[12] P. Wojtaszczyk, *The Franklin system is an unconditional basis in* H_1, Ark. Mat. 20 (1982), 293–300.
[13] —, *Hardy spaces on the complex ball are isomorphic to Hardy spaces on the disc*, $1 \leqslant p < \infty$, Ann. of Math. 118 (1983), 21–34.
[14] T. Wolniewicz, *On isomorphisms between Hardy spaces on complex balls*, preprint.

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Factors of coalescent automorphisms

by

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Abstract. The class of all ergodic coalescent automorphisms is not closed under taking factors, powers and inverse limits. Even if a T-invariant sub-σ-algebra of an ergodic coalescent automorphism T is completely invariant it need not be coalescent. However, if $\mathscr{C}$ is a completely invariant sub-σ-algebra of a simple automorphism T then it is canonical.

I. Introduction. Let T be an ergodic automorphism of a Lebesgue space $(X, \mathscr{B}, \mu)$. The *centralizer* of T, $C(T)$, is the semigroup of all endomorphisms $S\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ such that $ST = TS$. T is called *coalescent* if $C(T)$ is a group, or equivalently, if every endomorphism commuting with T is invertible ([8]). Another definition of coalescence is the following (see [5], [9]): if a T-invariant sub-σ-algebra $\mathscr{C} \subset \mathscr{B}$ has the property that $T\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ and $T\colon (X, \mathscr{C}, \mu) \to (X, \mathscr{C}, \mu)$ are isomorphic, i.e. $\mathscr{C}$ is isomorphic to $\mathscr{B}$, then $\mathscr{C} = \mathscr{B}$.

The basic problem connected with coalescence is whether or not it implies zero entropy. Observe that no Bernoulli automorphism is coalescent. Indeed, if τ is a Bernoulli automorphism then represent τ as $\tau_1 \times \tau_1$, where $\tau_1\colon (W, \mathscr{C}, \nu) \to (W, \mathscr{C}, \nu)$ is Bernoulli and

(1) $$h(\tau_1) = h(\tau)/2.$$

Then take the flip map $f(x, y) = (y, x)$ which is in the centralizer of $\tau_1 \times \tau_1$ and take the corresponding sub-σ-algebra $\mathscr{C}_f = \{A \in \mathscr{C} \otimes \mathscr{C}\colon fA = A \text{ a.e.}\}$. Then the factor

(2) $$\tau_1 \times \tau_1\colon (W \times W, \mathscr{C}_f, \nu \times \nu) \to (W \times W, \mathscr{C}_f, \nu \times \nu)$$

is again Bernoulli with the same entropy as τ. Hence τ and (2) are isomorphic and consequently τ is not coalescent (the original proof of that fact is due to Kamiński [5]).

Therefore to prove that coalescence implies zero entropy it is enough to show that the class of all ergodic coalescent automorphisms is closed under taking factors (then use Sinai's Weak Isomorphism Theorem). That is why the question on factors of coalescent automorphisms stated by Newton in [8] is important. However, in the present paper we provide a counterexample to that question. This makes the hypothesis that there are ergodic coalescent automorphisms with positive entropy quite plausible.

Let $\mathscr{C} \subset \mathscr{B}$ be a T-invariant sub-σ-algebra. Following [9], $\mathscr{C}$ is called

(3) *canonical*

provided that if $\mathscr{C}'$ is another T-invariant sub-σ-algebra which is isomorphic to $\mathscr{C}$, then $\mathscr{C}' = \mathscr{C}$ (of course, in that case $\mathscr{C}$ is coalescent);

(4) *completely invariant*

provided that $S^{-1}\mathscr{C} = \mathscr{C}$ for every $S \in C(T)$.

Newton's question [9] is whether these two notions coincide. We show that it is even possible for a completely invariant $\mathscr{C}$ to be noncoalescent. However, there is a class of ergodic automorphisms called simple transformations ([4]) for which (3) and (4) coincide.

In the last section we discuss the problem of coalescence of inverse limits of coalescent automorphisms.

The results of the paper are applications of [4], [11], [12].

