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On projections in H^1 and BMO

by

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Abstract. It is shown that BMO is primary. This property is derived from the finite-dimensional version. It is also shown that H^1 is primary. This is derived from the properties of subspaces of H^1 generated by infinite subsequences of the Haar basis in H^1. This solves a problem from [6]. A linear embedding i: $\mathrm{BMO}_N \to \mathrm{BMO}$ is constructed which satisfies the condition of order-inversion in the sense of [3].

Introduction. To the pair (n, i), $n \in N$, $0 \leqslant i \leqslant 2^n - 1$, we associate the dyadic interval $(2^{-n}i, 2^{-n}(i+1)]$ and the Haar function h_{ni}, which is 1 on the left half of $(2^{-n}i, 2^{-n}(i+1)]$, -1 on the right half and zero elsewhere.

Dyadic intervals are nested in the sense that if $I \cap J \neq 0$ then either $I \subset J$ or $J \subset I$. We order the set of dyadic intervals (n, i) lexicographically.

Given $f = \sum_{(n,i)} a_{ni} h_{ni}$ in $L^1(0, 1)$, we write

$$S(f) := \Big(\sum_{(n,i)} a_{ni}^2 h_{ni}^2\Big)^{1/2}, \qquad \|f\|_{H^1} := \int S(f).$$

Then $H^1 := \{f \in L^1 : \|f\|_{H^1} < \infty\}$. H_n^1 denotes the subspace of H^1 spanned by $\{h_{mj}: m \leqslant n,\ 0 \leqslant j \leqslant 2^m - 1\}$.

Given $f \in L^1(0, 1]$ and a dyadic interval I we write

$$f_I := |I|^{-1} \int_I f, \qquad \|f\|_{\mathrm{BMO}} := \sup\Big\{\Big(|I|^{-1} \int_I |f - f_I|^2\Big)^{1/2}: I \text{ dyadic interval}\Big\},$$

$$\mathrm{BMO} = \Big\{f \in L^1: \int f = 0,\ \|f\|_{\mathrm{BMO}} < \infty\Big\}.$$

The duality between BMO and H^1 is established by the following formula:

$$\|f\|_{H^1} = \sup\Big\{\Big|\int fg\Big|: \|g\|_{\mathrm{BMO}} = 1,\ g \in L^\infty\Big\}.$$

We frequently use the fact that for $f = \sum a_{ni} h_{ni}$ we can express the BMO norm of f in terms of the coefficients. In fact,

$$\|f\|_{\mathrm{BMO}} = \sup_{(n,i)} \Big(2^n \sum_{(m,j) \subset (n,i)} 2^{-m} a_{mj}^2\Big)^{1/2}.$$

BMO_n denotes the subspace of BMO generated by $\{h_{mj}: m \leqslant n,\ 0 \leqslant j \leqslant 2^m - 1\}$.

By the John–Nirenberg Theorem, an equivalent norm on BMO is given by the expression $\sup |I|^{-1} \int_I |f - f_I|$, where the supremum is extended over all dyadic intervals.

We use the following notation (cf. Jones [8]). Given a set $I \subset (0, 1]$ and collections $\mathscr{A}$, $\mathscr{B}$ of dyadic intervals we write $I \cap \mathscr{B}$ for the set $\{J \in \mathscr{B}: J \subset I\}$ and $\mathscr{A} \cap \mathscr{B}$ for $\{J: J \in \mathscr{A}, J \in \mathscr{B}\}$. Q_n denotes the collection of all dyadic intervals with length at least 2^{-n}.

Let J be a dyadic interval. Then $Q_n(J)$ denotes the collection of all dyadic intervals contained in J and having length at least $2^{-n}|J|$. We write

$$G_1(J, \mathscr{A}) = \{I \in \mathscr{A}: I \subset J \text{ and } J \text{ maximal}\},$$

$$G_n(J, \mathscr{A}) = \bigcup_{I \in G_{n-1}(J, \mathscr{A})} G_1(I, \mathscr{A}).$$

The following fact (cf. Garnett [7], Ch. X, Lemma 3.2) will be useful later. Given $0 < \gamma < 1$, $n \in N$ and a collection of dyadic intervals $\mathscr{A}$ so that

$$\sup_{I \in A} |I|^{-1} \sum_{J \in I \cap \mathscr{A}} |J| > \frac{n}{1-\gamma},$$

there exists $I_0 \in \mathscr{A}$ so that

$$\sum_{J \in G_n(I_0, \mathscr{A})} |J| \geqslant \gamma |I_0|.$$

We recall a few concepts from the theory of Banach spaces. An infinite-dimensional Banach space X is called *primary* iff for any bounded idempotent linear operator P on X, either the range of P or its kernel is isomorphic to X. Bounded idempotent linear operators are called *projections*. The range of a projection on X is by definition a *complemented subspace* of X. The fundamental link between projections and isomorphisms is given by the decomposition principle of Pełczyński: Let X be a Banach space isomorphic to $(\sum X)_{lp}$ for some $p \in [1, \infty]$, and suppose that a complemented subspace Y of X contains a subspace Z which is complemented in Y and isomorphic to X. Then Y is isomorphic to X.

Being primary is an isomorphism invariant. Hence Theorem 6 of this paper holds if H^1 is replaced there by any space X known to be isomorphic to H^1; for instance we can take:

– $H^1(B_n)$, $n \in N$, the Hardy space of bounded analytic functions on the n-dimensional ball (cf. Maurey [9], Carleson [5], Wojtaszczyk [12] for $n = 1$, Wojtaszczyk [13] and Wolniewicz [14] for $n > 1$).

