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Received December 12, 1986 (2254)

STUDIA MATHEMATICA, T. LXXXIX. (1988)

Left-invariant degenerate elliptic operators on semidirect extensions of homogeneous groups

by

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Abstract. Let a solvable Lie group S be the semidirect product of a nilpotent group N and an abelian subgroup A such that Ad_a, $a \in A$, are diagonalizable. For a class of second order left-invariant degenerate elliptic operators L on S we study bounded L-harmonic functions F. We describe L-boundaries of S and prove, for L hypoelliptic, the convergence of Poisson integrals to functions on the boundaries. The results of the paper imply theorems on admissible semirestricted convergence of classical Poisson integrals on symmetric spaces.

Introduction. This paper treats harmonic functions with respect to left-invariant degenerate elliptic operators L on a class of solvable Lie groups S. Our approach is motivated by the classical theory of harmonic functions with respect to the Laplace–Beltrami operator on a noncompact symmetric space $X = G/K$ considered as $\bar{N}A$, where $G = \bar{N}AK$ is the Iwasawa decomposition of the group of its isometries, G. We find a class of boundaries of S and study the Poisson integrals on them. Among them there is a maximal boundary in the sense that the Poisson integrals of bounded Borel functions on it reproduce all the L-harmonic functions.

Our main result is the almost everywhere convergence of Poisson integrals of L^p functions, $p > 1$. This gives a natural extension to the context of our spaces and operators of the admissible semirestricted convergence for symmetric spaces. The main problem we shall have to overcome is little information on Poisson kernels. We have no explicit formula; we are able, however, to prove enough properties of the Poisson kernel to obtain the convergence theorem. Before we sketch our results and techniques in greater detail we shall describe some of the background facts about harmonic functions on symmetric spaces.

Harmonic functions on symmetric spaces have been studied thoroughly. By a *harmonic function* on $G/K = \bar{N}A$ one means a function F such that

1985 *AMS Subject Classification*: Primary 22E30; Secondary 43A85, 58G20.
Key words and phrases: degenerate elliptic left-invariant operators, Poisson kernels, convergence to the boundary, homogeneous groups.

$DF = 0$ for all G-invariant operators D which annihilate constants. If F is bounded then it is harmonic if and only if $\Delta F = 0$, where Δ is the Laplace–Beltrami operator on G/K. Boundaries for symmetric spaces (in the sense of [16]) can be identified with some normal subgroups $\bar{N}_1$ of $\bar{N}$. These nilpotent groups have the additional structure of possessing "dilations". The Poisson integrals can then be realized as convolutions on these groups and this fact greatly facilitates their analysis. $\bar{N}$ is a maximal boundary, i.e. every bounded harmonic function on G/K is representable as the Poisson integral of a function $f \in L^\infty(\bar{N})$ [16]. Moreover, the Poisson integrals of functions $f \in L^p(\bar{N}_1)$, $p \geqslant 1$, and of bounded measures are harmonic.

For symmetric spaces two modes of convergence of Poisson integrals up to a boundary: restricted and admissible semirestricted, defined by A. Korányi, have been studied. Admissible convergence has been proved first for $p = \infty$ by A. Korányi [16] and A. W. Knapp and A. R. Williamson [15], then for p sufficiently large by A. W. Knapp, L. A. Lindahl and E. M. Stein [17] and finally for all $p > 1$ by P. Sjögren [22]. If $p = 1$ the admissible convergence does not hold. The product of discs is a counterexample [14], [18]. Moreover, we have restricted convergence proved by E. M. Stein [23]. A simpler proof has recently been found by P. Sjögren [22].

In this paper we consider a class of solvable Lie groups S satisfying the following conditions:

(i) The Lie algebra $\mathfrak{s}$ of S is a semidirect sum of a nilpotent algebra $\boldsymbol{n}$ and an abelian algebra $\boldsymbol{a}$.

(ii) The operators $\mathrm{ad}_H|_{\boldsymbol{n}}$, $H \in \boldsymbol{a}$, are diagonalizable.

(iii) There is an $H \in \boldsymbol{a}$ such that the operator ad_H has strictly positive eigenvalues.

Of course, this class includes in particular noncompact symmetric spaces $\bar{N}A$.

We study not only the Laplace–Beltrami operator but the class of left-invariant operators of the form

$$L = X_1^2 + \ldots + X_j^2 + X$$

where $X_1, \ldots, X_j$, $X \in \mathfrak{s}$. In the case of a symmetric space this covers second order elliptic degenerate operators which commute with the $\bar{N}A$ action and annihilate constants. The study of such an operator on an abelian group is not interesting because it is elliptic on a subgroup. On the other hand, if L is degenerate elliptic and G-invariant on a symmetric space then it must be elliptic in view of the action of K. In our situation a new phenomenon appears. If $j < \dim \mathfrak{s}$ and $X_1, \ldots, X_j$ generate $\mathfrak{s}$, the operator L is only hypoelliptic but for our purposes has as good properties as an elliptic one.

The operator L is the infinitesimal generator of a convolution semigroup of probability measures $\{\mu_t\}_{t>0}$, and for a bounded function F the equality $LF = 0$ (in the sense of distributions) is equivalent to $F * \mu_1 = F$, i.e. F is a μ-harmonic function in the sense of Azencott–Cartier [1] with $\mu = \check{\mu}_1$. Therefore we can apply the theory of boundaries for μ-harmonic functions developed by A. Raugi [21] and Y. Guivarc'h [9].

By a theorem of L. Birgé and A. Raugi [2] nontrivial bounded L-harmonic functions exist on S if and only if $\lambda(Z) < 0$ for a root λ of $\mathfrak{s}$, where $X = Y + Z$, $Y \in \boldsymbol{n}$, $Z \in \boldsymbol{a}$. Then the maximal boundary $N_1(L)$ is the subgroup of N with the Lie algebra being the sum of all the eigenspaces $\boldsymbol{n}^\lambda$ corresponding to λ such that $\lambda(Z) < 0$. The other boundaries are subgroups of N_1 called in this paper homogeneous subgroups. For the Laplace–Beltrami operator on a symmetric space we have $\lambda(Z) < 0$ for all λ, $N_1(L) = N$ and the boundaries in the sense of [16] are included in our class of boundaries.

The plan of this paper is as follows. Some basic facts concerning μ-harmonic functions, Lie groups and semigroups of probability measures are recalled in Section 1. Moreover, we describe there the class of solvable Lie groups which are the subject of the present paper, and their homogeneous subgroups. In Section 2 the theory of A. Raugi [21] is adapted to the group S, and μ-boundaries of S are described. This section is a preparation to Section 3 devoted to degenerate elliptic operators where we prove that the Poisson kernel ν corresponding to the operator L has a positive moment. This is all what can be proved without any additional assumptions on the operator. If $X_1, \ldots, X_j$, X generate $\mathfrak{s}$ then ν has a bounded smooth density P_0.

Our main result, the convergence theorem for operators such that $X_1, \ldots, X_j$ generate $\mathfrak{s}$, is included in Section 4. We prove it under the assumption that all roots of $\mathfrak{s}$ are rational combinations of a maximal linearly independent over $\boldsymbol{R}$ subset of them. The semigroup of measures corresponding to such an L has very good properties: it is smooth and decreases very rapidly at infinity. This makes it possible to prove that $P_0(y) \leqslant C(1 + \tau_{N_1}(y))^{-\varepsilon}$ for some C, $\varepsilon > 0$.

This paper is in some sense a continuation of [4] where the semidirect product of a Heisenberg type nilpotent group N and the group A of dilations of N is considered. The results concerning harmonic functions with respect to the Laplace–Beltrami operator are analogous to those for rank one symmetric spaces. However, due to the absence of the group K the proofs use different methods. These methods developed further are applicable in a still more general situation. This is the subject of the next paper by the author and A. Hulanicki, where the case $\dim A = 1$ will be studied in more detail, and, in particular, the admissible convergence of the Poisson integrals of L^1 functions will be proved. This yields the restricted convergence of such integrals on S.

The author is grateful to Andrzej Hulanicki for his ideas and helpful suggestions. Many results of this paper are in fact a joint work.

1. Preliminaries. This section is devoted to a brief presentation of the later needed basic facts concerning the following topics: μ-harmonic functions on a group, invariant Riemannian metrics on Lie groups and semi-groups of probability measures.

μ-harmonic functions. Let G be a separable locally compact group. We say that a locally compact space X is a *G-space* if there is a continuous map

$$G\times X\ni (s, x)\to sx\in X$$

such that $(ss')x = s(s'x)$. If μ, ν are bounded measures on G and X respectively, the *convolution* $\mu*\nu$ is a measure on X defined by

$$\langle f, \mu*\nu\rangle = \int f(sx)\,d\mu(s)\,d\nu(x), \quad f\in C_0(X).$$

The measure $\delta_s*\nu$ will be denoted by $s\nu$.

For a probability measure μ on G a Borel bounded function F on G is called *μ-harmonic* if

$$F(s) = \int F(ss')\,d\mu(s') = F*\check{\mu}(s) \tag{1.1}$$

where $\check{\mu}(M) = \mu(M^{-1})$ and the *convolution* of a function f on G and a measure μ on G is defined by

$$f*\mu(x) = \int f(xy^{-1})\,d\mu(y).$$

Let $X_1, X_2, \ldots$ be a sequence of independent G-valued random variables, each with distribution μ. Then for the sample space

$$\mathbf{G} = G\times G\times \ldots$$

with the product Borel σ-field and the product measure

$$\boldsymbol{\mu} = \mu\times\mu\times\ldots,$$

X_j is the projection

$$X_j\colon \mathbf{G}\ni \mathbf{s} = (s_1, s_2, \ldots)\to X_j(\mathbf{s}) = s_j\in G.$$

Let $s_n = X_1(\mathbf{s})\cdot\ldots\cdot X_n(\mathbf{s})$.

We say that a G-space X with a probability measure ν is a *μ-boundary of G* if

$$\mu*\nu = \nu \tag{1.2}$$

and for almost all $\mathbf{s}\in\mathbf{G}$ the measures $s_n\nu$ are $*$-weakly convergent to the point mass $\delta_{Z(\mathbf{s})}$ for a $Z(\mathbf{s})\in X$, i.e. if f is a bounded continuous function on X then

$$\lim_{n\to\infty}\langle f, s_n\nu\rangle = \langle f, \delta_{Z(\mathbf{s})}\rangle \quad \boldsymbol{\mu}\text{-a.e.} \tag{1.3}$$

The measure ν is called a *Poisson kernel* for μ.