II. A class of coalescent automorphisms with a factor which is not coalescent. Let $T: (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be an ergodic automorphism. Denote by $J(X, \ldots, X)$ the space of all *n-joinings* of T:

(5) $\lambda \in J(X, \ldots, X)$ if λ is a $T \times \ldots \times T$-invariant probability measure on $\mathscr{B}_1 \otimes \ldots \otimes \mathscr{B}_n$, $\mathscr{B}_i = \mathscr{B}$, such that $\lambda|_{\mathscr{B}_i} = \mu$.

A standard example of ergodic 2-joinings comes from the centralizer of T. More precisely, if $S \in C(T)$ then the measure given by

$$\mu_S(A \times B) = \mu(A \cap S^{-1} B) \tag{6}$$

is a 2-joining. Following [4], [12] we call T *2-fold simple* if each ergodic 2-joining of T either is on the graph of some $S \in C(T)$ or is the product measure $\mu \times \mu$. It is an easy observation that 2-fold simplicity implies coalescence (consider $\bar{\mu}_S(A \times B) = \mu(S^{-1} A \cap B)$).

Now, for $n > 1$ and any $S_i \in C(T)$, $i = 1, \ldots, n$, the measure (n-joining) $\mu_{S_1,\ldots,S_n}$ given by

$$\mu_{S_1,\ldots,S_n}(A_1 \times \ldots \times A_n) = \mu(S_1^{-1} A_1 \cap \ldots \cap S_n^{-1} A_n)$$

is called an *off-diagonal*.

T is said to be *simple* if $C(T)$ is a group and for every $n \geq 2$ and every n-joining λ the set $\{1, \ldots, n\}$ can be split into subsets $s_1, \ldots, s_n$ such that $\lambda|_{\mathscr{B}^{s_i}}$ is an off-diagonal and λ is the product of these off-diagonals ([4]). It is proved in [4] that if T is weakly mixing and simple then so is T^k, $k \neq 0$. If T is simple and $C(T)$ is *trivial*, i.e. $C(T) = \{T^i: i \in \mathbf{Z}\}$, then T has the *minimal self-joining* (MSJ) property.

For further properties of simple automorphisms we refer to [4].

Now, we fix $T: (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$. We require T to have the MSJ property.

Let G be an infinite metric abelian group such that

$$(\exists n > 0)\ (\forall g \in G) \quad g^n = 1. \tag{7}$$

We denote by μ_G the Haar measure on G. Let $\varphi: X \to G$ be measurable and such that the corresponding G-extension

$$T_\varphi: (Y, \tilde{\mu}) \to (Y, \tilde{\mu}), \quad Y = X \times G,\ \tilde{\mu} = \mu \times \mu_G, \quad T_\varphi(x, g) = (Tx, \varphi(x) g) \tag{8}$$

is weakly mixing. We recall that the paper [3] ensures the existence of an abundance of such φ's.

Take $g \in G$ and define $\sigma_g \in C(T)$ by

$$\sigma_g(x, h) = (x, hg), \quad h \in G. \tag{9}$$

Observe that the transformations $T_\varphi \sigma_g$, $g \in G$, are weakly mixing G-extensions of T. Indeed, $T_\varphi \sigma_g$ is an n-root of $(T_\varphi \sigma_g)^n = T_\varphi^n$ and the latter automorphism is weakly mixing.

Hence if $\{g_n\} \subset G$, $g_i \neq g_j$ whenever $i \neq j$, then

$$U = T_\varphi \sigma_{g_1} \times T_\varphi \sigma_{g_2} \times \ldots: \quad (Y \times Y \times \ldots, \tilde{\mu} \times \tilde{\mu} \times \ldots) \to (Y \times Y \times \ldots, \tilde{\mu} \times \tilde{\mu} \times \ldots) \tag{10}$$

is ergodic (actually, it is weakly mixing).

We intend to prove the following

THEOREM 1. $C(U) = C(T_\varphi \sigma_{g_1}) \times C(T_\varphi \sigma_{g_2}) \times \ldots = C(T_\varphi)^{\mathbf{N}}$.

This result will be a simple consequence of the lemma below. Although this lemma easily follows from Rudolph's considerations [11] it is stated neither in [11] nor in [4]. Therefore we give a proof.