– $H^1((\mathscr{F}_n), \Omega, P)$, the Hardy spaces of bounded martingales on Ω, provided $P(\bigcap_{\varepsilon > 0} \bigcup_{k \in N} A_k^\varepsilon) > 0$, where $A_k^\varepsilon := \{B: B \text{ is an atom in } \mathscr{F}_k \text{ and } P(B) < \varepsilon\}$ (cf. Müller [10]).

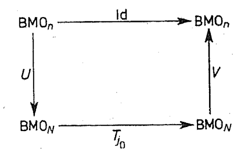
Consequently, Theorem 1 holds if BMO is replaced by the dual space of $H^1(B_n)$ resp. $H^1((\mathscr{F}_n), \Omega, P)$. See also the remark after the proof of Theorem 2.

A.

THEOREM 1. *If* BMO *is isomorphic to the direct sum of two spaces X and Y, then either X or Y is isomorphic to* BMO.

Theorem 1 will be derived from its finite-dimensional version.

THEOREM 2. *For any $n \in N$ and for any $c > 0$ there exists $N(n, c)$ such that if $N > N(n, c)$ and if T is any operator on BMO_N with norm less than c, then the identity on BMO_n factorizes boundedly either through T or through $T - \mathrm{Id}$, i.e. there exist operators U, V and $j_0 \in \{1, 2\}$ such that the diagram*



commutes, where $T_1 = T$ and $T_2 = \mathrm{Id} - T$, $\|U\|\,\|V\| \leqslant c$.

We begin by showing that Theorem 1 follows from Theorem 2.

LEMMA 3. *For any $\varepsilon > 0$ and any $n \in N$, there exists $N(\varepsilon, n)$ with the following property: If $N > N(\varepsilon, n)$ and E is an n-dimensional subspace of BMO_N, there exists a subspace F of BMO_N which is isometric to BMO_n and a projection Q from BMO_N on F such that $\|Q\| \leqslant 1$ and*

$$\|Qx\| \leqslant \varepsilon \|x\| \quad \textit{for } x \in E.$$

Proof. We first show that for any $x \in E$ with $\|x\| \leqslant 1$ and any $\varepsilon > 0$, $n \in N$, there is $N(\varepsilon, n)$ such that for any $N > N(\varepsilon, n)$ there exists $I \in Q_N$ with:

(Pa) $Q_n(I) \subset Q_N$.

(Pb) $\sup_{J \in Q_n(I)} |J|^{-1} \sum_{K \in J \cap Q_n(I)} |\langle x, h_K \rangle| < \varepsilon$.

Suppose to the contrary that this condition does not hold. Then we can find $x \in E$, $\|x\| \leqslant 1$, $\varepsilon_0 > 0$, $n_0 \in N$ such that for any $N \in N$ there exists $N_0 > N$ such that for any $I \in Q_{N_0}$ with $Q_{n_0}(I) \subset Q_{N_0}$ we obtain the inequality

$$\sup_{J \in Q_{n_0}(I)} |J|^{-1} \sum_{K \in J \cap Q_{n_0}(I)} |\langle x, h_K \rangle| > \varepsilon_0.$$

Let J be a dyadic interval and

$$b(J) := \sum_{K \in Q_{n_0}(J)} |\langle x, h_K \rangle|,$$

$$\mathscr{B} := \{J: b(J) \geqslant \varepsilon_0 |J|, Q_{n_0}(J) \subset Q_{N_0}\}.$$

By assumption we get

$$\sup_{I\in\mathscr{B}} |I|^{-1} \sum_{J\in I\cap\mathscr{B}} |J| = N_0 - n_0.$$

By the variant of [7], Ch. X, Lemma 3.2, mentioned in the introduction, there exist $I_1\in\mathscr{B}$, $m_1 \geqslant \frac{1}{2}(N_0-n_0)$ such that

$$\sum_{K\in G_{m_1}(I_1,\mathscr{B})} |K| \geqslant \tfrac{1}{2}|I_1|.$$

Consequently,

$$\sum_{j=1}^{m_1} \sum_{K\in G_j(I_1,\mathscr{B})} b(K) \geqslant \varepsilon_0 m_1 \tfrac{1}{2}|I_1|.$$

On the other hand, it follows from the expression of BMO norms in terms of coefficients that

$$\sum_{j=1}^{m_1} \sum_{K\in G_j(I_1,\mathscr{B})} b(K) \leqslant n_0 m_1^{1/2} |I_1|,$$

which leads to a contradiction for large enough N_0.

Using an Auerbach basis in E we may substitute for (Pb) the following inequality:

$$\text{(Pb}'\text{)} \qquad \sup_{\substack{x\in E\\ \|x\|\leqslant 1}} \sup_{I\in Q_n(I_0)} |I|^{-1} \sum_{J\in Q_n(I_0)\cap I} |\langle x, h_J\rangle| < \varepsilon.$$

Now we define

$$F := \operatorname{span}\{h_I\colon I\in Q_n(I_0)\},$$

$$Q\colon \mathrm{BMO}_N \to F, \quad f \to \sum_{I\in Q_n(I_0)} \langle f, h_I/|I|\rangle h_I.$$

Clearly, F is isometric to BMO_n and Q is a bounded idempotent map onto F with norm 1. It remains to estimate $\sup_{x\in E, \|x\|\leqslant 1} \|Q(x)\|$.

