ν_M) of (L, ν). Here, in Section 3, we prove that (L_M, ν_M) is equal to Stone's extension (L_S, ν_S) provided G is weakly σ-complete. (This order-completeness type notion was introduced and studied in [6] and [7].)

In Section 4 we define and investigate the generalized Daniell extension (L_D, ν_D) of (L, ν) under the assumption that ν has the Fatou property. In Section 5 we compare Stone's and Daniell's extensions. In particular, the two extensions are identical if ν has the Beppo Levi property, and "almost" identical if ν has the saturability property.

Some theorems are adaptations to this more abstract setting of the results known for function spaces: Theorem 1 – cf. [4], p. 913; Theorems 2 and 6 – cf. [2], part I, p. 163; Theorem 4 – cf. [5], p. 238.

2. Notation and terminology. Let G be a *commutative* lattice-ordered group ($=$ l-group). All joints (sup, $\vee$) and meets (inf, $\wedge$) are taken with respect to the whole of G. The notation $a \leqslant' \sup_n a_n$ means that $\inf_n (a-a_n)^+ = 0$. (Here $b^+ = b \vee 0$.) For $a_n \in G^+$,

$$a \leqslant' \sum_n a_n \text{ means that } a \leqslant' \sup_k \Big(\sum_{n \leqslant k} a_n\Big).$$

Similarly,

$$\inf_n a_n {}' \leqslant a \text{ means that } \inf_n (a_n - a)^+ = 0.$$

(The accents beside "$\leqslant$" mark that sup, $\sum$ or inf need not exist in G.) We write $a \sim \sum_n a_n$ if

$$|a - \sum_{n \leqslant k} a_n| \leqslant' \sum_{n>k} |a_n| \quad \text{for } k = 1, 2, \ldots$$

This expansion was studied in Section 2 of [7]. We say that G is *weakly σ-complete* if for every sequence $\{a_n\} \subset G$ there exists an element $a \in G$ such that $a \sim \sum_n a_n$. Such G need not be Archimedean; cf. [7], Section 3.

Let L be a subgroup-sublattice ($=$ l-subgroup) of G and let ν be an l-seminorm on L, i.e. a function on L into $[0, \infty]$ such that $\nu(0) = 0$, $\nu(a+b) \leqslant \nu(a) + \nu(b)$, and $\nu(a) \leqslant \nu(b)$ whenever $|a| \leqslant |b|$ $(a, b \in L)$. Throughout this paper we assume that ν is *finite* ($\nu(a) < \infty$ for all $a \in G$) and *σ-subadditive*, i.e.

$$\nu(a) \leqslant \sum_n \nu(a_n) \quad \text{whenever } a = \sum_n a_n \quad (a, a_n \in L^+).$$

We write $a \overset{\sigma}{\sim} \sum_n a_n$ if $a \sim \sum_n a_n$, $\{a_n\} \subset \operatorname{dom} \nu = L$ and $\sum_n \nu(a_n) < \infty$; the set of all elements $a \in G$ possessing such an expansion is denoted by L_M. The equality

$$\nu_M(a) = \lim_k \nu(\sum_{n \leqslant k} a_n) \quad \text{whenever } a \overset{\sigma}{\sim} \sum_n a_n$$

defines (correctly) a finite σ-subadditive l-seminorm ν_M on L_M which extends ν; cf. [7], Section 4.

3. The seminorm ν^* and the extension (L_S, ν_S). *The generalized Stone extension* (L_S, ν_S) of (L, ν) is defined as follows (cf. [3], [2], [4]). For each $a \in G$ we put

$$\nu^*(a) = \inf \{ \sum_{n=1}^{\infty} \nu(a_n) : \{a_n\} \subset L^+ \text{ and } |a| \leqslant' \sum_n a_n \},$$

where the convention $\inf \emptyset = \infty$ is adopted; ν^* is a σ-subadditive l-seminorm on G and $\nu^*(a) = \nu(a)$ for all $a \in L$ (cf. [6], Lemma 5). If G is weakly σ-complete (in its ordering, see Section 2), then G (endowed with ν^*) is metrically complete; cf. [6], Theorem 6. Let L_S denote the closure of L in G (endowed with the topology induced by ν^*) and put $\nu_S = \nu^* | L_S$; L_S is an l-subgroup of G and ν_S is a finite σ-subadditive l-seminorm on L_S; L_S is metrically complete provided G is weakly σ-complete.