LEMMA 1. *Let $\tau: (W, \mathscr{C}, \nu) \to (W, \mathscr{C}, \nu)$ be a weakly mixing simple transformation. Then for every $R \in C(\tau \times \tau \times \ldots)$ there are $\varphi: \mathbf{N} \to \mathbf{N}$ which is one-to-one (but not necessarily onto) and a sequence $\{S_n\} \subset C(\tau)$ such that*

$$R = (S_1 \times S_2 \times \ldots) \circ \hat{\varphi} \tag{11}$$

where

$$\hat{\varphi}(x_1, x_2, \ldots) = (x_{\varphi(1)}, x_{\varphi(2)}, \ldots). \tag{12}$$

Proof. Set $\bar{W} = W \times W \ldots$, $\bar{\nu} = \nu \times \nu \times \ldots$, $\bar{\tau} = \tau \times \tau \times \ldots: (\bar{W}, \bar{\nu}) \to (\bar{W}, \bar{\nu})$. Consider the 2-joining of $\bar{\tau}$ which lies on the graph of R, i.e.

$$\bar{\nu}_R(\bar{A} \times \bar{B}) = \bar{\nu}(\bar{A} \cap R^{-1} \bar{B}) \tag{13}$$

where

$$\bar{A} \in \bigotimes_{i=1}^{\infty} \mathscr{C}_i, \quad \bar{B} \in \bigotimes_{i=1}^{\infty} \mathscr{C}'_i, \quad \mathscr{C}_i = \mathscr{C}'_i = \mathscr{C}.$$

Denote by $\bar{\nu}_R^{(n)}$ the projection of $\bar{\nu}_R$ on

$$\mathscr{C}^{(n)} \otimes \mathscr{C}'^{(n)} = (\mathscr{C}_1 \otimes \ldots \otimes \mathscr{C}_n) \otimes (\mathscr{C}'_1 \otimes \ldots \otimes \mathscr{C}'_n).$$

Thus $\bar{\nu}_R^{(n)}$ is a $2n$-joining of τ and therefore $\bar{\nu}_R^{(n)}$ is a product of off-diagonals. But $\bar{\nu}_R^{(n)}|_{\mathscr{C}^{(n)}} = \bar{\nu}_R^{(n)}|_{\mathscr{C}'^{(n)}} = \nu \times \ldots \times \nu$, so the off-diagonals are either a single coordinate, or a pair: one from $\mathscr{C}^{(n)}$ and one from $\mathscr{C}'^{(n)}$.

Now, letting $n \to \infty$ we assert that

(14) $$(\forall r)\ (\exists t) \quad \bar{\nu}_R|_{\mathscr{C}_t \otimes \mathscr{C}'_r} = \nu_{S_r}$$

for some $S_r \in C(\tau)$. Indeed, otherwise there exists a coordinate, say $\mathscr{C}'_r$, which is independent of $\bigotimes_{i=1}^{\infty} \mathscr{C}_i$ with respect to $\bar{\nu}_R$. But from (13) it follows that

(15) $$(\forall \bar{C} \in \bigotimes_{i=1}^{\infty} \mathscr{C}'_i)\ (\exists \bar{D} \in \bigotimes_{i=1}^{\infty} \mathscr{C}_i) \quad \bar{\nu}_R(\bar{C} \triangle \bar{D}) = 0.$$

Therefore if (14) fails then by (15) the sets from $\mathscr{C}'_r$ are independent of one another (with respect to $\bar{\nu}_R$). This is an obvious contradiction because of the marginals of $\bar{\nu}_R$.

Now, using (14) we can define $\varphi\colon N \to N$ by $\varphi r = t$. This definition is correct and φ is one-to-one. Moreover, if we put

$$R' = (S_1 \times S_2 \times \ldots) \circ \hat{\varphi}$$

we see that $\bar{\nu}_R = \bar{\nu}_{R'}$ and therefore $R' = R$.