Let $L(X)$ be the set of bounded Borel functions on X. (1.2) implies that the *Poisson integral*

$$P^X f(s) = \int f(sx)\,d\nu(x), \quad f\in L(X), \tag{1.4}$$

is a μ-harmonic function. (X, ν) is called a *reproducing μ-boundary* of G if every bounded μ-harmonic function on G is the Poisson integral (1.4) for an f in $L(X)$.

Finally, we recall that a probability measure μ on G is called *spread out* if it satisfies the following equivalent conditions:

(i) There is an integer $n>0$ such that the n-fold convolution μ^{*n} is nonsingular with respect to the right-invariant Haar measure m.

(ii) There is an integer n and a nonempty open set U in G such that $\mu^{*n}(Z)\geqslant m(Z)$ for every Borel subset Z of U.

Lie groups. Let now G be a connected Lie group with a right-invariant Haar measure m. A nonnegative Borel function ψ on G is called *subadditive* if it is bounded on compact sets and

(i) $\quad \psi(xy)\leqslant\psi(x)+\psi(y)$ for x, y in G,

(ii) $\quad \psi(x^{-1}) = \psi(x)$, $x\in G$.

If instead of (i) we have

(i') $\quad \psi(xy)\leqslant\psi(x)\psi(y)$ for $x, y\in G$

and also $\psi(x)\geqslant 1$, we say that ψ is *submultiplicative.*

Let $\|\ \|$ be a left-invariant Riemannian metric on G and τ_G the corresponding distance (from the identity), i.e.

$$\tau_G(x) = \inf\int_0^1 \|\dot{\sigma}(t)\|_{\sigma(t)}\,dt$$

where the infimum is over all C^1 curves σ in G such that $\sigma(0) = e$, $\sigma(1) = x$ (cf. e.g. [11]). τ_G is subadditive, and for every nonnegative function ψ on G which is bounded on compact sets and satisfies (i), there is a constant C such that

$$\psi(x)\leqslant C\big(\tau_G(x)+1\big) \quad \text{for all } x \text{ in } G \tag{1.5}$$

(cf. Proposition 1.2 of [11]). Consequently, for every submultiplicative function ψ on G there is a constant C such that

$$\psi(x)\leqslant e^{C(\tau_G(x)+1)}, \quad x\in G. \tag{1.6}$$

Let $U = \{x\colon \tau_G(x)<1\}$ and let $\Phi\in C_c^\infty(U)$ be a nonnegative function such that $\int\Phi(x)\,dm(x) = 1$. Then for left-invariant vector fields X and Y on G

we have

$$\tau_G(x)-1 \leqslant \tau_G * \Phi(x) \leqslant \tau_G(x)+1,$$

(1.7) $$|X(\tau_G * \Phi)(x)| \leqslant \int |\Phi(y)|\, \|\mathrm{Ad}_y X\|\, dm(y),$$

$$|XY(\tau_G * \Phi)(x)| \leqslant \int |Y\Phi(y)|\, \|\mathrm{Ad}_y X\|\, dm(y) \quad [11],$$

where $\tau_G * \Phi(x) = \int \tau_G(xy^{-1})\Phi(y)\, dm(y)$.

If B is an automorphism of G and B_x^* the differential of B at $x \in G$ then we write

$$\|B_e^*\| = \sup\{\|B_e^*(w)\|: w \in T_e G,\ \|w\| = 1\}.$$

We shall use the following simple

(1.8) PROPOSITION. *For any automorphism B of G we have*

$$\tau_G(B(x)) \leqslant \|B_e^*\|\, \tau_G(x).$$

Proof. If Y is a left-invariant vector field on G and $x \in G$, we write Y_x for the corresponding element of $T_x G$. Let for $x \in G$

$$L_x\colon G \ni y \to xy \in G.$$

Since $L_{B(x)} \circ B = B \circ L_x$, we have

(1.9) $$\|B_x^*(Y_x)\|_{B(x)} = \|B_e^*(Y_e)\|_e.$$

Let $w \in T_x G$ and let Y be the left-invariant vector field such that $Y_x = w$. Then by (1.9)

$$\|B_x^*(w)\|_{B(x)} = \|B_e^*(Y_e)\|_e \leqslant \|B_e^*\|\, \|Y_e\|_e = \|w\|.$$

If γ is a C^1 curve such that $\gamma(0) = e$, $\gamma(1) = x$ and $\sigma(t) = B(\gamma(t))$ then

$$\tau_G(B(x)) \leqslant \int_0^1 \|\dot\sigma(t)\|_{\sigma(t)}\, dt$$

$$= \int_0^1 \|B_{\gamma(t)}^*(\dot\gamma(t))\|_{B(\gamma(t))}\, dt \leqslant \|B_e^*\| \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}\, dt,$$

and the proof is complete.

Finally we recall that a simply connected nilpotent Lie group N is called *homogeneous* [6] if there is a basis $E_1, \ldots, E_n$ of the Lie algebra of N and numbers $1 = d_1 \leqslant \ldots \leqslant d_n$ such that for $t > 0$ the map

$$E_j \to t^{d_j} E_j$$

extends to an automorphism δ_t of the Lie algebra. For $x = \exp X$ in N we write $\delta_t(x) = \exp(\delta_t X)$. Of course, δ_t is an automorphism of N called a *dilation* of N.

(1.10) PROPOSITION. *Let τ_N be a left-invariant distance and f a nonnegative function on the homogeneous group N such that:*

(i) $f(\delta_t(x)) \leqslant t^d f(x)$ *for a* $d \geqslant 0$ *and all* $t \geqslant 1$.

(ii) $U = \{x: f(x) \leqslant 1\}$ *is a bounded set containing an open neighbourhood of* e.

Then $f(x) \leqslant C(1+\tau_N(x))^\beta$ *for some* $C, \beta > 0$.

Proof. The proof is essentially due to E. M. Stein (unpublished). Let $\tau_U(x) = \inf\{n: x \in U^n\}$. τ_U is nonnegative, bounded on compact sets and

$$\tau_U(xy) \leqslant \tau_U(x) + \tau_U(y).$$

By (1.5) it is sufficient to prove

(1.11) $$f(x) \leqslant C' \tau_U(x)^{\beta'} \quad \text{for some } C', \beta' > 0.$$

Let $t_1 \geqslant 1$ be such that $U^2 \subset \delta_{t_1}(U)$. Since δ_{t_1} is an automorphism, we have

(1.12) $$U^{2^m} \subset (\delta_{t_1})^m(U), \qquad m = 1, 2, \ldots,$$

and $f(x) \leqslant t_1^{dm}$ for $x \in (\delta_{t_1})^m(U)$. Let $\beta' = d \log_2 t_1$ and

$$t_1^{dm} < f(x) \leqslant t_1^{d(m+1)}.$$

Then by (1.12), $x \notin U^{2^m}$, hence $\tau_U(x) > 2^m$. Consequently, $\tau_U(x)^{\beta'} > t_1^{dm}$ and $f(x) \leqslant t_1^d \tau_U(x)^{\beta'}$, which yields (1.11).

Semigroups of probability measures. A one-parameter family of measures $\{\mu_t\}_{t>0}$ on a Lie group G is called a *semigroup* if

$$\mu_s * \mu_t = \mu_{s+t}, \qquad \lim_{t \to 0} \|f * \mu_t - f\|_{C_0} = 0 \quad \text{for } f \in C_0(G).$$

The *infinitesimal generator* A of $\{\mu_t\}_{t>0}$ is defined by

$$Af = \lim_{t \to 0} t^{-1}(f * \mu_t - f),$$

where the domain $D(A)$ of A is the set of functions in $C_0(G)$ for which the limit exists in the C_0 norm. We also define $D_1(A)$ as the set of f in $L^1(m)$ for which the limit exists in the $L^1(m)$ norm.

The following facts are due to G. Hunt and well known (cf. e.g. [12]). Both $D(A)$ and $D_1(A)$ contain $C_c^\infty(G)$, and A on both $D(A)$ and $D_1(A)$ is the closure of A on $C_c^\infty(G)$. For every $\lambda > 0$ the operator $\lambda - A$ maps $D_1(A)$ onto $L^1(m)$ and the inverse map $(\lambda - A)^{-1} = K_\lambda$ maps boundedly $L^1(m)$ onto $D_1(A)$. We have

$$K_\lambda f = f * k_\lambda,$$

where k_λ is a nonnegative bounded measure and

(1.13) $$\lim_{n\to\infty} \|f*((n/t)k_{n/t})^{*n} - f*\mu_t\|_{L^1(m)} = 0$$

for f in $L^1(m)$.

For $f\in L^p(m)$, $g\in L^{p'}(m)$, $p'^{-1}+p^{-1}=1$, $1\leqslant p\leqslant\infty$, we write $\langle f, g\rangle = \int fg\,dm$. The semigroup of probability measures $\{\check{\mu}_t\}_{t>0}$, where $\check{\mu}_t(M) = \mu_t(M^{-1})$, has the infinitesimal generator A', where

(1.14) $$\langle A'\varphi, \psi\rangle = \langle\varphi, A\psi\rangle, \quad \varphi, \psi\in C_c^\infty(G).$$

Let L be a second order left-invariant differential operator on G of the form

$$L = \sum_{1\leqslant i,j\leqslant n} a_{ij}X_iX_j + \sum_{j=1}^{n} b_jX_j,$$

where $X_1, \ldots, X_n$ is a basis of the Lie algebra $\mathfrak{g}$ of G, the matrix (a_{ij}) is positive-semidefinite and $b_1, \ldots, b_n$ are arbitrary constants. Such an operator is called *degenerate elliptic*. The closure $\bar{L}$ of L (restricted to $C_c^\infty(G)$) in C_0 is the infinitesimal generator of a semigroup of probability measures $\{\mu_t\}_{t>0}$ on G (cf. e.g. [12]).

Moreover, for all $\alpha>0$ and $T>0$ we have

(1.15) $$\int e^{\alpha\tau_G(x)}\,d\mu_t(x) \leqslant C_{\alpha,T}, \quad t\in(0, T]$$

(cf. [12] for a simple proof).