According to the general definition of the expansion, as recalled in Section 2, $a \overset{\nu^*}{\sim} \sum_n a_n$ means that $a \sim \sum_n a_n$, $\{a_n\} \subset \operatorname{dom} \nu^* = G$ and $\sum_{n=1}^{\infty} \nu^*(a_n) < \infty$.

LEMMA 1. *If $a \overset{\nu^*}{\sim} \sum_n a_n$, then $\lim_k \nu^*(a - \sum_{n \leqslant k} a_n) = 0$.*

Proof. $\nu^*(a - \sum_{n \leqslant k} a_n) \leqslant \sum_{n>k} \nu^*(a_n) \to 0$ as $k \to \infty$.

PROPOSITION 1. *We have $L_M \subset L_S$ and $\nu_M(a) = \nu_S(a)$ for all $a \in L_M$.*

Proof. Let $a \overset{\nu}{\sim} \sum_n a_n$. By Lemma 1, $a \in L_S$ and

$$\nu_M(a) = \lim_k \nu(\sum_{n \leqslant k} a_n) = \nu^*(a).$$

Remark 1. In Theorem 5 of [7] we had to prove that the definition of ν_M is correct and that ν_M is a σ-subadditive l-seminorm; all this follows immediately from Proposition 1 and the properties of ν^*.

LEMMA 2. *If $\nu^*(a) = 0$, then $a \in L_M$.*

Proof. Choose $a_{(m,n)} \in L^+$ so that

$$|a| \leqslant' \sum_n a_{(m,n)}, \quad \sum_n \nu(a_{(m,n)}) < 2^{-m}, \quad m \in N.$$

Let f be a one-to-one mapping of N onto $N \times N$ and define

$$b_1 = a_{f(1)}, \quad b_2 = -b_1, \quad b_3 = a_{f(2)}, \quad b_4 = -b_3, \quad \ldots$$

Given $k \in N$, there is an index m such that

$$\{(m, n) : n \in N\} \subset f(\{k+1, k+2, \ldots\}),$$

and consequently

$$|a| \leqslant' \sum_{n > 2k} |b_n|.$$

Hence

$$|a - \sum_{n \leqslant 2k} b_n| = |a| \leqslant' \sum_{n > 2k} |b_n|,$$

$$|a - \sum_{n \leqslant 2k-1} b_n| = |a - b_{2k-1}| \leqslant |b_{2k-1}| + |a| = |b_{2k}| + |a| \leqslant' \sum_{n > 2k-1} |b_n|.$$

This shows that $a \overset{\nu}{\sim} \sum_n b_n$.

THEOREM 1. *If G is weakly σ-complete, then the extensions (L_M, ν_M) and (L_S, ν_S) are identical.*

Proof. Let $a \in L_S$. Choose $a_n \in L$ so that

$$\lim_k \nu^*(a - \sum_{n \leqslant k} a_n) = 0, \quad \sum_n \nu(a_n) < \infty.$$

Since G is weakly σ-complete, there exists $b \in G$ satisfying $b \overset{\nu}{\sim} \sum_n a_n$; we have $b \in L_M$ and Lemma 1 implies that $\nu^*(a - b) = 0$. By Theorem 4 of [7] and Lemma 2, $a = (a-b) + b \in L_M$.

In Section 2 of [7], $L_\sim$ was defined as the set of all elements $a \in G$ possessing an expansion $a \sim \sum_n a_n$ with some $\{a_n\} \subset L$. Let us consider

$\nu = 0$. For this ν we have $L_M = L_\sim$, $\nu^*(G) \subset \{0, \infty\}$, $L_S = \{a \in G\colon \nu^*(a) = 0\}$, and Lemma 2 proves that $L_\sim = L_S$. Thus we have obtained

COROLLARY 1. *An element $a \in G$ belongs to $L_\sim$ if and only if $|a| \leqslant' \sum_n a_n$ for some sequence $\{a_n\} \subset L^+$ (or, equivalently, $|a| \leqslant' \sup_n b_n$ for some sequence $\{b_n\} \subset L^+$).*

Remark 2. For ν infinite, Stone's procedure works equally well, while MacNeille–Mikusiński's procedure must be slightly modified. Namely, in the definition of the expansion $a \overset{\nu}{\sim} \sum_n a_n$ the condition $\sum_{n=1}^{\infty} \nu(a_n) < \infty$ must be replaced with $\sum_{n=2}^{\infty} \nu(a_n) < \infty$. Then L_M contains L, and Theorems 4–6 of [7] as well as the results of this section remain valid.