Given $x\in E$, $\|x\|\leqslant 1$, we have

$$\|Q(x)\|_{\mathrm{BMO}} \leqslant 2\sup_J |J|^{-1} \sum_{K\in J\cap Q_n(I_0)} |\langle x, h_K/|K|\rangle|\,|K|$$

$$\leqslant 2 \sup_{J\in Q_n(I_0)} |J|^{-1} \sum_{K\in J\cap Q_n(I_0)} |\langle x, h_K\rangle| \leqslant 2\varepsilon. \ \blacksquare$$

In order to prove Theorem 1 we use the isomorphism

$$\mathrm{BMO} \sim \big(\textstyle\sum \mathrm{BMO}_N\big)_{l^\infty}$$

which is obtained (implicitly) in Wojtaszczyk [9], p. 153. We then complete the proof exactly as in Bourgain [3], Lemma 2.

The following lemma and proposition are needed for the proof of Theorem 2.

Notation. For a collection $\mathscr{A}$ of dyadic intervals we write $\mathscr{A}^* := \left|\bigcup_{I\in\mathscr{A}} I\right|$, $\max\mathscr{A} := G_1((0,1],\mathscr{A})$.

LEMMA 4. *Let $\mathscr{A}$ be a collection of dyadic intervals. Given $l\in N$, $\varepsilon>0$, $x\in\mathrm{BMO}$, $y\in H^1$ we put*

$$\mathscr{B} := \{K\in\mathscr{A}\colon |K|^{-1}(|\langle x, h_K\rangle| + |\langle y, h_K\rangle|) > \varepsilon\}.$$

Then there exists a collection $\mathscr{D}$ of pairwise disjoint dyadic intervals such that:

(a) *$I\in\mathscr{D}$ implies $I\notin\mathscr{B}$.*

(b) *$\mathscr{D}^* > \mathscr{A}^*(1-1/l)$.*

(c) *$\inf_{I\in\max\mathscr{A}} \{|J|/|I|\colon J\in I\cap\mathscr{D}\} > 2^{-(l(\|x\|+\|y\|)/(\varepsilon\mathscr{A}^*))^2}$.*

Proof. Suppose not; then for

$$\mathscr{D}_j := \bigcup_{I\in\max\mathscr{A}} \{J\subset I\colon |J| = 2^{-j}|I|\}\cap\mathscr{B}$$

we get $\mathscr{D}_j^* > \mathscr{A}^*(1/l)$ for $j=1,\ldots,i_0$ where

$$i_0 = \left[\frac{1+l(\|x\|+\|y\|)}{\varepsilon\mathscr{A}^*}\right]^2.$$

Now

$$\sum_{j=1}^{i_0}\sum_{L\in\mathscr{D}_j} (|\langle x, h_L\rangle| + |\langle y, h_L\rangle|) > \sum_{j=1}^{i_0}\sum_{L\in\mathscr{D}_j} \varepsilon|L| > i_0|\mathscr{A}^*|\,\varepsilon/l.$$

On the other hand,

$$\sum_{j=1}^{i_0}\sum_{L\in\mathscr{D}_j} (|\langle x, h_L\rangle| + |\langle y, h_L\rangle|) \leqslant \sqrt{i_0}\,\|x\|_{\mathrm{BMO}}|\mathscr{A}^*|/l + \sqrt{i_0}\,\|y\|_{H^1}$$

$$\leqslant \sqrt{i_0}\,(\|x\|+\|y\|).$$

Hence $\sqrt{i_0} < l(\|x\|+\|y\|)/(\varepsilon\mathscr{A}^*)$, contradicting the choice of i_0. $\blacksquare$

PROPOSITION 5. *For any $n\in N$, $\varepsilon_I>0$, $I\in Q_n$, and $c>0$ there exists $N(n,(\varepsilon_I),c)$ such that if $N>N(n,(\varepsilon_I),c)$ and if T is any operator on BMO_N with norm less than c, there exist collections of pairwise disjoint intervals $(E_I)_{I\in Q_n}$ such that:*

(i) *If $I, J\in Q_n$ and if $I\subset J$, then either*

$$E_I \cap \{t\colon \sum_{K\in E_J} h_K(t) = 1\} = E_I$$

or

$$E_I \cap \{t\colon \sum_{K\in E_J} h_K(t) = -1\} = E_I.$$

(ii) $|I| \geqslant \sum_{K \in E_I} |K| \geqslant |I|(1-4^{-n})$ *for* $I \in Q_n$.

