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Basic sequences and smooth norms in Banach spaces

by

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Abstract. This paper is concerned with the existence of basic sequences with small basis constants. We also study smoothness properties of norms.

Preliminaries. Let E be a Banach space. The *characteristic* of a subspace X of the dual E^* is the number

$$r(X) = \inf_{x\in E\setminus\{0\}} \sup_{f\in X\setminus\{0\}} \frac{|f(x)|}{\|f\|\,\|x\|}.$$

We refer the reader to [9], [16] for the basic facts on this notion. Obviously $0 \leqslant r(X) \leqslant 1$.

We define

$$\chi(E) = \sup\{r(X):\ X \subsetneqq E^*\}.$$

For a nonreflexive Banach space E, one has $\frac{1}{2} \leqslant \chi(E) \leqslant 1$. We notice that if E is a Banach space with a basis (x_n) and X is the subspace of E^* spanned by the coefficient functionals then we have $r(X) \geqslant K(x_n)^{-1} > 0$ (where $K(x_n)$ is the basis constant of (x_n)).

We get the following

LEMMA 0.1. *If (x_n) is a basis of E with basis constant strictly less than $\chi(E)^{-1}$ then (x_n) is shrinking.*

We will show that a kind of converse of this lemma is valid: namely, if E is any Banach space, there exists, for any $\varepsilon > 0$, a nonshrinking basic sequence (x_n) in E with $K(x_n) \leqslant \chi(E)^{-1} + \varepsilon$. In this direction, A. Pełczyński has shown the existence of a nonshrinking basic sequence in a nonreflexive Banach space [14]. We also mention the result of M. Zippin: if E is a Banach space with a basis, then E is reflexive if and only if each basis of E is shrinking [17]. In both results, there are no estimates of the basis constants. We end the first chapter by some applications, in particular we get an extension of a result of [7].

In the second chapter, we introduce a class of Banach spaces satisfying smoothness conditions and we get $\chi(E) < 1$ for any Banach space E in this class (Theorem 2.3). As a consequence, if X and Y are superreflexive Banach spaces then $\chi(K(X, Y))$ is stricly less than 1, where $K(X, Y)$ is the space of compact operators from X to Y. We also present some questions and in particular a renorming problem.

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Notation. If E is a Banach space, the closed unit ball of E is denoted by $B(E)$, the unit sphere by $S(E)$. The dual of E is E^*. The basis constant of a basic sequence (x_n) is denoted by $K(x_n)$. For a set A, $\overline{\operatorname{conv}}(A)$ is the norm-closed convex hull of A and $[A]$ is the span of A. $K(X, Y)$ denotes the space of compact operators from X to Y (with the operator norm). For the definitions and basic facts on basic sequences the reader should consult [1].

1. Construction of basic sequences with a given constant. The main result of this chapter is the following:

THEOREM 1.1. *Let E be a nonreflexive Banach space. For every $\varepsilon > 0$, there exists a nonshrinking basic sequence (x_n) such that $K(x_n) \leqslant \chi(E)^{-1}+\varepsilon$ and there exists a nonboundedly complete basic sequence (y_n) such that $K(y_n) \leqslant 1+\varepsilon$.*

Let us first note that there is no restriction on $\chi(E)$. Hence any nonreflexive Banach space contains a nonshrinking basic sequence (x_n) with $K(x_n) \leqslant 2+\varepsilon$; any nonreflexive dual space contains a nonshrinking basic sequence (y_n) with $K(y_n) \leqslant 1+\varepsilon$. The proof of the theorem does not use the methods of A. Pełczyński [14] or M. Zippin [17] but the local reflexivity principle [12].