We can diagonalize the quadratic form $\sum_{1\leqslant i,j\leqslant n} a_{ij}\xi_i\xi_j$ to obtain a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ such that the operator L is of the form

$$L = X_1^2 + \ldots + X_m^2 + X,$$

where $X\in\mathfrak{g}$. If $X_1, \ldots, X_m$, X generate $\mathfrak{g}$ (as a Lie algebra) we have

(1.16) Weak Harnack Inequality [3]. *For an open set U, a compact set $K\subset U$ and every differential operator ∂ with continuous coefficients there is a constant C such that*

$$\sup_{x\in K} |\partial F(x)| \leqslant C\sup_{x\in U} F(x)$$

for every nonnegative function F satisfying $LF=0$ in U.

Under the assumption that $X_1, \ldots, X_m$ generate $\mathfrak{g}$ the following Harnack inequality holds.

(1.17) Harnack Inequality [3]. *For an open set U, a point $x_0\in U$ and a compact set $K\subset U$ and every differential operator ∂ with continuous coefficients there is a constant C such that*

$$\sup_{x\in K} |\partial F(x)| \leqslant CF(x_0)$$

for every nonnegative function F satisfying $LF=0$.

If $X_1, \ldots, X_m$ generate $\mathfrak{g}$ then, of course, $X_1, \ldots, X_m$, d/dt generate the Lie algebra of the group $G\times\boldsymbol{R}$. Then by Hörmander's theorem [10] the operator

$$-\frac{d}{dt}+L = -\frac{d}{dt}+X+X_1^2+\ldots+X_m^2$$

is hypoelliptic on $G\times\boldsymbol{R}$ and so the measures μ_t are absolutely continuous: $d\mu_t(x) = p_t(x)\,dm(x)$, $t>0$. The function

$$G\times\boldsymbol{R}^+ \ni (x, t)\to p_t(x)\in\boldsymbol{R}$$

is C^∞ and harmonic with respect to the operator $-d/dt+L$. If $f\in C_0(G)$ and $u(x, t) = f*p_t(x)$, $t>0$, then also $(-d/dt+L)u = 0$.

The left-invariant Haar measure on G will be denoted by m_l. We define the *convolution* of a measure μ and a function f by

$$\mu*f(x) = \int f(y^{-1}x)\,d\mu(y).$$

Let

$$-\Delta = Y_1^2+\ldots+Y_n^2,$$

where $Y_1, \ldots, Y_n$ is a basis of $\mathfrak{g}$, and let

$$-\check{\Delta} = \check{Y}_1^2+\ldots+\check{Y}_n^2,$$

where $\check{Y}_j(x) = (d/dt)f(\exp tY_j\cdot x)|_{t=0}$. Then of course

$$\Delta\varphi(x^{-1}) = (\Delta\varphi)^\vee(x) = \check{\Delta}\check{\varphi}(x).$$

By [20], $\check{\Delta}$ is essentially selfadjoint on $C_c^\infty\subset L^2(m_l)$, $(1+\check{\Delta})^{-1}f = k*f$, $k\in L^1(m_l)$ and for $s>n/4$, $k^{*s}\in L^2(m_l)$. Moreover, $\langle\check{\varphi}, k\rangle_{m_l} = \langle\varphi, k\rangle_{m_l} = \int\varphi(x)k(x)\,dm_l(x)$. Consequently, there is a constant C such that for every $f\in D(\overline{(1+\check{\Delta})^s})\cap C^\infty(G)$ we have

(1.18) $$|f(x)| \leqslant C\left(\frac{dm}{dm_l}(x)\right)^{1/2} \|(1+\check{\Delta})^s f\|_{L^2(m_l)}$$

because

$$f(x) = k^{*s}*(1+\check{\Delta})^s f(x) = \int(1+\check{\Delta})^s f(yx)k^{*s}(y)\,dm_l(y).$$

If $\varphi \in L^2(m_l)$ then $\varphi * p_t \in L^2(m_l)$ and $\|\varphi * p_t\|_{L^2(m_l)} \leqslant C_T \|\varphi\|_{L^2(m_l)}$, where

$$C_T = \sup_{t\in[0,T]} \int \left(\frac{dm}{dm_l}(y^{-1})\right)^{1/2} dp_t(y),$$

which is finite by (1.15).

Now let $l \geqslant s$ and $\varphi \in C_c^\infty$. In view of (1.16) and (1.18), for a compact neighbourhood K of e in G and $\varepsilon > 0$, we have

$$\begin{aligned}
&|\langle (1+\Delta)^{l-s}\varphi, p_t\rangle| \\
&\quad = \left|(1+\check{\Delta})^l\big(((1+\check{\Delta})^{-s}\check{\varphi}) * p_t\big)(e)\right| \\
&\quad \leqslant C_1 \sup_{x\in K,\, r\in[t-\varepsilon, t+\varepsilon]} \left|((1+\check{\Delta})^{-s}\check{\varphi}) * p_r(x)\right| \\
&\quad \leqslant C_2 \left(\frac{dm}{dm_l}(x)\right)^{1/2} \sup_{r\in[t-\varepsilon, t+\varepsilon]} \|\check{\varphi} * p_r\|_{L^2(m_l)} \leqslant C_3 \|\varphi\|_{L^2(m)}.
\end{aligned}$$

Hence $p_t \in D\left(\overline{(1+\Delta)^N}\right)$ for all $N = 0, 1, \ldots$ Consequently, since $p_t \in C^\infty(G)$,

$$(1.19) \qquad |p_t(x)| \leqslant C\left(\frac{dm_l}{dm}(x)\right)^{1/2} \|(1+\Delta)^s p_t\|_{L^2(m)},$$

and by (1.15) for every submultiplicative function ψ on G

$$(1.20) \qquad \int p_t^2(x)\,\psi(m)\,dm(x) < +\infty.$$

Let $X^I = X_1^{i_1} \ldots X_n^{i_n}$ where I is a multiindex, $I = (i_1, \ldots, i_n)$. It is well known [19] that for every I there is an N and a constant C such that

$$\|X^I f\|_{L^2(m)} \leqslant C \|\overline{(1+\Delta)}^N f\|_{L^2(m)}$$

for every $f \in \overline{\left((1+\Delta)^N\right)}$. Now we prove the following proposition which for nilpotent Lie groups is proved in [13].

(1.21) PROPOSITION. *For every real number α and every multiindex I we have*

$$\sup\{|X^I p_t(x)|\, e^{\alpha\tau_G(x)}\colon x \in G\} < \infty.$$

Proof. Let $\tau_G * \Phi = f$ be as in (1.7). It suffices to show

$$\sup\{|X^I p_t(x)|\, e^{\alpha f(x)}\colon x \in G\} < \infty$$

and this is implied by

$$(1.22) \qquad (X^I p_t)\, e^{\alpha f} \in L^2(m) \quad \text{for all } I \text{ and } \alpha > 0.$$

In fact, since by (1.16) and (1.19), $X^I p_{t/2}$ is bounded for every t, we have

$$X^I p_t(x) = p_{t/2} * X^I p_{t/2}(x)$$

whence

$$\begin{aligned}
|X^I p_t(x)|\, e^{\alpha f(x)} &\leqslant C \int p_{t/2}(y^{-1})\, e^{\alpha f(y^{-1})} |X^I p_{t/2}(yx)|\, e^{\alpha f(yx)}\, dm(y) \\
&\leqslant C \|p_{t/2}\, e^{\alpha f} (d\check{m}/dm)^{1/2}\|_{L^2(m)} \|X^I p_{t/2} \cdot e^{\alpha f}\|_{L^2(m)}.
\end{aligned}$$

The right side of the inequality is finite because $d\check{m}/dm$ is multiplicative and in view of (1.6), (1.7), $d\check{m}/dm \leqslant e^{\beta f}$ for some $\beta > 0$.

We notice that by repeated application of (1.7) we obtain

$$(1.23) \qquad |X^I e^{\alpha f(x)}| \leqslant C e^{\alpha f(x)}$$

and we prove (1.22) by induction on the length of the multiindex I. Suppose that for an X in $\mathfrak{g}$, $X^I = XX^J$ with $|I| > |J|$ and $(X^J p_t)\, e^{\alpha f} \in L^2(m)$ for all $\alpha > 0$. Let $\{f_n\}$ be a sequence of nonnegative functions in $C_c^\infty(G)$ such that $f_n(x) \leqslant 1$, for every $Y \in \mathfrak{g}$ the sequence $\sup_G |Yf_n(x)|$ is bounded and

$$\lim_{n\to\infty} f_n(x) = 1 \quad \text{uniformly on compact sets,}$$

$$\lim_{n\to\infty} Yf_n(x) = 0 \quad \text{uniformly on compact sets}$$

([11], p. 267). Then, for every n, in virtue of (1.23) with $g = e^{\alpha f}$,

$$\begin{aligned}
&\langle f_n (X^I p_t)^2, g^2\rangle = \langle f_n X^I p_t, (X^I p_t) g^2\rangle \\
&\quad \leqslant |\langle X(f_n X^I p_t), (X^J p_t) g^2\rangle| + |\langle f_n X^I p_t, (X^J p_t) X g^2\rangle| \\
&\quad \leqslant |\langle f_n X X^I p_t, (X^J p_t) g^2\rangle| + C\langle f_n |X^I p_t|, |X^J p_t| g^2\rangle \\
&\qquad + |\langle X f_n X^I p_t, (X^J p_t) g^2\rangle| \leqslant \|f_n X X^I p_t\|_{L^2(m)} \|(X^J p_t) g^2\|_{L^2(m)} \\
&\qquad + C\|f_n X^I p_t\|_{L^2(m)} \|(X^J p_t) g^2\|_{L^2(m)} + \|(Xf_n) X^I p_t\|_{L^2(m)} \|(X^J p_t) g^2\|_{L^2(m)},
\end{aligned}$$

whence, letting $n \to \infty$, we obtain (1.22).

A class of solvable Lie groups. Let $\mathfrak{s}$ be a real solvable Lie algebra of the form

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$$

where $\mathfrak{n}$ is a nilpotent and $\mathfrak{a}$ an abelian algebra. We assume that the operators $\{ad_H\colon H \in \mathfrak{a}\}$ are diagonalizable, i.e. in a basis $E_1, \ldots, E_n$ of $\mathfrak{n}$: $\mathrm{ad}_H(E_i) = \lambda_i(H) E_i$.

We require that

(1.24) There is an H in $\mathfrak{a}$ such that $\lambda_i(H) > 0$, $i = 1, \ldots, n$.