Proof of Theorem 1. Let $V \in C(U)$. Then by (7)

$$V \in C(U^n) = C\big((T_\varphi \sigma_{g_1})^n \times (T_\varphi \sigma_{g_2})^n \times \ldots\big) = C(T_\varphi^n \times T_\varphi^n \times \ldots).$$

Since T_φ is simple as a weakly mixing group extension of an MSJ-automorphism [4], so is T_φ^n. Now, by Lemma 1

$$V = (S_1 \times S_2 \times \ldots) \circ \hat{\psi}$$

where $S_i \in C(T_\varphi^n)$, $\psi\colon N \to N$ is one-to-one. By [4] it follows that

$$C(T_\varphi) = C(T_\varphi^n) = C\big((T_\varphi \sigma_g)^n\big) = C(T_\varphi \sigma_g), \quad g \in G.$$

Hence $S_i \in C(T_\varphi)$. All we have to show is that $\psi = \mathrm{id}$. Observe that

$$C(T_\varphi) \ni (S_1^{-1} \times S_2^{-1} \times \ldots) \circ V = \hat{\psi}.$$

Moreover, from (12) it follows that

$$U \circ \hat{\psi}\big((x_1, h_1), (x_2, h_2), \ldots\big) = \big(T_\varphi \sigma_{g_1}(x_{\psi(1)}, h_{\psi(1)}),\ T_\varphi \sigma_{g_2}(x_{\psi(2)}, h_{\psi(2)}), \ldots\big),$$

$$\hat{\psi} \circ U\big((x_1, h_1), (x_2, h_2), \ldots\big) = \big(T_\varphi \sigma_{g_{\psi(1)}}(x_{\psi(1)}, h_{\psi(1)}),\ T_\varphi \sigma_{g_{\psi(2)}}(x_{\psi(2)}, h_{\psi(2)}), \ldots\big).$$

Hence $\psi = \mathrm{id}$ and Theorem 1 is proved.

Now, we list some consequences of Theorem 1.

Remark 1. The class of ergodic coalescent automorphisms is not closed under taking factors. This is so because U has $T \times T \times \ldots$ as a factor and the latter automorphism, obviously, is not coalescent.

Remark 2. The class of ergodic coalescent automorphisms is closed under taking roots, but not powers. Indeed, U is coalescent but its nth power is not.

Remark 3. Using the same methods as in the proof of Theorem 1 one can show that $T_\varphi \sigma_{g_1} \times T_\varphi \sigma_{g_2} \times \ldots$ and $T_\varphi \sigma_{g'_1} \times T_\varphi \sigma_{g'_2} \times \ldots$ are isomorphic iff the sets $\{g_i\}$ and $\{g'_i\}$ are the same.

Remark 4. Because $T_\varphi \sigma_g$ and $T_\varphi \sigma_h$, $g \neq h$, are not isomorphic, using the considerations above and Rudolph's arguments [11] one can show that the factors $T_\varphi \sigma_g$ are canonical factors of U. Notice that this gives a new proof for both Theorem 1 and Remark 3.

Remark 5. D. Newton in [9] asked whether every completely invariant sub-σ-algebra has to appear as canonical. We see that for our example U, the σ-algebra generated by

(16) $$\{A_1 \times G \times A_2 \times G \times \ldots\colon\ A_i \in \mathscr{B}\}$$

is completely invariant. Indeed, this is a consequence of Theorem 1 and of the following fact:

$$C(T_\varphi \sigma_{g_i}) = C(T_\varphi) = \{T_\varphi^n \sigma_h\colon\ n \in \mathbf{Z},\ h \in G\} \quad ([4]).$$

But the σ-algebra (16) is not even coalescent and consequently cannot be canonical.

In the next section we provide a class of ergodic automorphisms for which the answer to Newton's question is affirmative.

III. If $\mathscr{C}$ is completely invariant then it is canonical for simple transformations. Let $T\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be an ergodic coalescent automorphism.

If H is a subgroup of $C(T)$, then the σ-algebra

$$\mathscr{C}(H) = \{A \in \mathscr{B}\colon\ (\forall S \in H)\ SA = A \text{ a.e.}\}$$

defines a factor of T.

If $\mathscr{C}$ is a T-invariant sub-σ-algebra of $\mathscr{B}$, then

$$H(\mathscr{C}) = \{S \in C(T)\colon\ (\forall A \in \mathscr{C})\ SA = A \text{ a.e.}\}$$

is a subgroup of $C(T)$.

We will use the following result.