4. The seminorm ν^0 and the extension (L_D, ν_D). In this section we define a generalized Daniell extension of (L, ν) (cf. [1], [2]). For each $a \in G$ we put

$$\nu^0(a) = \inf \{\sup_n \uparrow \nu(a_n)\colon \ \{a_n\} \subset L^+ \text{ and } |a| \leqslant' \sup_n \uparrow a_n\},$$

where $\inf \emptyset = \infty$. It is easy to verify that ν^0 itself is an l-seminorm on G.

PROPOSITION 2. *The l-seminorm ν^0 extends ν if and only if ν has the Fatou property:*

$$\text{(F)} \qquad a = \sup_n \uparrow a_n \text{ implies } \nu(a) = \lim_n \uparrow \nu(a_n) \quad (a, a_n \in L^+).$$

Proof. Condition (F) is equivalent to

$$\text{(F)}' \qquad a \leqslant' \sup_n \uparrow a_n \quad \text{implies} \quad \nu(a) \leqslant \sup_n \uparrow \nu(a_n) \quad (a, a_n \in L^+).$$

Remark 3. Every l-seminorm satisfying (F) is σ-subadditive.

From now on we assume that ν has the *Fatou property*.

THEOREM 2. *The l-seminorm ν^0 is σ-subadditive.*

Proof. Let $a, a_n \in G^+$, $a = \sum_n a_n$, $\sum_{n=1}^{\infty} \nu^0(a_n) < \infty$, $\varepsilon > 0$. Choose $a_{m,n} \in L^+$ so that for each index n

$$a_n \leqslant' \sup_m \uparrow a_{m,n}, \qquad \lim_m \uparrow \nu(a_{m,n}) < \nu^0(a_n) + \varepsilon 2^{-n}.$$

Put $b_k = \sum_{n \leqslant k} a_{k,n}$; we have $a \leqslant' \sup_k \uparrow b_k$ (because $\sum_{n \leqslant p} a_n \leqslant' \sup_k \uparrow b_k$ for each p), and so

$$\nu^0(a) \leqslant \sup_k \uparrow \nu(b_k) \leqslant \sup_k \uparrow \Big(\sum_{n \leqslant k} \nu(a_{k,n})\Big) \leqslant \sum_{n \leqslant k} \sup_m \uparrow \nu(a_{m,n})$$

$$\leqslant \sum_{n=1}^{\infty} [\nu^0(a_n) + \varepsilon 2^{-n}] = \sum_n \nu^0(a_n) + \varepsilon.$$

Since ν^0 is σ-subadditive, Theorem 5 of [6] yields

THEOREM 3. *If G is weakly σ-complete, then the seminormed space (G, ν^0) is metrically complete.*

Now L_D is defined as the closure of L in G (endowed with ν^0) and ν_D as the restriction of ν^0 to L_D. Clearly, L_D is an l-subgroup of G and ν_D is a finite σ-subadditive l-seminorm on L_D which extends ν; L_D is metrically complete provided G is weakly σ-complete.

The classical Daniell extension of (L, ν), constructed for additive ν, turns out to be identical with (L_D, ν_D). This follows from

PROPOSITION 3. *An element $a \in G$ belongs to L_D if and only if for each $\varepsilon > 0$ there are $b_n, c_n \in L$ such that $\nu(c_n - b_n) < \varepsilon$ for all n and*

$$\lim_n \downarrow b_n \leqslant' a \leqslant' \lim_n \uparrow c_n.$$

In this case b_n, c_n can be chosen so that $b_n \leqslant c_n$, $\nu_D(a - b_n) < \varepsilon$ and $\nu_D(a - c_n) < \varepsilon$ for all n.