This guarantees the existence of a *nonnegative basis* $H_1, \ldots, H_k$ for Δ in $\mathfrak{a}$, i.e. a basis such that $\lambda(H_r) \geqslant 0$ for all $\lambda \in \Delta$ and $r = 1, \ldots, n$. Let A be the connected and simply connected Lie group with Lie algebra $\boldsymbol{a}$. The inverse of exp: $\boldsymbol{a} \to A$ will be denoted by log. We say that $\log s \to -\infty$, $a \in A$, iff $\lambda(\log a) \to -\infty$, $\lambda \in \Delta$. We also notice that if $H_1, \ldots, H_k$ is an arbitrary non-negative basis, $\log a = \sum_{i=1}^{k} (\log a)_i H_i$ and $(\log a)_i \to -\infty$ then $\log a \to -\infty$.

Putting $\lambda_j < \lambda_i$ if and only if $\lambda_j(H_{r_0}) < \lambda_i(H_{r_0})$ for $r_0 = \min\{r: \lambda_j(H_r) \neq \lambda_i(H_r)\}$ we introduce a lexicographical order in Δ. Then $\lambda_j + \lambda_i > \lambda_i$ whenever $\lambda_j + \lambda_i \in \Delta$. Thus for a suitable numbering of the λ's we have $\lambda_1 \leqslant \ldots \leqslant \lambda_n$. Let

$$\boldsymbol{n}^\lambda = \{X \in \boldsymbol{n}: \ \forall H \in \boldsymbol{a} \ \operatorname{ad}_H(X) = \lambda(H) X\}, \qquad \lambda \in \Delta.$$

Since

$$(1.25) \qquad [\boldsymbol{n}^{\lambda_i}, \boldsymbol{n}^{\lambda_j}] \begin{cases} \subset \boldsymbol{n}^{\lambda_i+\lambda_j} & \text{if } \lambda_i+\lambda_j \in \Delta, \\ = \{0\} & \text{otherwise,} \end{cases}$$

$\operatorname{lin}(E_i, E_{i+1}, \ldots, E_n)$ is an ideal in $\boldsymbol{n}$.

Let S, N be connected and simply connected Lie groups corresponding to the algebras $\boldsymbol{s}$, $\boldsymbol{n}$ respectively. Of course, S is a semidirect product of N and A. N is a normal subgroup of S and the mapping $S \ni xa \to a \in A$ is a homomorphism. Since exp: $\boldsymbol{n} \to N$ is a global diffeomorphism every $x \in N$ can be written as $x = \exp(\sum_{j=1}^{n} x_j E_j)$. Then

$$axa^{-1} = x^a = \exp\big(\sum_{j=1}^{n} e^{\lambda_j(\log a)} x_j E_j\big), \qquad a \in A.$$

For the element H of $\boldsymbol{a}$, as defined by (1.24), we introduce dilations in N by

$$\delta_t(x) = x^{\exp((\log t) H)}, \qquad t > 0,$$

making N a homogeneous group.

A subgroup N_1 of N will be called a *homogeneous subgroup* if its Lie algebra $\boldsymbol{n}_1$ satisfies the following conditions:

(i) $\boldsymbol{n}_1$ is invariant under the action of ad_H, $H \in \boldsymbol{a}$.

(ii) There is a subalgebra $\boldsymbol{n}_0$ invariant under the action of ad_H, $H \in \boldsymbol{a}$, such that $\boldsymbol{n} = \boldsymbol{n}_1 \oplus \boldsymbol{n}_0$.

The subgroup corresponding to $\boldsymbol{n}_0$ is denoted by N_0. It is easy to see that $\boldsymbol{n}_i = \bigoplus_{\lambda \in \Delta} (\boldsymbol{n}_i \cap \boldsymbol{n}^\lambda)$, $i = 0, 1$.

The next lemma is a particular case of a more general proposition. However, the proof here is very simple.

(1.26) LEMMA. *ζ: $N_1 \times N_0 \ni (y, z) \to yz \in N$ is a diffeomorphism.*

Proof. Let U_1, U_0, U be neighbourhoods of e in N_1, N_0, N respectively such that ζ: $U_1 \times U_0 \to U$ is a diffeomorphism. If $x \in N$ then $x = \delta_t(x')$ for an $x' \in U$ and a $t > 0$. Thus $x' = y'z'$, $y' \in N_1$, $z' \in N_0$, and $x = \delta_t(y') \cdot \delta_t(z')$. Moreover, if $y \in \delta_t(U_1)$, $z \in \delta_t(U_0)$ then

$$\zeta(y, z) = \delta_t\big(\zeta|_{U_1 \times U_0}(\delta_{t^{-1}}(y), \delta_{t^{-1}}(z))\big)$$

which completes the proof.

If τ_G is an invariant Riemannian distance and $\| \ \|$ the corresponding norm in $\boldsymbol{h}$ then

$$(1.27) \qquad \tau_G(\exp X) \leqslant \|X\|.$$

On a homogeneous group N we have an inverse inequality, i.e. there are $C > 0$, $\beta \geqslant 0$ such that

$$(1.28) \qquad \|X\| \leqslant C\big(1 + \tau_N(\exp X)\big)^\beta.$$

The proof of (1.28) follows immediately from Proposition (1.10) applied to the function $f(\exp X) = \|X\|$.

Let G' be a subgroup of G. Since $\tau_G|_{G'}$ is a subadditive function on G', in view of (1.5) we have

$$(1.29) \qquad \tau_G(x) \leqslant C\big(\tau_{G'}(x) + 1\big), \qquad x \in G',$$

for a constant $C > 0$.

In our case the homogeneous structures of N, N_1, N_0 yield an inverse estimate.

(1.30) LEMMA. *Let τ_{N_1}, τ_{N_0}, τ_N be invariant Riemannian distances on N_1, N_0, N respectively. Then there are $C > 0$ and $\beta \geqslant 0$ such that*

$$(1.31) \qquad \tau_{N_1}(y) + \tau_{N_0}(z) \leqslant C\big(1 + \tau_N(yz)\big)^\beta$$

for all y in N_1 and z in N_0.

Proof. Let $f(yz) = \tau_{N_1}(y) + \tau_{N_0}(z)$. By Proposition (1.8)

$$f\big(\delta_t(yz)\big) \leqslant (\delta_t|_{N_1})_e^* \| \tau_{N_1}(y) + \|(\delta_t|_{N_0})_e^*\| \tau_{N_0}(z).$$

Hence there is a $\beta_1 \geqslant 0$ such that $f\big(\delta_t(yz)\big) \leqslant t^{\beta_1} f(yz)$ for $t \geqslant 1$. Moreover, the set $\{yz: f(yz) \leqslant 1\}$ is bounded and contains a neighbourhood of the identity. Consequently, (1.31) follows from Proposition (1.10).

Obviously, $S = N_1 N_0 A$ in the sense that

$$N_1 \times N_0 \times A \ni (y, z, a) \to yza \in S$$

is a diffeomorphism. We shall write $s = y(s) \cdot z(s) \cdot a(s)$ and $y(s) = \pi_{N_1}(s)$, $z(s) = \pi_{N_0}(s)$, $a(s) = \pi_A(s)$, $x(s) = y(s) \cdot z(s) = \pi_N(s)$.

Let τ_{N_1}, τ_{N_0A}, τ_S be arbitrary invariant Riemannian distances on N_1, $N_0 A$, S respectively. If in addition N_1 is a normal subgroup then there is a constant C such that

(1.32) $$\log\left(1+\tau_{N_1}(y(s))\right)+\tau_{N_0A}(z(s)\cdot a(s)) \leqslant C(\tau_S(s)+1).$$

To prove (1.32) we write

$$\log\left(1+\tau_{N_1}(y(s))\right) \leqslant \gamma(s) = \sup_{s'\in S}\left[\log\left(1+\tau_{N_1}(y(ss'))\right)-\log\left(1+\tau_{N_1}(y(s'))\right)\right].$$

By Proposition (1.8)

$$\gamma(yza) \leqslant \log\left(1+\tau_{N_1}(y)\right)+\log\left(1+\|\mathrm{Ad}_{za}|_{\mathfrak{n}_1}\|\right)$$

and we can apply (1.5) to γ. (See also the proof of Proposition 3 in [8].)

Finally, let us remark that since every nilpotent Lie group is of polynomial growth [5], there is a large α such that

(1.33) $$\int_{N_1}\left(1+\tau_{N_1}(y)\right)^{-\alpha}dy < \infty.$$

2. μ-boundaries of S. Let

$$\Delta_1(\mu) = \{\lambda\in\Delta:\ \int_S \lambda(\log a(s))\,d\mu(s) < 0\},$$

$$\mathfrak{n}_1(\mu) = \bigoplus_{\lambda\in\Delta_1(\mu)}\mathfrak{n}^\lambda, \qquad \mathfrak{n}_0(\mu) = \bigoplus_{\lambda\in\Delta\setminus\Delta_1(\mu)}\mathfrak{n}^\lambda,$$

and let $N_1(\mu)$, $N_0(\mu)$ be the subgroups of N corresponding to $\mathfrak{n}_1(\mu)$, $\mathfrak{n}_0(\mu)$. The aim of this section is to prove the following

(2.1) THEOREM. *For every probability measure μ on S such that*

$$\int\tau_S(s)\,d\mu(s) < \infty$$

every homogeneous subgroup N_1 of $N_1(\mu)$ is a μ-boundary of S.

The proof of the theorem is a simplification of the proof of a similar theorem by A. Raugi [21] where only "maximal boundaries" $N_1(\mu)$ are considered. The generalization is motivated by the theory of harmonic functions on symmetric spaces where some homogeneous subgroups of $N_1(\mu)$ are boundaries [16]. Our simplification of the proof is due to the homogeneity of the groups involved, otherwise we adopt the ideas of A. Raugi [21].

The proof of Theorem (2.1) uses a representation of S on certain spaces of polynomials on N. Such representations are well known (cf. e.g. [7]). Our representation is similar to the one given by A. Raugi.

Let $\mathfrak{m}_1 = \mathfrak{n}, \ldots, \mathfrak{m}_{l+1} = [\mathfrak{n}, \mathfrak{m}_l], \ldots, \mathfrak{m}_{p+1} = \{0\}$. Since ad_H, $H\in\mathfrak{a}$, preserve $\mathfrak{m}_l$, $\mathfrak{m}_l = \bigoplus_{\lambda\in\Delta}(\mathfrak{m}_l\cap\mathfrak{n}^\lambda)$ and there is a basis $E_1, \ldots, E_n$ of the algebra $\mathfrak{n}$ such that:

(i) $E_1, \ldots, E_n$ are eigenvectors of ad_H, $H\in\mathfrak{a}$.