(iii) *For* $\tilde{h}_I := \sum_{K \in E_I} h_K$ *the following inequality holds:*

$$\sum_{I \neq J} |\langle T\tilde{h}_I, \tilde{h}_J\rangle| < \varepsilon_J \quad \text{for each } J \in Q_n.$$

(iv) *If we set* $T_1 := T$, $T_2 := \mathrm{Id} - T$, $\mathscr{A}_j := \{I: \langle \tilde{h}_I, T_j \tilde{h}_I\rangle > \frac{1}{2}|I|\}$, $X_j := \operatorname{span}\{\tilde{h}_I: I \in \mathscr{A}_j\}$,

$$i_j: X_j \to \mathrm{BMO}_N, \quad \tilde{h}_I \to \tilde{h}_I,$$

$$P_j: \mathrm{BMO}_N \to X_j, \quad f \to \sum_{I \in A_j} \frac{\langle f, \tilde{h}_I\rangle \tilde{h}_I}{|I|} \cdot \frac{|I|}{\langle T_j \tilde{h}_I, \tilde{h}_I\rangle},$$

then:

(a) $\|P_j T_j i_j f - f\|_{\mathrm{BMO}} \leqslant 4\varepsilon \|f\|_{\mathrm{BMO}}$, $f \in X_j$, *where*

$$\varepsilon = \sup_I (|I|^{-1} \sum_{J \subset I} \varepsilon_J^2 |J|^{-1})^{1/2}.$$

(b) $\|P_j\| \leqslant 4$.

Proof of Proposition 5.

Construction. Fix $\gamma < 1$ such that $\gamma^{2^n} > 1 - 8^{-n}$ and choose $\delta_I > 0$ with $I \in Q_n$ such that $\delta_I < \varepsilon_I/2$ and $\sum_{J > I} \delta_J < \varepsilon_I/2$.

Step 00. $E_{00} := (0, 1)$, $\tilde{h}_{00} := h_{00}$, $\mathscr{D}_{00} := (0, 1]$.

Step 10. Consider

$$\mathscr{B} := \{L \subset (0, 1]: |L|^{-1}(|\langle T\tilde{h}_{00}, h_L\rangle| + |\langle T^* \tilde{h}_{00}, h_L\rangle|) > \delta_{00}\}.$$

By Lemma 4 there exists a collection $\mathscr{D}_{10}$ of pairwise disjoint dyadic intervals such that:

(a) $I \in \mathscr{D}_{10}$ implies $I \notin \mathscr{B}$.
(b) $\mathscr{D}_{10}^* > \gamma \mathscr{D}_{00}^*$.
(c) $\inf\{|I|: I \in \mathscr{D}_{10}\} > 2^{-(2\|T\|/(\delta_{00}(1-\gamma)))^2}$.

Now put

$$E_{10} := \{J \in \mathscr{D}_{10}: J \subset (0, \tfrac{1}{2}]\}, \quad \tilde{h}_{10} := \sum_{J \in E_{10}} h_J.$$

Step (m, i) *for* $i \neq 0$. We are given: $\tilde{h}_{00}, \ldots, \tilde{h}_{m,i-1}$, $E_{00}, \ldots, E_{m,i-1}$ and $\mathscr{D}_{00}, \ldots, \mathscr{D}_{m,i-1}$. Define

$$\mathscr{A} := \{J: \exists I \in \mathscr{D}_{m,i-1} \text{ such that } J \subset I\}.$$

$$\mathscr{B} := \{L \in \mathscr{A}: |L|^{-1} (\sum_{(k,j) < (m,i-1)} (|\langle T\tilde{h}_{kj}, h_L\rangle| + |\langle T^* \tilde{h}_{kj}, h_L\rangle|)) > \delta_{mi}\}.$$

By Lemma 4 there exists a collection $\mathscr{D}_{mi}$ of pairwise disjoint dyadic intervals such that:

(a) $I \in \mathscr{D}_{mi}$ implies $I \notin \mathscr{B}$.
(b) $\mathscr{D}_{mi}^* > \gamma \mathscr{D}_{m,i-1}^*$.
(c) $\inf_{J \in \mathscr{D}_{m,i-1}} \{|I|/|J|: I \in \mathscr{D}_{m,i} \cap J\} > 2^{-((2^m + 2^m)\|T\|/(\delta_{mi}(1-\gamma)\gamma^m))^2}$.

Finally, we define

$$S_{mi} := \begin{cases} \{t: \tilde{h}_{m-1,i/2}(t) = 1\} & \text{for } i \text{ even}, \\ \{t: \tilde{h}_{m-1,(i-1)/2}(t) = -1\} & \text{for } i \text{ odd}, \end{cases}$$

$$E_{mi} := \{J \in \mathscr{D}_{mi}: I \subset S_{mi}\}, \quad \tilde{h}_{mi} := \sum_{J \in E_{mi}} h_J.$$

Step (m, i) *for* $i = 0$. We replace $(m, i-1)$ by $(m-1, 2^{m-1})$, but otherwise we verbatim repeat the construction.

We stop the process at step $(n, 2^n)$. A possible estimate of $N(n, (\varepsilon_I), c)$ is given by the number

$$2^{(2^{n+1}c/((1-\gamma)\gamma^n \delta_{(n,2^n)})^{2^n}}.$$

Verification of (i)–(iv).