Proof. Necessity. Let $a \in L_D$ and $\varepsilon > 0$. There exist $d \in L$ and $d_n \in L^+$ such that

$$|a - d| \leqslant' \sup_n \uparrow d_n, \qquad \lim_n \uparrow \nu(d_n) < \varepsilon/2.$$

Define $b_n = d - d_n$ and $c_n = d + d_n$. We have $\nu_D(a - b_n) \leqslant \nu_D(a - d) + \nu(d_n) < \varepsilon$, and similarly for c_n.

Sufficiency. We have $|a - b_1| \leqslant' \sup_n \uparrow (c_n - b_n)$, and so $\nu^0(a - b_1) \leqslant \varepsilon$. Thus $a \in L_D$.

Let us consider the so-called *Daniell property* of ν:

$$\text{(D)} \qquad a_n \searrow 0 \text{ implies } \nu(a_n) \searrow 0 \quad (a_n \in L^+).$$

Here are equivalent forms of this property:

$$\text{(D)}' \qquad a_n \nearrow a \text{ implies } \nu(a - a_n) \searrow 0 \quad (a, a_n \in L^+);$$

$$\text{(D)}'' \qquad a_n \searrow a \text{ implies } \nu(a - a_n) \searrow 0 \quad (a, a_n \in L^+);$$

$$\text{(D)}''' \qquad \inf_n \downarrow a_n \leqslant' a \text{ implies } \lim_n \downarrow \nu(a_n) \leqslant \nu(a) \quad (a, a_n \in L^+).$$

The Daniell property (see (D)′) implies the Fatou property.

LEMMA 3. *Suppose ν has the Daniell property. If $a_n \in L^+$, $a \in G^+$ and $\inf_n \downarrow a_n \leqslant' a$, then $\lim_n \downarrow \nu(a_n) \leqslant \nu^0(a)$.*

Proof. Let $b_n \in L^+$ and $a \leqslant' \sup_n \uparrow b_n$. Since $(a_n - b_n)^+ \searrow 0$, we infer that $\nu((a_n - b_n)^+) \searrow 0$. The inequality

$$\nu(a_n) \leqslant \nu(b_n) + \nu((a_n - b_n)^+)$$

implies

$$\lim_n \downarrow \nu(a_n) \leqslant \lim_n \uparrow \nu(b_n).$$

This justifies the assertion.

Proposition 3 and Lemma 3 yield

THEOREM 4. *If ν has the Daniell property, then for each element $a \in L_D$ we have the equality*

$$\nu_D(a) = \sup \{\lim_n \downarrow \nu(b_n): \{b_n\} \subset L^+ \text{ and } \inf_n \downarrow b_n \grave{\leqslant} |a|\}.$$

5. Comparison of (L_S, ν_S) and (L_D, ν_D). In this section we continue to assume that ν has the *Fatou property*.

PROPOSITION 4. *We have $\nu^0(a) \leqslant \nu^*(a)$ for all $a \in G$. Hence $L_S \subset L_D$ and $\nu_S(a) = \nu_D(a)$ for all $a \in L_S$.*

Proof. Let $a_n \in L^+$ and $|a| \grave{\leqslant} \sum_n a_n$. Then

$$\nu^0(a) \leqslant \sup_k \uparrow \nu(\sum_{n \leqslant k} a_n) \leqslant \sup_k \uparrow \sum_{n \leqslant k} \nu(a_n) = \sum_n \nu(a_n).$$

THEOREM 5. *Let G be weakly σ-complete. For each element $a \in L_D$ there exists $b \in L_S$ with $\nu^0(a-b) = 0$. Hence the equality $L_S = L_D$ holds if and only if*

(1) $$\nu^0(c) = 0 \text{ implies } \nu^*(c) = 0 \quad (c \in G).$$

Proof. Given $a \in L_D$, there are $a_n \in L$ with $\lim_n \nu^0(a-a_n) = 0$. Since $\{a_n\}$ is a Cauchy sequence in $L \subset L_S$ and (L_S, ν_S) is complete, there exists an element $b \in L_S$ satisfying $\lim_n \nu^*(b-a_n) = 0$. Thus $\lim_n \nu^0(b-a_n) = 0$ (Proposition 4), and so $\nu^0(a-b) = 0$. The second assertion is a consequence of the first one and Proposition 4.