THEOREM 2 ([4], [12]). *Let T be a weakly mixing 2-fold simple automorphism. If $\mathscr{C}$ is a nontrivial T-invariant sub-σ-algebra, then $H(\mathscr{C})$ is compact and*

$$\mathscr{C} = \mathscr{C}(H(\mathscr{C})). \tag{17}$$

Now, we are able to prove the following

THEOREM 3. *Let T be a weakly mixing simple automorphism and $\mathscr{C} \subset \mathscr{B}$ a nontrivial T-invariant sub-σ-algebra. The following are equivalent*:
(i) *$\mathscr{C}$ is canonical.*
(ii) *$\mathscr{C}$ is completely invariant.*
(iii) *$\mathscr{C}$ is simple.*
(iv) *$H(\mathscr{C})$ is normal.*

Proof. (iii)$\Leftrightarrow$(iv) is proved in [4], (i)$\Rightarrow$(ii) is always true.

(ii)$\Rightarrow$(iv). Let $U \in C(T)$. We wish to show

$$UH(\mathscr{C})U^{-1} = H(\mathscr{C}). \tag{18}$$

To this end take $S \in H(\mathscr{C})$. To prove $USU^{-1} \in H(\mathscr{C})$ it is necessary to show $USU^{-1}A = A$ a.e. for $A \in \mathscr{C}$. But $U^{-1}A \in \mathscr{C}$ because $\mathscr{C}$ is completely invariant. Hence $SU^{-1}A = U^{-1}A$ a.e. On the other hand, if $S \in H(\mathscr{C})$ then $S = U(U^{-1}SU)U^{-1}$ and $U^{-1}SU \in H(\mathscr{C})$ from the foregoing argument. Therefore (18) holds.

(iv)$\Rightarrow$(ii). Let $H(\mathscr{C})$ be normal and take $S \in C(T)$. We have to show

$$S\mathscr{C} = \mathscr{C}. \tag{19}$$

From (17) we have

(20) $SA \in \mathscr{C}$ iff $SA \in \mathscr{C}(H(\mathscr{C}))$ iff $U(SA) = SA$ a.e. for every $U \in H(\mathscr{C})$.

But $US = SU'$ with $U' \in H(\mathscr{C})$ because $H(\mathscr{C})$ is normal, and therefore $USA = SU'A = SA$ a.e., so $S\mathscr{C} \subset \mathscr{C}$. Analogously $S^{-1}\mathscr{C} \subset \mathscr{C}$, and hence (19) holds.

(ii)$\Rightarrow$(i). Let $\varphi\colon \mathscr{C} \to \mathscr{C}'$ establish an isomorphism between two T-invariant sub-σ-algebras.

Take the relatively independent joining over the common factor, i.e. the measure $\bar{\mu}$ on $\mathscr{B}\otimes\mathscr{B}$ given by

$$\bar{\mu}(A \times B) = \int_X E(A|\mathscr{C})\,\varphi^{-1}E(B|\mathscr{C})\,d\mu \tag{21}$$

where $E(\cdot|\mathscr{C})$ denotes the conditional expectation. This measure need not be ergodic so we can decompose it as

$$\bar{\mu} = \int \bar{\mu}_\gamma\, d\nu(\gamma)$$

where the $\bar{\mu}_\gamma$ are ergodic 2-joinings of T and ν is a probability measure (see [4]).

Now, from (21) it follows that

(22)
$$A \in \mathscr{C} \quad \text{iff} \quad \bar{\mu}_\gamma(A \times X \triangle X \times \varphi A) = 0 = \bar{\mu}_\gamma(A \times \varphi A^c \cup A^c \times \varphi A) \quad \gamma\text{-a.e.}$$

Hence, for a.e. γ, $\bar{\mu}_\gamma$ is not the product measure and from the simplicity of T we get

$$\bar{\mu}_\gamma(A \times B) = \mu(A \cap S_\gamma^{-1}B), \qquad S_\gamma \in C(T). \tag{23}$$

Combining (22) and (23) we have

$$0 = \bar{\mu}_\gamma(A \times \varphi A^c) = \mu(A \cap S_\gamma^{-1}\varphi A^c) \quad \text{for } A \in \mathscr{C}.$$

Thus $S_\gamma^{-1}\varphi A^c \subset A^c$ a.e. for $A \in \mathscr{C}$. But the measures of these two sets are the same, so $S_\gamma^{-1}\varphi A^c = A^c$ a.e. for $A \in \mathscr{C}$. Consequently, $S_\gamma A = \varphi A$ a.e. for every $A \in \mathscr{C}$. This implies $\mathscr{C}' = \varphi\mathscr{C} = S_\gamma\mathscr{C} = \mathscr{C}$, since $\mathscr{C}$ is completely invariant. We have thus proved that $\mathscr{C}$ is canonical.