(i) is clear.
(ii) follows from the choice of γ and from property (b) of $\mathscr{D}_I$, $I \in Q_n$.
(iii) First, for I fixed we have $\sum_{J<I} |\langle T\tilde{h}_J, \tilde{h}_I\rangle| < \delta_I$, and

$$|\langle T\tilde{h}_J, \tilde{h}_I\rangle| = |\langle \tilde{h}_J, T^* \tilde{h}_I\rangle| < \delta_J \quad \text{for } J > I.$$

This follows from property (a) of $\mathscr{D}_I$. Finally,

$$\sum_{\substack{J \\ J \neq I}} |\langle T\tilde{h}_J, \tilde{h}_I\rangle| = \sum_{J<I} |\langle T\tilde{h}_J, \tilde{h}_I\rangle| + \sum_{J>I} |\langle T\tilde{h}_J, \tilde{h}_I\rangle| \leqslant \delta_I + \sum_{J>I} \delta_J < \varepsilon_I.$$

(iv)(a) By the construction of $(\tilde{h}_I)$ the operator i: $\mathrm{BMO}_N \to \mathrm{BMO}$, $h_I \to \tilde{h}_I$ is bounded with norm less than 2 (cf. [10], Theorem 0). Next fix $j \in \{1, 2\}$ and put $\alpha_I := (\langle T_j \tilde{h}_I, \tilde{h}_I\rangle)^{-1}$. Take $f = \sum a_I \tilde{h}_I$ with $\|f\|_{\mathrm{BMO}} = 1$. Then we have

$$P_j T_j f = \sum_{I \in \mathscr{A}_j} \langle T_j f, \tilde{h}_I\rangle \tilde{h}_I \alpha_I$$

$$= \sum_{I \in \mathscr{A}_j} a_I \langle T_j \tilde{h}_I, \tilde{h}_I\rangle \tilde{h}_I \alpha_I + \sum_{I \in \mathscr{A}_j} \sum_{\substack{J \\ J \neq I}} a_J \langle T_j \tilde{h}_J, \tilde{h}_I\rangle \tilde{h}_I \alpha_I.$$

Hence we obtain the estimate

$$\|P_j T_j f-f\|^2_{\mathrm{BMO}} = \Big\|\sum_{I\in\mathscr{A}_j}\sum_{\substack{J\\ J\neq I}} a_J\langle T_j\tilde h_J, \tilde h_I\rangle \tilde h_I\alpha_I\Big\|_{\mathrm{BMO}}$$

$$\leqslant 8\sup_{I\in\mathscr{A}_j}|I|^{-1}\sum_{K\subset I}\Big|\sum_{\substack{J\\ J\neq K}} a_K\langle T_j\tilde h_J, \tilde h_K\rangle\Big|^2|K|^{-1}$$

$$\leqslant 16\sup_{I}|I|^{-1}\sum_{K\subset I}\varepsilon_K^2|K|^{-1}.$$

(iv)(b) First we recall that for f given

$$\|f\|^2_{\mathrm{BMO}} = \sup\{|I|^{-1}\int_I|f-f_I|^2\colon \langle f, h_I\rangle\neq 0\}$$

(cf. [8], p. 855). Fix now $x\in\mathrm{BMO}_N$; by the definition of P_j, $\langle P_j x, h_I\rangle = 0$ for $I\notin\bigcup\{J\colon J\in E_K,\ K\in Q_n\}$. Hence in order to estimate $\|P_j x\|_{\mathrm{BMO}}$ it suffices to take $K\in Q_n$, $J\in E_K$, and to estimate as follows:

$$|J|^{-1}\int_J|P_j x-(P_j x)_J|^2$$

$$\leqslant 2|J|^{-1}\sum_{\substack{I\in\mathscr{A}_j\\ I\subset K}}|\langle x,\tilde h_I\rangle|^2|\tilde h_I|^2|\langle T_j\tilde h_I,\tilde h_I\rangle|^{-2}$$

$$\leqslant 4|J|^{-1}\sum_{\substack{I\in\mathscr{A}_j\\ I\subset K}}\frac{|\langle x,\tilde h_I\rangle|^2}{|I|^2}\int_J|\tilde h_I|^2$$

$$\leqslant 4|J|^{-1}\sum_{\substack{I\in\mathscr{A}_j\\ I\subset K}}\frac{|\langle x,\tilde h_I\rangle|^2}{|I|^2}\cdot\frac{|J|\cdot|I|}{|K|} = 4|K|^{-1}\sum_{\substack{I\in\mathscr{A}_j\\ I\subset K}}|\langle x,\tilde h_I\rangle|^2|I|^{-1}$$

$$\leqslant 4|K|^{-1}\sum_{\substack{I\in\mathscr{A}_j\\ I\subset K}}|I|^{-1}\Big(\sum_{J\in E_I}|\langle x, h_J\rangle|\Big)^2$$

$$\leqslant 4|K|^{-1}\sum_{\substack{I\in\mathscr{A}_j\\ I\subset K}}|I|^{-1}\Big(\sum_{J\in E_I}|J|\Big)\Big(\sum_{J\in E_I}|\langle x, h_J\rangle|^2|J|^{-1}\Big)$$

$$\leqslant 8|K|^{-1}\sum_{\substack{I\in\mathscr{A}_j\\ I\subset K}}\sum_{J\in E_I}|\langle x, h_J\rangle|^2|J|^{-1}\leqslant 8|K|^{-1}\sum_{L\in E_K}\sum_{J\subset L}|\langle x, h_J\rangle|^2|J|^{-1}$$

$$\leqslant 8|K|^{-1}\Big(\sum_{L\in E_K}|L|\Big)\|x\|^2_{\mathrm{BMO}}.$$

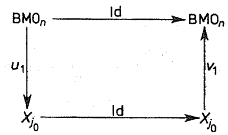
Proof of Theorem 2. Fix $m_1 > n\cdot 4^{n+1}$, $1>\gamma>1-4^{-n}$ and $(\varepsilon_J)_{J\in Q_{m_1}}$ such that

$$\sup_{I\in Q_{m_1}}|I|^{-1}\sum_{J\subset I}|J|^{-1}\varepsilon_J^2 < \tfrac{1}{100}.$$