LEMMA 4. *Suppose ν has the Beppo Levi property*:

(BL) $$\sup_k \uparrow \nu(\sum_{n \leqslant k} a_n) < \infty \text{ implies } \lim_n \nu(a_n) = 0 \quad (a_n \in L^+).$$

Let $a_n \in L^+$ and $\sup_k \uparrow \nu(\sum_{n \leqslant k} a_n) = \alpha < \infty$. Then

(2) *Given $\varepsilon > 0$, there are indices $n_1 < n_2 < \ldots$ such that $\sum_{i=1}^{\infty} \nu(\bar{a}_i) < \alpha + \varepsilon$, where $\bar{a}_i = a_{n_{i-1}+1} + \ldots + a_{n_i}$ for $i = 1, 2, \ldots$ $(n_0 = 0)$.*

Proof. Observe that the series $\sum_n a_n$ is ν-Cauchy (otherwise $\nu(\bar{a}_i) > \delta$ for some $\{n_i\}$ and $\delta > 0$, which contradicts (BL)). Hence there are indices $n_1 < n_2 < \ldots$ such that $\nu(\bar{a}_i) < \varepsilon 2^{-i}$ for $i \geqslant 2$, and we have

$$\sum_i \nu(\bar{a}_i) < \nu(\bar{a}_1) + \sum_{i \geqslant 2} \varepsilon 2^{-i} < \nu(\bar{a}_1) + \varepsilon \leqslant \alpha + \varepsilon.$$

THEOREM 6. *If ν has the Beppo Levi property, then $\nu^0 = \nu^*$.*

Proof. Assume that $a \in G$, $a_n \in L^+$, $|a| \leqslant' \sum_n a_n$,

$$\sup_k \uparrow \nu(\sum_{n \leqslant k} a_n) = \alpha < \infty,$$

$\varepsilon > 0$. Let $\bar{a}_i$ be as in Lemma 4; since $|a| \leqslant' \sum_i \bar{a}_i$, we infer that

$$\nu^*(a) \leqslant \sum_i \nu(\bar{a}_i) < \alpha + \varepsilon.$$

This shows that $\nu^*(a) \leqslant \alpha$, and consequently $\nu^*(a) \leqslant \nu^0(a)$ whenever $\nu^0(a) < \infty$ (because $\nu^0(a)$ is the least upper bound of such numbers α).

An important special case is when ν is additive, i.e.

(A) $$\nu(a+b) = \nu(a) + \nu(b) \quad (a, b \in L^+);$$

then

$$\sup_k \nu(\sum_{n \leqslant k} a_n) = \sum_{n=1}^{\infty} \nu(a_n) \quad (a_n \in L^+)$$

and so ν has the Beppo Levi property.

Now let us consider the so-called *saturability property* of ν, which is weaker than the Beppo Levi property:

(S) $$\text{If } \sum_{n \leqslant k} a_n \leqslant a \text{ for all } k, \text{ then } \lim_n \nu(a_n) = 0 \quad (a, a_n \in L^+).$$

We will write

$$\sum_n a_n \grave{\leqslant}' \sum_p c_p$$

if $\sum_{n \leqslant k} a_n \leqslant' \sum_p c_p$ for all k $(a_n, c_p \in G^+)$.

LEMMA 5. *Suppose ν has the saturability property, $a_n \in L^+$ and*

$$\alpha = \sup_k \uparrow \nu(\sum_{n \leqslant k} a_n).$$

If there exists a sequence $\{c_p\} \subset G^+$ such that

$$\sum_n a_n \grave{\leqslant}' \sum_p c_p, \quad \sum_p \nu^*(c_p) < \infty,$$

then $\lim_n \nu(a_n) = 0$ and condition (2) holds.