(ii) $E_{i_{j-1}+1}, \ldots, E_{i_j}$, $j = 1, \ldots, p$, $i_0 = 0$, $i_p = n$, is a basis of $\mathfrak{m}_j/\mathfrak{m}_{j+1}$.

We choose the coordinates in N writing $x = \exp(x_1E_1+\ldots+x_nE_n)$.

We define the degree of a polynomial on N as follows:

(i) The degree of a constant polynomial is 0.

(ii) $\mathrm{dg}\,x_i = j$ for $i_{j-1} < i \leqslant i_j$.

(iii) If $I = (I_1, \ldots, I_n)$ is a multiindex then

$$\mathrm{dg}\,x^I = \sum_{j=1}^n I_j\,\mathrm{dg}\,x_j.$$

(iv) For an arbitrary polynomial $T = \sum_I \alpha_I x^I$

$$\mathrm{dg}\,T = \max_I \mathrm{dg}\,x^I.$$

Let $x'T$, aT, $x'\in N$, $a\in A$, be polynomials on N defined by $x'T(x) = T(x'x)$, $aT(x) = T(x^a)$. Then

$$aT = \sum_I \alpha_I e^{I\cdot\lambda(\log a)}x^I,$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $I\cdot\boldsymbol{\lambda} = I_1\lambda_1+\ldots+I_n\lambda_n$. Hence

(2.2) $$\mathrm{dg}\,aT = \mathrm{dg}\,T.$$

If $(x'x)_i$ is the ith coordinate of $x'x$ and $i_{j-1} < i \leqslant i_j$ then $(x'x)_i = x_i+x_i'+T_i(x_1', \ldots, x_{i_{j-1}}', x_1, \ldots, x_{i_{j-1}})$, where T_i is a polynomial whose degree as a polynomial of x is at most $j-1$. Therefore $x'T = T+T_1$ where $\mathrm{dg}\,T_1 < \mathrm{dg}\,T$ and

(2.3) $$\mathrm{dg}\,x'T = \mathrm{dg}\,T.$$

Let J_r be the space of polynomials T of degree at most r such that $T(yz) = T(y)$ for $y\in N_1 = \exp\mathfrak{n}_1$, $z\in N_0 = \exp\mathfrak{n}_0$, where N_1, N_0 are homogeneous subgroups and $\mathfrak{n} = \mathfrak{n}_1\oplus\mathfrak{n}_0$. Since the inclusion $N_1\ni y\to y\in N$ and the projection π_{N_1} are polynomial mappings, $T|_{N_1}$ is a polynomial on N_1 and $T = T_{N_1}\circ\pi_{N_1}$. It follows that J_r is the set of polynomials of the form $T'\circ\pi_{N_1}$, where T' is a polynomial on N_1 and $\mathrm{dg}(T'\circ\pi_{N_1}) \leqslant r$. For $s = x'a$ and $T\in J_r$ we write

(2.4) $$sT(x) = T(\pi_N(sx)) = T(x'x^a).$$

By (2.2)–(2.4) it is easy to see that if $T\in J_r$ then $sT\in J_r$ and $(ss')T = s'(sT)$. Hence S acts on J_r.

Let $Y_1, \ldots, Y_\chi$, $\chi = \dim N_1$, be a basis in $\mathfrak{n}_1$ consisting of eigenvectors of ad_H, $H\in\mathfrak{a}$. Writing

(2.5) $$y = \exp(y_1Y_1+\ldots+y_\chi Y_\chi)$$

we have coordinates on N_1. We choose r such that $y_1^2\circ\pi_{N_1}, \ldots, y_\chi^2\circ\pi_{N_1}\in J_r$. The polynomials $y^I\circ\pi_{N_1}$, $\mathrm{dg}(y^I\circ\pi_{N_1}) \leqslant r$, form a basis Q of J_r (the polynomial 1 corresponds to $I = (0, \ldots, 0)$). Let $W = \{I\colon y^I \circ \pi_{N_1} \in Q\}$ and $|W| = \dim J_r = d$. We order the elements of Q according to the increasing degree. Then in view of (2.2) and (2.3) the matrix $M(s)$ of the operator $T \to sT$ is upper-triangular and has $e^{I\cdot\lambda(\log a)}$ on the diagonal, $I = (I_1, \ldots, I_\chi)$ and

(2.6) $$\lambda = (\lambda_{i_1}, \ldots, \lambda_{i_\chi})$$

where $\mathrm{ad}_H(Y_j) = \lambda_{i_j}(H) Y_j$.

In J_r we choose an inner product such that the basis Q is orthonormal. If B is a linear operator defined on a subspace of J_r then $\|B\|$ denotes the norm of B in the sense of this inner product. Let $M(s) = [\alpha_{ij}(s)]$, $1 \leqslant i, j \leqslant d$,

(2.7) $$B(s) = [\alpha_{1j}(s)], \quad 2 \leqslant j \leqslant d, \qquad Q(s) = [\alpha_{ij}(s)], \quad 2 \leqslant i, j \leqslant d.$$

Obviously $\|Q(s)\| \leqslant \|M(s)\|$ and $\|B(s)\| \leqslant \|M(s)\|$. Finally, we have $M(s_m) = M(s_m) \ldots M(s_1)$,

(2.8) $$Q(s_m) = Q(s_m) \ldots Q(s_1),$$

(2.9) $$B(s_m) = \sum_{i=1}^{m} B(s_i) Q(s_{i-1}),$$

where $s_m = s_1 \ldots s_m$ and $Q(s_0) = I$.

Proof of Theorem (2.1). We shall prove that the sequence $M(s_m)$ converges μ-a.e. to a matrix $[\alpha_{ij}]$ such that $\alpha_{ij} = 0$ for $i = 2, \ldots, d$, $j = 1, \ldots, d$.

In view of (2.8) and (2.9) it suffices to prove that

(2.10) $$\lim_{m\to\infty} \|Q(s_m)\|^{1/m} < 1 \quad \mu\text{-a.e.},$$

(2.11) $$\limsup_{m\to\infty} \|B(s_m)\|^{1/m} \leqslant 1 \quad \mu\text{-a.e.}$$

We have $\int \log(1 + \|M(s)\|)\, d\mu(s) < \infty$, because $\log(1 + \|M(s)\|)$ is a subadditive function. Therefore $\limsup_{m\to\infty} \|M(s_m)\|^{1/m} \leqslant 1$ μ-a.e. (see e.g. [21], p. 69). Consequently, we have (2.11) and

(2.12) $$\limsup_{m\to\infty} \|Q(s_m)\|^{1/m} \leqslant 1 \quad \mu\text{-a.e.}$$

Let λ be as in (26). The strong law of large numbers yields

$$(1/m)\, I \cdot \lambda(\log a_1 \ldots a_m) \to \int_S I \cdot \lambda(\log a(s))\, d\mu(s).$$

The integral on the right is negative for $I \neq 0$ and finite by (1.5) because the function $s \to |I \cdot \lambda(\log a(s))|$ is subadditive. Hence

(2.13) $$0 < \lim_{m\to\infty} (e^{I\cdot\lambda(\log a_1 \ldots a_m)})^{1/m} < 1 \quad \mu\text{-a.e.}$$

Now (2.12), (2.13) and Lemma (9.4) of [21] imply (2.10).

S acts on N_1 by $s(y) = \pi_{N_1}(sy)$. We have proved that if $T \in J_r$ and $m \to \infty$ then $T(s_m(y))$ converges to a constant independent of y. Putting $T = y_i \circ \pi_{N_1}$, $i = 1, \ldots, \chi$, we see that $\lim_{m\to\infty} s_m(y)$ exists and is independent of y. Therefore we may write

$$Z(s) = \lim_{m\to\infty} s_m(y).$$

Consequently, for every probability measure α on N_1

$$\lim_{m\to\infty} s_m \alpha = \delta_{Z(s)} \quad \mu\text{-a.e.}$$

Let ν_m be the measure on N_1 defined by

$$\langle f, \nu_m \rangle = \int f(s_m(e))\, d\mu$$

for every bounded continuous function f on N_1. Then, of course,

(2.14) $$\mu * \nu_m = \nu_{m+1},$$

(2.15) $$\lim_{m\to\infty} \langle f, \nu_m \rangle = \langle f, \nu \rangle,$$

where ν is the distribution law of $Z(s)$. Hence $\mu * \nu = \nu$ and the proof is complete.

(2.16) Corollary. *If μ, μ^{*2} satisfy the assumptions of the previous theorem then the corresponding Poisson kernels ν, ν' are equal.*

Proof. By (2.15), ν is the *-weak limit of ν_m and ν' is the *-weak limit of ν_{2m} so they are equal.

We conclude this section with a theorem describing the maximal μ-boundary of S.

(2.17) Theorem (A. Raugi [21]). *If a probability measure μ is spread out and if its support generates S and $\int \tau_S(s)\, d\mu(s) < \infty$, then $N_1(\mu)$ is a reproducing μ-boundary of S. The Poisson kernel ν is the unique μ-invariant (i.e. $\mu * \nu = \nu$) probability measure on N_1.*

Theorem (2.17) is a particular case of Theorems (8.4) and (12.9) of [21].

(2.18) Corollary. *If μ, μ^{*2} satisfy the assumptions of the previous theorem then F is μ-harmonic if and only if F is μ^{*2}-harmonic.*

3. Degenerate elliptic invariant operators on S. Let L be a second order degenerate elliptic left-invariant differential operator, i.e.

(3.1) $$L = X_1^2 + \ldots + X_j^2 + X$$

where $X_1, \ldots, X_j, X \in \mathfrak{s}$. Let $\{\mu_t\}_{t>0}$ be the semigroup of probability measures whose infinitesimal generator is L.

We say that a Borel function F is *L-harmonic* if $LF = 0$ in the sense of distributions, i.e.

$$\langle F, L^+ \varphi\rangle = 0 \quad \text{for } \varphi\in C_c^\infty(S)$$

where

$$L^+ = X_1^2 + \ldots + X_j^2 - X.$$

Of course

(3.2) $$\langle L\psi, \varphi\rangle = \langle\psi, L^+\varphi\rangle, \quad \varphi, \psi\in C_c^\infty(S).$$

We have the following characterization of bounded L-harmonic functions.