Take $N(m_1, (\varepsilon_J), c)$ large enough for Proposition 5 to hold. Let $\mathscr{A}_j$, X_j, $j\in\{1, 2\}$, be as in Proposition 5. There exists $j_0\in\{1, 2\}$ so that

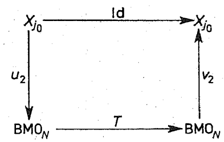
$$\sup_{I\in\mathscr{A}_{j_0}}|I|^{-1}\sum_{J\subset I}|J|\geqslant n\cdot 4^n\cdot 2.$$

Hence by the (dual version of) Main Lemma 2 in [10] there exist $u_1\colon \mathrm{BMO}_n\to X_{j_0}$ and $v_1\colon X_{j_0}\to\mathrm{BMO}_n$ such that $\|u_1\|\,\|v_1\|\leqslant 10$ and the diagram



commutes.

On the other hand, by a standard approximation theorem we obtain (from Proposition 5) $u_2\colon X_{j_0}\to\mathrm{BMO}_N$ and $v_2\colon\mathrm{BMO}_N\to X_{j_0}$ such that $\|u_2\|\,\|v_2\|\leqslant 10$ and the diagram



commutes. ∎

Remarks. Theorem 1 still holds if BMO is replaced by $(\sum H^1_n)_{l^1}$ or $(\sum\mathrm{BMO}_n)_{c_0}$. Dualization and Bochkarev's result [2] imply that in Theorem 2, BMO_n and $\mathrm{BMO}_{N(n)}$ can be replaced by L^1_n and $L^1_{N(n)}$ respectively, the spaces of trigonometric polynomials equipped with the L^1 norm. According to Bourgain this does not follow from the methods of [4].

B. By using two theorems from [10] we give an easy proof of the fact that H^1 is primary. This proof follows the ideas of Alspach–Enflo–Odell [1].

THEOREM 6. *For any operator $T\colon H^1\to H^1$ there exists a factorization of* Id: $H^1\to H^1$ *through T or* Id$-T$.

Proof. Fix a sequence (ε_J) of positive numbers indexed by the family of dyadic intervals so that $\|\sum\varepsilon_J h_J\| < 1/1000$; $(r_n, n\in N)$ denotes the sequence of Rademacher functions on $(0, 1]$. First we observe that there exists an increasing sequence $m((n,i))$ so that for $\tilde h_{ni} := \chi_{ni} r_{m((n,i))}$ we have the following estimate:

$$\sum_{\substack{J \\ I \neq J}} |\langle T\tilde h_J, \tilde h_I\rangle| \leqslant \varepsilon_I.$$

This follows from the fact that for any I, $r_n \chi_I$ tends weakly to zero as $n \to \infty$. Next define $T_1 := T$, $T_2 = \mathrm{Id} - T$, $j \in \{1, 2\}$,

$$\mathcal{A}_j := \{I\colon |\langle T_j \tilde h_I, \tilde h_I\rangle| \geqslant |I|/2\}, \qquad X_j := \operatorname{span}\{\tilde h_I\colon I \in \mathcal{A}_j\},$$

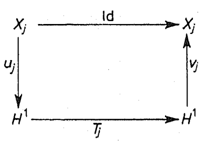
$$P_j\colon H^1 \to X_j, \quad f \to \sum_{I \in \mathcal{A}_j} \langle f, \tilde h_I\rangle \tilde h_I \frac{1}{\langle T_j \tilde h_I, \tilde h_I\rangle}.$$

Our choice of $(\tilde h_I)$ gives the following estimates (cf. [10], Theorem 0):

(a) $\|P_j\| \leqslant 4$.

(b) $\|P_j T_j f - f\| \leqslant \frac{1}{100}\|f\|$ for $f \in X_j$.

Hence there exist $u_j\colon X_j \to H^1$, $v_j\colon H^1 \to X_j$ such that $\|u_j\|\,\|v_j\| \leqslant 10$ and the diagram

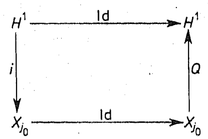


commutes.

On the other hand, there exists $j_0 \in \{1, 2\}$ so that

$$|\{t\colon t \in I \text{ for infinitely many } I \in \mathcal{A}_{j_0}\}| > 0.$$

Now we combine the selection process of the proof of Theorem 1a in [10] with Theorem 0 in [10] to conclude that there exists $i\colon H^1 \to X_{j_0}$, $Q\colon X_{j_0} \to H^1$ such that $\|i\|\,\|Q\| \leqslant 10$ and the diagram



commutes. ■

C.