Proof. (Notice that $\alpha < \infty$.) We may additionally assume that $c_p \in L^+$ (otherwise choose elements $c_{jp} \in L^+$ with $c_p \leqslant' \sum_j c_{jp}$ and $\sum_j \nu(c_{jp}) < \nu^*(c_p) + 2^{-p}$, and arrange them in a sequence). Let $\varepsilon > 0$. Fix p_0 so that

$\sum_{p>p_0} \nu(c_p) < \varepsilon/2$ and put $c = \sum_{p \leqslant p_0} c_p$. Define inductively $b_n \in L^+$ so that

$$\sum_{n \leqslant k} b_n = c \wedge \sum_{n \leqslant k} a_n \quad \text{for } k = 1, 2, \ldots$$

Clearly $b_n \leqslant a_n$ for all n and

$$c + \sum_{n \leqslant k} (a_n - b_n) = c + \sum_{n \leqslant k} a_n - c \wedge \sum_{n \leqslant k} a_n = c \vee \sum_{n \leqslant k} a_n \leqslant' \sum_{p=1}^{\infty} c_p.$$

Hence

$$\sum_{n \leqslant k} (a_n - b_n) \leqslant' \sum_{p>p_0} c_p,$$

which implies

$$\nu(a_n - b_n) \leqslant \sum_{p>p_0} \nu(c_p) < \varepsilon/2 \quad \text{for all } n.$$

Since $\sum_{n \leqslant k} b_n \leqslant c \in L^+$ for all k, we infer that $\lim_n \nu(b_n) = 0$, and so $\nu(a_n) < \varepsilon$ for n sufficiently large. Property (2) can be deduced as in the previous lemma.

THEOREM 7. *Let ν have the saturability property. Then for every $a \in G$ with $\nu^*(a) < \infty$ we have*:

(i) $\nu^0(a) = \nu^*(a)$.

(ii) *$a \in L_S$ if and only if $a \in L_D$.*

Hence the equality $\nu^0 = \nu^*$ holds if and only if

(3) $$\nu^0(a) < \infty \text{ implies } \nu^*(a) < \infty \quad (a \in G).$$

Proof. (i) There are $c_n \in L^+$ with $|a| \leqslant' \sum_n c_n$ and $\sum_n \nu(c_n) < \infty$. Let $b_n \in L^+$, $|a| \leqslant' \sup_n \uparrow b_n$. Define $a_1 = b_1 \wedge c_1$ and

$$a_n = b_n \wedge \sum_{i \leqslant n} c_n - b_{n-1} \wedge \sum_{i<n} c_n \quad \text{for } n = 2, 3, \ldots;$$

we have

$$\sum_{n \leqslant k} a_n = b_k \wedge \sum_{n \leqslant k} c_n \quad \text{for all } k, \quad |a| \leqslant' \sum_n a_n.$$

We are in a position to apply Lemma 5: given $\varepsilon > 0$, there are indices $n_1 < n_2 < \ldots$ such that

$$\nu^*(a) \leqslant \sum_i \nu(\bar{a}_i) < \alpha + \varepsilon \leqslant \sup_k \uparrow \nu(b_k) + \varepsilon.$$

Thus $\nu^*(a) \leqslant \nu^0(a)$.

(ii) Assume that $a \in L_D$ and $\varepsilon > 0$. There exists $b \in L$ such that $\nu^0(a-b) < \varepsilon$. Since $\nu^*(a-b) \leqslant \nu^*(a) + \nu(b) < \infty$, part (i) shows that $\nu^*(a-b) = \nu^0(a-b)$. Thus $a \in L_S$.

COROLLARY 2. *Let G be weakly σ-complete and let ν have the saturability property. The equality $L_S = L_D$ holds if and only if*

(4) $$\nu^0(a) = 0 \text{ implies } \nu^*(a) < \infty \quad (a \in G).$$

Proof. Condition (4) and Theorem 7 yield condition (1) of Theorem 5.

Finally, we notice that the saturability property implies the Daniell property; this can be proved as in [5], pp. 239–240 (cf. also [6], Theorem 2). Thus (A) $\Rightarrow$ (BL) $\Rightarrow$ (S) $\Rightarrow$ (D) $\Rightarrow$ (F); counterexamples to the converse implications are given in [5].

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INSTYTUT MATEMATYKI POLITECHNIKI WROCŁAWSKIEJ
INSTITUTE OF MATHEMATICS, WROCŁAW TECHNICAL UNIVERSITY
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

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