(3.3) THEOREM. *A bounded Borel function F is L-harmonic if and only if for every $t>0$*

(3.4) $$F*\mu_t = F.$$

Proof. By (1.14) and (3.2), $\overline{L^+}$ is the infinitesimal generator of $\{\check{\mu}_t\}_{t>0}$. Suppose F is a bounded L-harmonic function on S. Then by definition, for $\varphi\in C_c^\infty(S)$ we have $0=\langle F, L^+\varphi\rangle$ and consequently $0 = \langle F, \overline{L^+}\varphi\rangle$ for $\varphi\in D_1(\overline{L^+})$. Let f be in $L^1(m)$ and let $\varrho>0$. Then

$$\langle F, f\rangle = \langle F, (\varrho - \overline{L^+})K_\varrho f\rangle = \langle F, \varrho K_\varrho f\rangle,$$

because $K_\varrho f$ belongs to $D_1(\overline{L^+})$. Consequently, for all $n=1, 2, \ldots$, $\langle F, f\rangle = \langle F, (\varrho K_\varrho)^n f\rangle$ and so, by (1.13), putting $\varrho = n/t$ and letting $n\to\infty$ we obtain

$$\langle F, f\rangle = \langle F, f*\check{\mu}_t\rangle = \langle F*\mu_t, f\rangle.$$

The rest of the proof is trivial.

Theorem (3.3) shows that bounded L-harmonic functions on the whole of S are $\check{\mu}_1$-harmonic and the theory of Guivarc'h–Raugi applies. In what follows we shall specify the conditions on L such that the measures μ_t satisfy the conditions of Theorem (2.1).

Let L^A be the operator on A defined by $L^A f(a) = L\tilde{f}(a)$ where $f\in C^\infty(A)$ and $\tilde{f}(xa) = f(a)$. Then

(3.5) $$L^A = \sum_{i=1}^{l} Z_i^2 + Z$$

for a basis $Z_1, \ldots, Z_k$ of $\boldsymbol{a}$ and $Z\in\boldsymbol{a}$. Since N is a normal subgroup of S, Z is the image of X by the mapping $s\to s/\boldsymbol{n} = \boldsymbol{a}$.

(3.6) LEMMA. *For $\lambda\in\Delta$ the following conditions are equivalent:*
(i) *There is a $t>0$ such that $\int\lambda(\log a(s))\,d\check{\mu}_t(s) < 0$.*
(ii) *For every $t>0$, $\int\lambda(\log a(s))\,d\check{\mu}_t(s)<0$.*
(iii) $\lambda(Z)<0$.

Proof. Since the map $s\to a(s)$ is a homomorphism, L^A is the infinitesimal generator of the semigroup $\mu_t^A = \pi_A(\mu_t)$. In the coordinates $a = \exp(\sum_{i=1}^k x_i Z_i)$ the measure μ_t^A is

$$(4\pi t)^{l/2}\exp\Big(-(4t)^{-1}\sum_{i=1}^{l}(x_i+\alpha_i t)^2\Big)dx_1\ldots dx_l\times\delta_{(-\alpha_{l+1},\ldots,-\alpha_k)t},$$

where $Z = \sum_{i=1}^k \alpha_i Z_i$. Consequently,

$$\begin{aligned}\int\lambda(\log a(s))\,d\check{\mu}_t(s) &= -\int\lambda(\log a)\,d\mu_t^A(a)\\ &= -(4\pi t)^{-l/2}\int\Big(\sum_{i=1}^{k}x_i\lambda(Z_i)\Big)\exp\Big(-(4t)^{-1}\sum_{i=1}^{l}(x_i+\alpha_i t)^2\Big)dx_1\ldots dx_l\\ &\qquad\times d\delta_{-t\alpha_{l+1}}(x_{l+1})\ldots d\delta_{-t\alpha_k}(x_k) = \sum_{i=1}^{k}\lambda(Z_i)\alpha_i t = t\lambda(Z).\end{aligned}$$

Now we are ready to reformulate Theorem (2.1) for L-harmonic functions. We use the same notation as in Section 2. Moreover, we put $\boldsymbol{n}_1(L) = \boldsymbol{n}_1(\check{\mu}_1)$, $N_1(L) = N_1(\check{\mu}_1)$, $N_0(L) = N_0(\check{\mu}_1)$.

(3.7) THEOREM. *If $N_1\subset N_1(L)$ is a homogeneous subgroup, then there is a probability measure ν on N_1 called the Poisson kernel such that the functions*

$$P^{N_1}f(s) = \int f(s(y))\,d\nu(y),$$

where f is a Borel bounded function on N_1, are L-harmonic. ν is defined by (2.15) for $\mu = \check{\mu}_1$.

Proof. Since $\lambda(Z)<0$, $\lambda\in\Delta_1 = \{\lambda\colon \boldsymbol{n}_1\cap\boldsymbol{n}_1^\lambda \neq\{0\}\}$ and by (1.15), $\int\tau_S(s)\,d\check{\mu}_t(s)<\infty$, it follows that N_1 is, in view of Theorem (2.1), a $\check{\mu}_{1/2^n}$-boundary of S, $n=1, 2, \ldots$ Let ν be defined by (2.15) for $\mu=\check{\mu}_1$. In view of Corollary (2.16), ν is the Poisson kernel for $\check{\mu}_{1/2^n}$. Therefore $P^{N_1}f*\mu_{1/2^n} = P^{N_1}f$, $n=1, 2, \ldots$, which implies $L(P^{N_1}f) = 0$.

Now we shall prove a number of properties of the Poisson kernel ν corresponding to the operator L.

In what follows $\mu = \check{\mu}_1$.

(3.8) LEMMA. *Let $\lambda\in\boldsymbol{a}^*$, $\lambda(Z)<0$, $\mu^A = \pi_A(\mu)$ and*

$$\eta\sum_{i=1}^{l}\lambda(Z_i)^2 < -\lambda(Z).$$

Then

(3.9) $$\int e^{\eta\lambda(\log a)}\, d\mu^A(a) < 1.$$

Proof. L^A is the infinitesimal generator of $\mu_t^A = \pi_A(\mu_t)$. Therefore if we choose the coordinates $a = (x_1, \ldots, x_k)$, $a = \exp\left(\sum_{i=1}^k x_i Z_i\right)$, then the left-hand side of (3.9) is equal to

$$(4\pi)^{-l/2} \int e^{\eta\lambda(a)} \exp\left(-\tfrac{1}{4}\sum_{i=1}^{l}(x_i-\alpha_i)^2\right) dx_1 \ldots dx_l \times \delta_{(\alpha_{l+1}, \ldots, \alpha_k)}$$
$$= \exp\left(\eta\lambda(Z) + \eta^2 \sum_{i=1}^{l} \lambda(Z_i)^2\right),$$

which completes the proof.

(3.10) THEOREM. *Let ν be the Poisson kernel given by Theorem (3.7). Let τ_{N_1} be an arbitrary Riemannian distance on a boundary N_1, λ as in (2.6) and $\eta < 1$. Assume that*

$$\eta \sum_{i=1}^{l} \left(I\cdot\lambda(Z_i)\right)^2 < -I\cdot\lambda(Z)$$

for all $I \in W$, $I \neq (0, \ldots, 0)$. Then

$$\int \tau_{N_1}(y)^\eta\, d\nu(y) < \infty.$$

Proof. Let the coordinates $y_1, \ldots, y_\chi$ in N_1 be as in (2.5) and $T = y_i^2 \circ \pi_{N_1}$. In view of (1.27) it is sufficient to prove that there is a constant C such that for every m

$$\int T(s_m(e))^{\eta/2}\, d\mu(s_1)\ldots d\mu(s_m) < C.$$

Since $T(s_m(e)) = (M(s_m)T)(e) \leqslant \|B(s_m)\|$ (see (2.7)) we have to prove

(3.11) $$\sum_{m=1}^{\infty} \int \|B(s_m)\|^{\eta/2} \|Q(s_{m-1})\|^{\eta/2}\, d\mu(s_1)\ldots d\mu(s_m) < \infty.$$

First,

$$\int \|B(s)\|^{\eta/2}\, d\mu(s) < \infty,$$

because $\|B(s)\|^{\eta/2}$ is dominated by the submultiplicative function

$$(1+\|M(s)\|)^{\eta/2}(1+\|M(s^{-1})\|)^{\eta/2}.$$

On the other hand, by (2.8)

$$Q(s_{m-1}) = \prod_{i=m-1}^{l} Q(a_i)\, Q(x_i).$$

Let $x_i = \exp X_i$. Then $Q(x_i) = e^{R(X_i)}$ where $R(X_i)$ is an upper triangular matrix with zeros on the diagonal and $Q(a)$ is a diagonal matrix with eigenvalues $e^{I\cdot\lambda(\log a)}$, $I \in W' = W \setminus \{(0, \ldots, 0)\}$, λ as in (2.6). Therefore $Q(s_{m-1})$ is a linear combination of

$$Q(a_{m-1}\ldots a_{l_r})\, R(X_{l_r})^{j_r}\, Q(a_{l_r-1}\ldots a_{l_{r-1}})\, R(X_{l_{r-1}})^{j_{r-1}}$$
$$\ldots Q(a_{l_2-1}\ldots a_1)\, R(X_{l_1})^{j_1}$$

with coefficients less than or equal to 1. Let d be as in (2.7). Then $0 \leqslant r \leqslant d-2$, $1 \leqslant j_2, \ldots, j_r \leqslant d-2$, $0 \leqslant j_1 \leqslant d-2$ and the number of summands is dominated by a constant depending only on d multiplied by $(m-1)^{d-2}$.