COROLLARY 1. *If T is simple and $C(T)$ is abelian then every factor of T is canonical. In particular, for every weakly mixing G-extension T_φ of an MSJ-automorphism T, where G is compact abelian, every factor of T_φ is canonical.*

Remark 6. From [4] it follows that every factor of a simple automorphism is coalescent.

IV. An application to finite rank automorphisms. For the definition and properties of *finite rank* (FR-) automorphisms we refer to [10].

We are interested in the following problem: is every factor of an ergodic FR-automorphism canonical? The answer is affirmative in the case of rank 1 because these automorphisms have simple spectrum (see [1], [9]). In this section we prove that in the class of all ergodic FR-automorphisms the answer is negative.

The following lemma can be easily proved.

LEMMA 2. *Let $T\colon (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be of rank 1 and weakly mixing and let G be any finite group. Then there exists a measurable $\varphi\colon X \to G$ such that the corresponding G-extension T_φ is weakly mixing and has rank at most the order of G, i.e. $\operatorname{rk}(T_\varphi) \leqslant |G|$.*

Now, let T have the MSJ property and rank 1. Let G be a finite group with a subgroup $H \subset G$ which is not normal. Take a $\varphi\colon X \to G$ such that T_φ is weakly mixing and $\operatorname{rk}(T_\varphi) \leqslant |G|$ (Lemma 2). Thus

$$T_\varphi \text{ is simple and } C(T_\varphi) = \{T_\varphi^n\sigma_g\colon n \in \mathbf{Z},\ g \in G\} \quad ([4]). \tag{24}$$

Set $\hat{H} = \{\sigma_h\colon h \in H\}$ (see (9)). Then $\hat{H}$ is not normal in $C(T_\varphi)$. Hence combining (24) with Theorem 3 we see that the factor determined by $\hat{H}$ is not canonical.

However, in this case if $\mathscr{C}'$ and $\mathscr{C}$ induce isomorphic factors then they are linked by an element $S\in C(T_\varphi)$ (i.e. $\mathscr{C}'=S\mathscr{C}$). It is natural to ask whether in the class of FR-automorphisms this is the only reason for two different sub-σ-algebras to determine isomorphic factors.

V. Inverse limits of coalescent automorphisms. It is not hard to see that the class of all coalescent automorphisms is not closed under taking inverse limits. Indeed, if T is simple and weakly mixing, then

$$C(T\times\ldots\times T)=\{(S_1\times\ldots\times S_n)\circ\hat{\varphi}:\ S_i\in C(T) \text{ and } \varphi \text{ is a coordinate permutation}\}.$$

Therefore $T\times\ldots\times T=T^{(n)}$ is coalescent but $\lim\operatorname{inv} T^{(n)}=T\times T\times\ldots$ is not.

The situation quite changes if we deal with simple automorphisms.

THEOREM 4. *If $T\colon (X,\mathscr{B},\mu)\to(X,\mathscr{B},\mu)$ is an ergodic automorphism and $\mathscr{B}_n\subset\mathscr{B}$ are T-invariant sub-σ-algebras such that $\mathscr{B}_n\nearrow\mathscr{B}$ and $T\colon (X,\mathscr{B}_n,\mu)\to(X,\mathscr{B}_n,\mu)$ is 2-fold simple (simple) then so is $T\colon\mathscr{B}\to\mathscr{B}$.*

In particular, inverse limits of 2-fold simple automorphisms are coalescent.