Definition (cf. Bourgain [3], p. 47). A linear map $i\colon \mathrm{BMO}_N \to \mathrm{BMO}$ (or $i\colon H^1_N \to H^1$) is called *order-inversing* if

$$i(h_{ni}) \in \operatorname{span}\{h_I\colon I \in S_{ni}\},$$

where $(S_{ni})_{(n,i)\in Q_N}$ denotes pairwise disjoint collections of dyadic intervals such that $(n, i) < (m, j)$ implies

$$\inf\{I\colon I \in S_{ni}\} > \sup\{J\colon J \in S_{mj}\}.$$

In Bourgain [3] it was shown that given an order-inversing embedding $i\colon H^1_N \to H^1$, there does not exist an order-inversing linear map $i^*\colon \mathrm{BMO}_N \to \mathrm{BMO}$ satisfying $\langle i(h_{ni}), i^*(h_{mj})\rangle = \langle h_{ni}, h_{mj}\rangle$ and

$$\|i^*\| < C \quad \text{(independent of } N\text{)}.$$

Examples of order-inversing embeddings $i\colon H^1_N \to H^1$ are easily constructed, and there exist many of them. This leads to the following question: Does there exist $C > 0$ so that for any $N \in \mathbf{N}$ there exists an order-inversing linear map $i^*\colon \mathrm{BMO}_N \to \mathrm{BMO}$ satisfying $(1/C)\|f\| \leqslant \|i^*(f)\|$ for $f \in \mathrm{BMO}_N$ such that

$$\|i^*\| < C?$$

I learned about this question in discussions with P. Wojtaszczyk for which I express my appreciation. Unexpectedly, the answer is affirmative.

Construction of the operator. Fix $N \in \mathbf{N}$ and start with

Step 0: $E_{00} := (0, \frac{1}{2}]$, $t_1 := \frac{1}{2}$,

$$g_1(t) := \begin{cases} 1, & t \in (0, \frac{1}{2}], \\ 0 & \text{elsewhere}. \end{cases}$$

Having constructed $E_{00}, \ldots, E_{n,2^n-1}, t_1, \ldots, t_n, g_1, \ldots, g_n$ for $n < N$ we continue as follows:

Step n: $t_{n+1} := t_n/2 + \frac{1}{2}$. The interval $(t_n, t_{n+1}]$ is then divided into 2^{n+1} dyadic intervals $(I_{n+1,j})_{0 \leqslant j \leqslant 2^{n+1}-1}$ of length $2^{-n-1}(t_{n+1} - t_n)$, and we set

$$E_{n+1,2i+j} := E_{ni} \cup I_{n+1,2i+j}, \quad j \in \{0, 1\},\ 0 \leqslant i \leqslant 2^n - 1,$$

$$g_{n+1}(t) := \begin{cases} 1, & t \in \bigcup_{j=0}^{2^{n+1}} I_{n+1,j}, \\ 2^{-1/2} g_n(t), & t \in \bigcup_{j=0}^{2^n-1} E_{nj}, \\ 0, & t \in (t_{n+1}, 1]. \end{cases}$$

Observe that

$$\bigcup_{j=0}^{2^{n+1}} I_{n+1,j} = (t_n, t_{n+1}], \qquad \bigcup_{j=0}^{2^n-1} E_{nj} = (0, t_n].$$

Hence g_{n+1} is well defined on $(0, 1]$.

We stop this process after having defined g_N. Starting with $E_{N,2^N-1}$ we choose inductively pairwise disjoint collections of dyadic intervals $(S_{ni})_{(n,i)\in Q_n}$ which satisfy the conditions in the definition of order-inversion and

$$\sum_{I\in S_{ni}} \chi_I = \chi_{E_{ni}}, \qquad (n, i)\in Q_N.$$

This implies in particular that for $m < n$, $I\in S_{ni}$ and $J\in S_{mj}$,

$$I\cap J \neq \emptyset \quad \text{implies} \quad I \supset J.$$

Next we define

$$\tilde{h}_{ni}(t) := \Big(\sum_{I\in S_{ni}} h_I\Big)(t)\, g_n(t), \qquad (n, i)\in Q_N.$$

LEMMA 7. *For any* $a_{ni}\in \mathbf{C}$, $(n, i)\in Q_N$ *we have the estimate*

$$\Big\|\sum a_{ni} h_{ni}\Big\|_{\mathrm{BMO}} \leqslant \Big\|\sum a_{ni} \tilde{h}_{ni}\Big\|_{\mathrm{BMO}} \leqslant \sqrt{2}\,\Big\|\sum a_{ni} h_{ni}\Big\|_{\mathrm{BMO}}.$$

Remark. This means that i^*: BMO $\to$ BMO, $h_{ni}\to \tilde{h}_{ni}$ is the operator we are looking for.

Proof of Lemma 7. We start with the right-hand inequality. Fix $(n_0, i_0)\in Q_N$, $I\in S_{n_0 i_0}$. Then $g^2_{n_0}(t) = 2^{-m_0}$ for any $t\in I$ and some $m_0 \leqslant n_0$. Let j_0 be the (well-defined) integer so that $(n_0 - m_0, j_0) \supset (n_0, i_0)$.

Let us analyse the behaviour of $(\tilde{h}_{ni})$, $(n, i)\in Q_N$, on the interval I:

Case 1: $n_0 - m_0 \leqslant n \leqslant n_0$ *and* $(n, i) \subset (n_0 - m_0, j_0)$. Then we have $\tilde{h}^2_{ni}(t) = 2^{-m_0+n_0-n}$ for any $t\in I$.

Case 2: $n_0 - m_0 \leqslant n \leqslant n_0$ *and* $(n, i) \cap (n_0 - m_0, j_0) = \emptyset$. Then we have $\tilde{h}^2_{ni}(t) = 0$ for any $t\in I$.

Case 3: $n \leqslant n_0 - m_0$ *and* (n, i) *arbitrary*. Then we have $\tilde{h}^2_{ni}(t) = 0$ for $t\in I$.