By the Schwarz inequality

$$\int \|Q(a_{m-1}\ldots a_{l_r}) R(X_{l_r})^{j_r} \ldots Q(a_{l_2-1}\ldots a_1) R(X_{l_1})^{j_1}\|^\eta\, d\mu(s_1)\ldots d\mu(s_{m-1})$$
$$\leqslant \left(\int \|R(X_{l_r})^{j_r}\|^\eta \ldots \|R(X_{l_1})^{j_1}\|^\eta\, d\mu(s_1)\ldots d\mu(s_{m-1})\right)^{1/2}$$
$$\times \left(\int \|Q(a_{m-1}\ldots a_{l_r})\|^\eta \ldots \|Q(a_{l_2-1}\ldots a_1)\|^\eta\, d\mu(s_1)\ldots d\mu(s_{m-1})\right)^{1/2}.$$

Let now

$$q = \max_{I\in W'} \int e^{\eta I\cdot\lambda(\log a(s))}\, d\mu(s).$$

Since

$$\|Q(a_{l_j-1}\ldots a_{l_{j-1}})\|^\eta \leqslant \sum_{I\in W'} e^{\eta(I\cdot\lambda(\log(a_{l_j-1}\ldots a_{l_{j-1}}))},$$

we have

(3.12) $$\int \|Q(a_{l_j-1}\ldots a_{l_{j-1}})\|^\eta\, d\mu(s_{l_j-1})\ldots d\mu(s_{l_{j-1}}) \leqslant (d-1)\, q^{l_j - l_{j-1}}.$$

On the other hand, $R(tX) = tR(X)$ so there is a constant $C' > 0$ such that $\|R(X)\| \leqslant C'\|X\|$. Consequently, by (1.28)

$$\|R(X)\| \leqslant C\left(1+\tau_N(\exp X)\right)^\beta$$

for some C, $\beta > 0$ and in view of (1.15), (1.32)

(3.13) $$b = \max_{1\leqslant j\leqslant d-2} \int \|R(X)^j\|^\eta\, d\mu(s) < \infty.$$

Therefore by (3.12), (3.13) there is a constant $C(d, b)$ depending on d and b such that

$$\int \|B(s_m)\|^{\eta/2} \|Q(s_{m-1})\|^{\eta/2}\, d\mu(s_1)\ldots d\mu(s_m)$$
$$\leqslant C(d, b)(m-1)^{d-2}\, q^{(m-1)/2} \int \|B(s)\|^{\eta/2}\, d\mu(s),$$

which yields (3.11).

If ν is absolutely continuous with respect to the Haar measure dx on a boundary N_1 with density P_0, then we obtain the following

(3.14) COROLLARY. *There is a $\sigma > 0$ such that*

$$\int P_0(y)^{1-\sigma}\,dy < \infty.$$

Proof. Let α be such that $\int(1+\tau_{N_1}(y))^{-\alpha}\,dy < \infty$, η as in Theorem (3.10), $\varepsilon = \eta\alpha/(\alpha+\eta)$, $\sigma = \eta/(\alpha+\eta)$. Then

$$\int P_0(y)^{1-\sigma}\,dy \leqslant \Big(\int\big(P_0(y)^{1-\sigma}(1+\tau_{N_1}(y))^{\varepsilon}\big)^{1/(1-\sigma)}\,dy\Big)^{1-\sigma}\Big(\int(1+\tau_{N_1}(y))^{-\varepsilon/\sigma}\,dy\Big)^{\sigma},$$

which is finite in view of the previous theorem.

From now on we assume that $X_1, \ldots, X_j$, X in (3.1) generate s as a Lie algebra. Under this assumption we have

(3.15) THEOREM. *Let $Y_1', \ldots, Y_\chi'$ be a basis of right-invariant vector fields on a boundary N_1. Then the Poisson kernel ν on N_1 and all the right-invariant derivatives $Y'^I\nu$ of ν are bounded smooth functions.*

Proof. Let $s = yza$, $F(s) = P^{N_1}f(s)$, $f \in C_c^\infty(N_1)$, and $g_{za}(y) = F(yza) \in C^\infty(N_1)$. Then

$$Y'^I g_{za}(y) = \int (Y'^I f)(s(u))\,d\nu(u).$$

If $f \geqslant 0$ then the Harnack inequality (1.16) yields

$$|(Y'^I g_e)(e)| \leqslant d_I \|f\|_{L^\infty}.$$

Consequently,

$$|\langle Y'^I f, \nu\rangle| \leqslant d_I \|f\|_{L^\infty} \quad \text{for } f \in C_c^\infty(N_1).$$

Let $\Delta' = -\sum_{i=1}^{\chi} Y_i'^2$. As in (1.18) there are $C, p > 0$ such that

$$\|f\|_{L^\infty} \leqslant C\|(1+\Delta')^p f\|_{L^2(dy)}.$$

Hence for every l

$$|\langle(1+\Delta')^l f, \nu\rangle| \leqslant C_l \|(1+\Delta')^p f\|_{L^2(dy)}.$$

Let $\psi = (1+\Delta')^p f$. Then

$$|\langle(1+\Delta')^{l-p}\psi, \nu\rangle| \leqslant C_l \|\psi\|_{L^2(dy)}$$

and $(1+\Delta')^l \nu \in L^2(dy)$ for all $l = 0, 1, \ldots$, which completes the proof.

The density of ν will be denoted by P_0. Let

$$P_s(u) = P_0(s^{-1}(u))\,D(a^{-1}) \tag{3.16}$$

where $D(a) = \exp(\lambda_{i_1}(\log a) + \ldots + \lambda_{i_\chi}(\log a))$. Then

$$\int f(s(u))\,P_0(u)\,du = \int f(u)\,P_s(u)\,du,$$

$$\|P_s\|_{L^p} \leqslant D(a^{-1})^{(p-1)/p}\|P_0\|_{L^\infty}^{(p-1)/p}.$$

(3.17) THEOREM. *For every $f \in L^p(N_1)$, $p \geqslant 1$, the function $P^{N_1}f$ is L-harmonic.*

Proof. Let $H_1, \ldots, H_k$ be a nonnegative basis for Δ, $a = \exp(\sum_{i=1}^k a_i H_i)$, $b = \exp(\sum_{i=1}^k b_i H_i)$ and $S_b = \{xa\colon x \in N,\ a_i \geqslant b_i,\ i = 1, \ldots, k\}$. We choose $g \in C_c(N_1)$ such that $\|f-g\|_{L^p} < \varepsilon$. Then $P^{N_1}g$ is L-harmonic and

$$\sup_{s\in S_b}|P^{N_1}g(s) - P^{N_1}f(s)| \leqslant \|f-g\|_{L^p}\sup_{s\in S_b}\|P_s\|_{L^q} < C(b)\,\varepsilon,$$

$p^{-1}+q^{-1} = 1$, which yields that $P^{N_1}f$ is L-harmonic.

4. Hypoelliptic invariant operators. In this section we continue our study of the properties of ν and the Poisson integrals $P^{N_1}f$ to prove finally a number of results concerning the convergence of $P^{N_1}f$ to f as $\log a(s) \to -\infty$. We are able to do it, however, only under the additional assumption that $X_1, \ldots, X_j$ in (3.1) generate s as a Lie algebra. Then the measures μ_t are absolutely continuous with respect to the Haar measure and Theorems (2.17), (3.15) yield

(4.1) THEOREM. *A bounded function F is L-harmonic if and only if*

$$F = P^{N_1(L)}f \tag{4.2}$$

for an $f \in L^\infty(N_1(L))$.

Since, by Theorem (3.15), $P_0 \in L^p(N_1)$, $p \geqslant 1$, and as will be shown in Theorem (4.4), $P_0 \in C_0(N_1)$, we also have a version of Theorem (4.1) for harmonic functions satisfying an L^p condition. Indeed, repeating the arguments of A. W. Knapp and R. E. Williamson (Theorem (3.2) of [15]) we obtain

(4.3) THEOREM. *Let F be an L-harmonic function and $F_{za}(y) = F(yza)$. If $\sup\{\|F_{za}\|_{L^p(N_1(L))}\colon za \in N_0(L)A\} < \infty$ for a $p \geqslant 1$ then F is the Poisson integral of:*

(i) *a finite signed measure if $p = 1$;*
(ii) *an L^p function if $p > 1$.*

Now we shall prove the required properties of P_0.

(4.4) THEOREM. *There are C, $\varepsilon > 0$ such that*

$$P_0(y) \leqslant C(1+\tau_{N_1}(y))^{-\varepsilon}.$$

Proof. Let $P(s, y) = P_s(y)$ where P_s is as in (3.16). Putting $F(s) = \int f(y) P(s, y)\, dy$ into (3.4) we obtain

$$\int f(y) P(s, y)\, dy = \int f(y) P(ss_1, y)\, dy\, d\mu(s_1).$$

Now since $P \in C^\infty(S \times N_1)$, the Tonelli theorem yields

$$P(s, e) = \int P(ss_1, e)\, d\mu(s_1). \tag{4.5}$$

Let q be the density of μ with respect to the right-invariant Haar measure. Then $\mu = q(uza) W(u, z)\, du\, dz\, da$, where du, dz, da are the Haar measures on N_1, N_0, A respectively and $W(u, z)$ is a polynomial. By (1.28)

$$|W(u, z)| \leqslant c_1 (1+\tau_N(uz))^\gamma \quad \text{for some } c_1, \gamma > 0. \tag{4.6}$$

Let

$$\int P_0(y)(1+\tau_{N_1}(y))^\eta\, dy = c_2 < \infty, \tag{4.7}$$

$P(s) = P(s, e)$ and $\varepsilon = \eta/(2\beta)$ where β is as in (1.31). Then by (4.5)

$$\begin{aligned} \check{P}_0(y)(1+\tau_{N_1}(y))^\varepsilon &= \int P(ys)(1+\tau_{N_1}(y))^\varepsilon\, d\mu(s) \\ &\leqslant \int P(yuza)(1+\tau_{N_1}(yu))^\varepsilon (1+\tau_N(u))^\varepsilon q(uza) W(u, z)\, du\, dz\, da = W. \end{aligned}$$

We apply the Schwarz inequality to W. In view of

$$P(uza) = P_0\big(a^{-1}(z^{-1}(u^{-1}))\big) D(a^{-1})$$

we may write

$$\begin{aligned} W_1 &= \int P(yuza)^2 (1+\tau_{N_1}(yu))^{2\varepsilon}\, du = \int P(uza)^2 (1+\tau_{N_1}(u))^{2\varepsilon}\, du \\ &= \int P_0\big(a^{-1}(z^{-1}(u))\big)^2 (1+\tau_{N_1}(u))^{2\varepsilon} D(a^{-1})^2\, du \\ &= \int P_0(a^{-1}(u))^2 \big(1+\tau_{N_1}(z(u))\big)^{2\varepsilon} D(a^{-1})^2\, du. \end{aligned}$$

Now by (1.29) and (1.31)

$$W_1 \leqslant c_3 (1+\tau_N(z))^\eta \int P_0(u)^2 D(a^{-1})\big(1+\tau_{N_1}(a(u))\big)^\eta\, du.$$

Therefore (4.7) and Proposition (1.8) yield

$$W_1 \leqslant c_4 (1+\tau_N(z))^\eta D(a^{-1})(1+\|\mathrm{Ad}_a|_{\mathfrak{n}_1}\|)^\eta.$$

Let

$$\psi(a) = (1+D(a^{-1}))^{1/2}(1+D(a))^{1/2}(1+\|\mathrm{Ad}_{a^{-1}}|_{\mathfrak{n}_1}\|)^{\eta/2}(1+\|\mathrm{Ad}_a|_{\mathfrak{n}_1}\|)^{\eta/2}.$$

In view of (1.6), $\psi(a) \leqslant \exp\big(c_5(\tau_A(a)+1)\big)$ for a $c_5 > 0$. Therefore by the Schwarz inequality

$$W \leqslant c_6 \int (1+\tau_N(z))^{\eta/2} e^{c_5\tau_A(a)} \Big(\int (1+\tau_{N_1}(u))^{2\varepsilon} q(uza)^2 W(u, z)^2\, du\Big)^{1/2} dz\, da.$$

If α is as in (1.33) then

$$\begin{aligned} W \leqslant c_6 &\Big(\int (1+\tau_{N_0}(z))^{-2\alpha}\, dz\Big)^{1/2} \\ &\times \int \Big(\int (1+\tau_{N_0}(z))^{2\alpha}(1+\tau_N(z))^\eta (1+\tau_{N_1}(u))^{2\varepsilon} q(uza)^2 W(u, z)^2\, du\, dz\Big)^{1/2} \\ &\times \exp\big(c_5\tau_A(a)\big)\, da. \end{aligned}$$

By (1.29), (1.31), (4.6) and Proposition (1.21)

$$W \leqslant c_7 \int \Big(\int (1+\tau_N(uz))^\gamma e^{-\sigma\tau_S(uza)}\, du\, dz\Big)^{1/2} e^{c_5\tau_A(a)}\, da$$

for a $\gamma > 0$ and every $\sigma > 0$.