Proof. Take any ergodic $\lambda\in J(X,X)$ which is not the product measure. In order to prove that λ is on the graph of some $S\in C(T)$ is is enough to show (see [7]) that

$$(\forall A\in\mathscr{B})\ (\exists B\in\mathscr{B})\quad \lambda(A\times X\triangle X\times B)=0. \tag{25}$$

First, observe that

$$\lambda|_{\mathscr{B}_n\otimes\mathscr{B}_n} \text{ is not the product measure for } n \text{ large enough.} \tag{26}$$

Indeed, since λ is not the product measure on $\mathscr{B}\otimes\mathscr{B}$, there must exist two sets $A,B\in\mathscr{B}$ such that

$$\lambda(A\times B)\neq\mu(A)\mu(B). \tag{27}$$

Let $A_n,B_n\in\mathscr{B}_n$, $n\geqslant 1$, be chosen so that $\mu(A\triangle A_n)\to 0$ and $\mu(B\triangle B_n)\to 0$ as $n\to\infty$. Then the conditions

$$\lambda(A_n\times X\triangle A\times X)=\mu(A\triangle A_n),\qquad \lambda(X\times B_n\triangle X\times B)=\mu(B\triangle B_n)$$

imply

$$\lambda(A_n\times X\cap X\times B_n\triangle A\times X\cap X\times B)\to 0\quad \text{as } n\to\infty.$$

This together with (27) implies (26).

Since $\lambda|_{\mathscr{B}_n\otimes\mathscr{B}_n}$ is not the product measure, by the simplicity of $T\colon\mathscr{B}_n\to\mathscr{B}_n$, λ is on the graph of some $S_n\in C(T,\mathscr{B}_n)$. In other words,

$$(\forall A\in\mathscr{B}_n)\ (\exists B\in\mathscr{B}_n)\quad \lambda(A\times X\triangle X\times B)=0. \tag{28}$$

Now, fix any $A\in\mathscr{B}$ and select a sequence $A_n\in\mathscr{B}_n$, $n=1,2,\ldots$, such that

$$\mu(A\triangle A_n)\to 0\quad \text{as } n\to\infty. \tag{29}$$

In view of (28) there is a sequence $B_n\in\mathscr{B}_n$, $n=1,2,\ldots$, such that

$$\lambda(A_n\times X\triangle X\times B_n)=0. \tag{30}$$

As an immediate consequence of the inequality

$$\lambda(X\times B_n\triangle X\times B_{n+k})\leqslant\lambda(X\times B_n\triangle A_n\times X)+\lambda(A_n\times X\triangle A_{n+k}\times X)+\lambda(A_{n+k}\times X\triangle X\times B_{n+k})$$

we find that $\{B_n\}$ is a Cauchy sequence. Combining this with (30) and (29) we conclude that (25) holds and hence $T\colon\mathscr{B}\to\mathscr{B}$ is 2-fold simple.

To finish the proof we have to show that if $n\geqslant 2$ and $\lambda\in J(X,\ldots,X)$, $\lambda|_{\mathscr{B}^{(i)}\otimes\mathscr{B}^{(j)}}=\mu\times\mu$, $i\neq j$, $\mathscr{B}^{(i)}=\mathscr{B}$, $i,j=1,\ldots,n$, then $\lambda=\mu\times\ldots\times\mu$ (n factors) (i.e. we have to show that $T\colon\mathscr{B}\to\mathscr{B}$ is a pairwise independent process [4]). But if $\lambda|_{\mathscr{B}^{(i)}\otimes\mathscr{B}^{(j)}}=\mu\times\mu$ then

$$\lambda|_{\mathscr{B}_k^{(i)}\otimes\mathscr{B}_k^{(j)}}=\mu\times\mu,\qquad k\geqslant 1,$$

and by the simplicity of $T\colon\mathscr{B}_k\to\mathscr{B}_k$,

$$\lambda|_{\mathscr{B}_k^{(1)}\otimes\ldots\otimes\mathscr{B}_k^{(n)}}=\mu\times\ldots\times\mu.$$

Hence letting $k\to\infty$ and applying the arguments which were used to prove (26) we get $\lambda=\mu\times\ldots\times\mu$, and the proof is complete.

Professor J. P. Thouvenot asked in a conversation whether or not every $\bar{d}$-limit of ergodic FR-automorphisms is coalescent.