This analysis helps us now to estimate:

$$|I|^{-1}\int_I \Big(\sum a_{ni}\tilde{h}_{ni} - \Big(\sum a_{ni}\tilde{h}_{ni}\Big)_I\Big)^2$$

$$= |I|^{-1}\int_I \Big(\sum_{n=n_0-m_0}^{n_0}\sum_{i=1}^{2^n-1} a_{ni}\tilde{h}_{ni} - \Big(\sum_{n=n_0-m_0}^{n_0}\sum_{i=1}^{2^n-1} a_{ni}\tilde{h}_{ni}\Big)_I\Big)^2$$

$$\leqslant 2|I|^{-1}\int_I \Big(\sum_{n=n_0-m_0}^{n_0}\sum_{i=0}^{2^n-1} a_{ni}\tilde{h}_{ni}\Big)^2$$

$$= 2|I|^{-1}\sum_{n=n_0-m_0}^{n_0}\sum_{i=0}^{2^n-1} a^2_{ni}\int_I \tilde{h}^2_{ni}(t)\,dt$$

$$= 2|I|^{-1}\sum_{n=n_0-m_0}^{n_0}\ \sum_{(n,i)\subset(n_0-m_0,j_0)} a^2_{ni}\,2^{-m_0+n_0-n}|I|$$

$$= 2\cdot 2^{-m_0+n_0}\sum_{n=n_0-m_0}^{n_0}\ \sum_{(n,i)\subset(n_0-m_0,j_0)} a^2_{ni}\,2^{-n}$$

$$\leqslant 2\Big\|\sum a_{ni}h_{ni}\Big\|^2_{\mathrm{BMO}}.$$

To prove the left-hand inequality fix $(m, j)\in Q_N$ such that

$$2^m \sum_{(n,i)\subset(m,j)} a^2_{ni}\,2^{-n} = \Big\|\sum a_{ni}h_{ni}\Big\|_{\mathrm{BMO}}.$$

Choose $i_0 \leqslant 2^N - 1$ so that $(N, i_0)\subset (m, j)$ and $J\in S_{Ni_0}$ so that $J\subset (m, j)$. Observe that then $g^2_N(t) = 2^{m-N}$ for $t\in J$ and $|J|^{-1}\int_J \sum a_{ni}\tilde{h}_{ni} = 0$. Hence

$$|J|^{-1}\int_J \Big(\sum a_{ni}\tilde{h}_{ni} - \Big(\sum a_{ni}h_{ni}\Big)_J\Big)^2 = |J|^{-1}\int_J \Big(\sum a_{ni}\tilde{h}_{ni}\Big)^2$$

$$= |J|^{-1}\sum_{(n,i)\subset(m,j)} a^2_{ni}\,2^{-n}\,2^m|J|$$

$$= \Big\|\sum a_{ni}h_{ni}\Big\|^2_{\mathrm{BMO}}.$$

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Factors of coalescent automorphisms

by

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Abstract. The class of all ergodic coalescent automorphisms is not closed under taking factors, powers and inverse limits. Even if a T-invariant sub-σ-algebra of an ergodic coalescent automorphism T is completely invariant it need not be coalescent. However, if $\mathscr{C}$ is a completely invariant sub-σ-algebra of a simple automorphism T then it is canonical.

I. Introduction. Let T be an ergodic automorphism of a Lebesgue space $(X, \mathscr{B}, \mu)$. The *centralizer* of T, $C(T)$, is the semigroup of all endomorphisms S: $(X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ such that $ST = TS$. T is called *coalescent* if $C(T)$ is a group, or equivalently, if every endomorphism commuting with T is invertible ([8]). Another definition of coalescence is the following (see [5], [9]): if a T-invariant sub-σ-algebra $\mathscr{C} \subset \mathscr{B}$ has the property that T: $(X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ and T: $(X, \mathscr{C}, \mu) \to (X, \mathscr{C}, \mu)$ are isomorphic, i.e. $\mathscr{C}$ is isomorphic to $\mathscr{B}$, then $\mathscr{C} = \mathscr{B}$.

The basic problem connected with coalescence is whether or not it implies zero entropy. Observe that no Bernoulli automorphism is coalescent. Indeed, if τ is a Bernoulli automorphism then represent τ as $\tau_1 \times \tau_1$, where τ_1: $(W, \mathscr{C}, \nu) \to (W, \mathscr{C}, \nu)$ is Bernoulli and

$$h(\tau_1) = h(\tau)/2. \tag{1}$$

Then take the flip map $f(x, y) = (y, x)$ which is in the centralizer of $\tau_1 \times \tau_1$ and take the corresponding sub-σ-algebra $\mathscr{C}_f = \{A \in \mathscr{C} \otimes \mathscr{C} \colon fA = A \text{ a.e.}\}$. Then the factor

$$\tau_1 \times \tau_1 \colon (W \times W, \mathscr{C}_f, \nu \times \nu) \to (W \times W, \mathscr{C}_f, \nu \times \nu) \tag{2}$$

is again Bernoulli with the same entropy as τ. Hence τ and (2) are isomorphic and consequently τ is not coalescent (the original proof of that fact is due to Kamiński [5]).

Therefore to prove that coalescence implies zero entropy it is enough to show that the class of all ergodic coalescent automorphisms is closed under taking factors (then use Sinai's Weak Isomorphism Theorem). That is why the question on factors of coalescent automorphisms stated by Newton in [8] is important. However, in the present paper we provide a counterexample to