Now taking σ large enough and applying (1.32), (1.33) and Proposition (1.5) of [11] we see that W is finite.

Let $Y_1, \ldots, Y_\chi$ be a basis of $\mathfrak{n}_1$ such that $\mathrm{ad}_H(Y_j) = \lambda_{i_j}(H) Y_j$ and $\lambda_{i_1} \leqslant \ldots \leqslant \lambda_{i_\chi}$. Then $\mathrm{lin}(Y_i, \ldots, Y_\chi)$ is an ideal of $\mathfrak{n}_1$, $i = 1, \ldots, \chi$ (see (1.25)) and therefore every $y \in N_1$ can be written as

$$y = \prod_{i=1}^{\chi} \exp(y_i Y_i)$$

(cf. Lemma (3.1) of [16], [22]). $y_1, \ldots, y_\chi$ are called *canonical coordinates*.

(4.8) COROLLARY. *There are C, $\gamma > 0$ such that*

$$P_0(y) \leqslant C \min(1, |y_1|^{-\gamma}, \ldots, |y_\chi|^{-\gamma}).$$

Proof. Since N_1 is a homogeneous group (see (1.24)), the corollary follows immediately from Proposition (1.10) applied to the function $f(x) = \max_{i=1,\ldots,\chi} |y_i|$.

(4.9) LEMMA. *There are constants C, $\sigma > 0$ such that if $u, y \in N_1$ then*

$$|P_0(u) - P_0(y)| \leqslant C\|u-y\|(1+\|u\|+\|u-y\|)^\sigma$$

where $\|y\| = (\sum_{i=1}^{\chi} |y_i|^2)^{1/2}$.

Proof. Let $Y_1', \ldots, Y_\chi'$ be a basis of right-invariant vector fields on N_1. Since $\partial_i = \sum_{j=1}^{\chi} W_{ij} Y_j'$ with W_{ij} being polynomials and by Theorem (3.15), $Y_j' P_0$, $j = 1, \ldots, \chi$, are bounded, we have

$$\|\nabla P_0(y)\| \leqslant C(1+\|y\|)^\beta \tag{4.10}$$

for some constants C, $\beta > 0$.

Let γ be the curve given by $\gamma(t)_i = u_i + t(y_i - u_i)$, $t \in [0, 1]$. We then have

$$|P_0(u) - P_0(y)| \leqslant \sup_t \|\nabla P_0(\gamma(t))\|\, \|y-u\|,$$

and by (4.10) the assertion follows.

Lemma (4.9) completes the list of properties of the Poisson kernel which are necessary to prove Theorem (4.13) below. The proof by P. Sjögren [22] which we adopt requires the assumption that all $\lambda \in \Delta$ are combinations with rational coefficients of a maximal linearly independent over $\boldsymbol{R}$ subset of Δ. A subset of a vector space having this property will be called *rational*. Δ is rational in the case of symmetric spaces [24].

Now let $\boldsymbol{a}_1$ be a subspace of $\boldsymbol{a}$ with the following properties:

(4.11) There is an $H \in \boldsymbol{a}_1$ such that $\lambda(H) > 0$ for every $\lambda \in \Delta_1 = \{\lambda\colon \boldsymbol{n}_1 \cap \boldsymbol{n}^\lambda \neq \{0\}\}$.

(4.12) The set $\Delta_1|_{\boldsymbol{a}_1} = \{\lambda|_{\boldsymbol{a}_1}\colon \lambda \in \Delta_1\}$ is rational.

Every linear complement V to $\Delta_1^\perp = \{H\colon \forall \lambda \in \Delta_1 \ \lambda(H) = 0\}$ is an example of such a subspace. Indeed, if H is such that $\lambda(H) > 0$ for $\lambda \in \Delta$ and $H = H' + H''$, $H' \in V$, $H'' \in \Delta_1^\perp$, then H' satisfies (4.11). Since Δ_1 is rational and the mapping

$$\mathrm{lin}_{\boldsymbol{R}}(\Delta_1) \ni \lambda \to \lambda|_V \in \mathrm{lin}_{\boldsymbol{R}}(\Delta_1|_V)$$

is an isomorphism, $\Delta_1|_V$ satisfies (4.12).

We say that $\log a \to -\infty$, $a \in A_1 = \exp \boldsymbol{a}_1$, if and only if for every $\lambda \in \Delta_1$ we have $\lambda(\log a) \to -\infty$. Now we are ready to prove the convergence theorem.

(4.13) THEOREM. *Let $\boldsymbol{a}_0$ be a linear complement of $\boldsymbol{a}_1$ in $\boldsymbol{a}$, $A_0 = \exp \boldsymbol{a}_0$ and let K_1, K_0 be compact sets in N_1 and $N_0 A_0$ respectively. If $f \in L^p(N_1)$ for a $p > 1$ and $\log a_1 \to -\infty$, $a_1 \in \boldsymbol{a}_1$, then for a.e. $y_1 \in N_1$*

$$P^{N_1} f(y_1 a_1 yza_0) \to f(y_1) \tag{4.14}$$

uniformly with respect to $y \in K_1$, $za_0 \in K_0$.

Remarks. 1. "Uniformly" means that there is a measurable set $M \subset N_1$ such that $N_1 \setminus M$ has Lebesgue measure 0 and for all $y_1 \in M$, $\varepsilon > 0$ and any compact sets K_1, K_0, $K_1 \subset N_1$, $K_0 \subset N_0 A_0$,

$$|P^{N_1} f(y_1 a_1 yza_0) - f(y_1)| < \varepsilon$$

whenever $y \in K_1$, $za_0 \in K_0$ and $\lambda(a_1) \leqslant \lambda(a_1')$ for every $\lambda \in \Delta_1$ and an a_1' depending on y_1, ε, K_1, K_0.

2. Our theorem is a generalization of the semirestricted admissible convergence of Poisson integrals on symmetric spaces (see [16], [22]). In the case of a symmetric space $\boldsymbol{a}_1 = \{H\colon \lambda(H) = 0, \lambda \in \Pi'\}$ where Π' is a subset of the set Π consisting of simple roots and $\boldsymbol{n}_1 = \bigoplus_{\lambda \in \Delta_1'} \boldsymbol{n}^\lambda$, $\Delta_1' = \{\lambda \in \Delta\colon \lambda|_{\boldsymbol{a}_1} \neq 0\}$. Then the corresponding subgroup N_1 is normal and $\boldsymbol{a}_1$ is the linear subspace of $\boldsymbol{a}$ spanned by the elements dual to $\Pi \setminus \Pi'$. Obviously, such an $\boldsymbol{a}_1$ satisfies the conditions (4.11) and (4.12). Theorem (4.13) for $\boldsymbol{a}_1$ and $\boldsymbol{n}_1$ as above has been proved by A. Korányi [16] for the L^∞ case and by P. Sjögren [22] for every $p > 1$.

Proof. Obviously we have (4.14) for $f \in C_c(N_1)$. Let

$$Mf(y_1) = \sup\left\{\int |f(y_1 a_1 yza_0(u))|\, P_0(u)\, du\colon\ a_1 \in A_1,\ y \in K_1,\ za_0 \in K_0\right\}. \tag{4.15}$$

We shall prove that for every $p > 1$ there is a constant C_p such that

$$\|Mf\|_{L^p(N_1)} \leqslant C_p \|f\|_{L^p(N_1)}, \quad f \in L^p(N_1); \tag{4.16}$$

the theorem then follows by a standard approximation argument.

By (4.12) there is a basis $H_1, \ldots, H_l$ of $\boldsymbol{a}_1$ such that $\lambda(H_i)$ is an integer for $\lambda \in \Delta_1$. For $a = \exp\left(\sum_{i=1}^l a_i H_i\right)$ we write $[a] = \exp\left(\sum_{i=1}^l [a_i] H_i\right)$. Let

$$K = \{a_1 yza_0\colon\ \forall i\ (a_1)_i \in [-1, 1],\ y \in K_1,\ za_0 \in K_0\}.$$

By the Harnack inequality (1.17) there is a constant C such that

$$\sup_{s \in K} F(s) \leqslant CF(e)$$

for every nonnegative harmonic F. Hence

$$P^{N_1} f(y_1 a_1 yza_0) \leqslant CP^{N_1} f(y_1 [a_1]), \quad f \in L^p(N_1),\ f \geqslant 0,$$

because L is left-invariant. Now it is sufficient to prove inequality (4.16) writing $M' f$ instead of Mf, where

$$M' f(y) = \sup_{h \in Z^\chi} \int |f(y\delta_{-h}(u))|\, P_0(u)\, du,$$

$$\delta_{-h}\Big(\prod_{i=1}^{\chi} \exp(u_i Y_i)\Big) = \prod_{i=1}^{\chi} \exp(e^{-h_i} u_i Y_i), \quad h = (h_1, \ldots, h_\chi).$$

The proof follows closely that of Proposition (5.1) of [22], because in view of Corollaries (3.14), (4.8) and Lemma (4.9), P_0 has the properties required by P. Sjögren for the kernel in (4.15).

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Received February 23, 1987 (2280)