We have been unable to solve this problem. However, if we restrict our attention to local rank 1 automorphisms ([2]), then the answer is negative. This is so because Katok in [6] proved that the class, say $\mathscr{K}_m$, of all automorphisms admitting linked approximation of the type $(n, n+1,\ldots, n+m-1)$ is of the second category (in the weak topology). Using arguments from the proof of Theorem 4.4 in [6] we see that for every $T\in\mathscr{K}_m$, $T^{(m)}$ is of local rank 1. Now, by taking $\mathscr{K}=\bigcap_{m=1}^{\infty}\mathscr{K}_m$ we see that the class of those T's for which $T^{(m)}$, $m\geqslant 1$, has local rank 1 is of the second category. As a conclusion, even the inverse limit of ergodic local rank 1 automorphisms need not be coalescent.

References

[1] J. R. Baxter, *A class of ergodic transformation having simple spectrum*, Proc. Amer. Math. Soc. 24 (1970), 275–279.

[2] S. Ferenczi, *Systèmes localement de rang un*, Ann. Inst. H. Poincaré 20 (1) (1984), 35–51.

[3] R. Jones and W. Parry, *Compact abelian group extensions of dynamical systems II*, Compositio Math. 25 (1972), 135–147.
[4] A. Del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory Dynamical Systems, to appear.
[5] B. Kamiński, *Some properties of coalescent automorphisms of a Lebesgue space*, Comment. Math. 21 (1979), 95–99.
[6] A. B. Katok, *Constructions in Ergodic Theory*, Progress Math., Birkhäuser, Boston, to appear.
[7] M. Lemańczyk, *Canonical factors on a Lebesgue space*, Bull. Polish Acad. Sci. Math., to appear.
[8] D. Newton, *Coalescence and spectrum of automorphisms of a Lebesgue space*, Z. Wahrsch. Verw. Geb. 19 (1971), 117–122.
[9] —, *On canonical factors of ergodic dynamical systems*, J. London Math. Soc. (2) 19 (1979), 129–136.
[10] D. Ornstein, D. Rudolph and B. Weiss, *Equivalence of measure-preserving transformations*, Mem. Amer. Math. Soc. 37 (262) (1982).
[11] D. Rudolph, *An example of a measure-preserving map with minimal self-joinings and applications*, J. Analyse Math. 35 (1979), 97–122.
[12] W. A. Veech, *A criterion for the process to be prime*, Monatsh. Math. 94 (1982), 335–341.

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Left-invariant degenerate elliptic operators on semidirect extensions of homogeneous groups

by

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Abstract. Let a solvable Lie group S be the semidirect product of a nilpotent group N and an abelian subgroup A such that Ad_a, $a \in A$, are diagonalizable. For a class of second order left-invariant degenerate elliptic operators L on S we study bounded L-harmonic functions F. We describe L-boundaries of S and prove, for L hypoelliptic, the convergence of Poisson integrals to functions on the boundaries. The results of the paper imply theorems on admissible semirestricted convergence of classical Poisson integrals on symmetric spaces.

Introduction. This paper treats harmonic functions with respect to left-invariant degenerate elliptic operators L on a class of solvable Lie groups S. Our approach is motivated by the classical theory of harmonic functions with respect to the Laplace–Beltrami operator on a noncompact symmetric space $X = G/K$ considered as $\bar{N}A$, where $G = \bar{N}AK$ is the Iwasawa decomposition of the group of its isometries, G. We find a class of boundaries of S and study the Poisson integrals on them. Among them there is a maximal boundary in the sense that the Poisson integrals of bounded Borel functions on it reproduce all the L-harmonic functions.

Our main result is the almost everywhere convergence of Poisson integrals of L^p functions, $p > 1$. This gives a natural extension to the context of our spaces and operators of the admissible semirestricted convergence for symmetric spaces. The main problem we shall have to overcome is little information on Poisson kernels. We have no explicit formula; we are able, however, to prove enough properties of the Poisson kernel to obtain the convergence theorem. Before we sketch our results and techniques in greater detail we shall describe some of the background facts about harmonic functions on symmetric spaces.

Harmonic functions on symmetric spaces have been studied thoroughly. By a *harmonic function* on $G/K = \bar{N}A$ one means a function F such that

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