Proof of the theorem. Observe first that $\chi(E)$ may also be computed by the following formula:

$$\chi(E)^{-1} = \inf_{x^{**}\in E^{**}\setminus E} \{\|\pi\|:\ \pi \text{ is the projection of } E\oplus \boldsymbol{R}x^{**} \text{ onto } E \text{ with kernel } \boldsymbol{R}x^{**}\}.$$

Let ε be any positive number. There is $x_1^{**}\in E^{**}\setminus E$, $\|x_1^{**}\| = 1$ and $\|\pi\| \leqslant \chi(E)^{-1}(1+\varepsilon)$, where π is the projection of $E\oplus \boldsymbol{R}x_1^{**}$ onto E with kernel $\boldsymbol{R}x_1^{**}$. We will construct a sequence (x_n) in E such that for any increasing sequence of integers (n_i):

(a) The sequence $(x_1^{**}, x_1^{**}-x_{n_i})$ is basic with basis constant near to 1.

(b) (x_{n_i}) is a nonshrinking basic sequence with basis constant near to $\chi(E)^{-1}$.

(c) $(x_{n_i}-x_{n_{i-1}})$ is a basic sequence with basis constant near to 1.

Choose $(\varepsilon_n)_{n\geqslant 0}$ so that $0<\varepsilon_n<1$ for all $n\geqslant 0$ and $\prod_{i=n}^{n+p-1}(1-\varepsilon_{l_i}) \geqslant 1-\varepsilon_0$ for any increasing sequence of integers (l_i). There is $f_1\in S(E^*)$ such that $x_1^{**}(f_1) \geqslant 1-\varepsilon_1$. Let F_1 denote the 1-dimensional subspace of E^{**} spanned by x_1^{**}. By the local reflexivity principle [12], there is a one-to-one operator T_1: $F_1 \to E$ so that:

1. $\|T_1\|\,\|T_1^{-1}\| \leqslant 1+\varepsilon_0$.
2. $f_1(T_1 x_1^{**}) = x_1^{**}(f_1)$.

We put $x_1 = T_1 x_1^{**}$.

Let F_2 denote the 2-dimensional subspace of E^{**} spanned by x_1^{**} and $x_1^{**}-x_1$. There is a set $Z_2 = \{e_1, \ldots, e_{N(2)}\} \subset S(F_2)$ which forms an $\varepsilon_2/2$-net for $S(F_2)$. Pick $e_1^*, \ldots, e_{N(2)}^*\in S(E^*)$ so that $e_i^*(e_i) > 1-\varepsilon_2/2$, and put

$$Z_2^* = \{e_1^*, \ldots, e_{N(2)}^*\}.$$

By the local reflexivity principle, there is a one-to-one operator T_2: $F_2\to E$ such that:

1. $\|T_2\|\,\|T_2^{-1}\| \leqslant 1+\varepsilon_0$.
2. $f_1(T_2 x^{**}) = x^{**}(f_1)$, $\quad\forall x^{**}\in F_2$.
3. $e_i^*(T_2 x_1^{**}) = x_1^{**}(e_i^*)$, $\quad\forall e_i^*\in Z_2^*$.
4. $T_2 x_1 = x_1$.

We put $x_2 = T_2 x_1^{**}$.

Notice that for any scalar a, we get

$$\|x_1^{**}+a(x_1^{**}-x_1)\| \geqslant 1-\varepsilon_1.$$

Indeed,

$$\|x_1^{**}+a(x_1^{**}-x_1)\| \geqslant |f_1(x_1^{**}+a(x_1^{**}-x_1))| \geqslant |f_1(x_1^{**})|-|a|\,|f_1(x_1^{**}-x_1)| \geqslant 1-\varepsilon_1.$$

We repeat the above procedure. Inductively, we find for all $n\geqslant 2$

$$F_n = [x_1^{**}, x_1^{**}-x_1, \ldots, x_1^{**}-x_{n-1}],$$

$$Z_n = \{e_1, \ldots, e_{N(n)}\} \quad \text{an } \varepsilon_n/2\text{-net for } S(F_n),$$

$$Z_n^* = \{e_1^*, \ldots, e_{N(n)}^*\} \subset S(E^*) \quad \text{such that} \quad e_i^*(e_i) > 1-\varepsilon_n/2,$$

and a 1-1 operator T_n: $F_n\to E$ with:

1. $\|T_n\|\,\|T_n^{-1}\| \leqslant 1+\varepsilon_0$.
2. $f_1(T_n x^{**}) = x^{**}(f_1)$, $\quad\forall x^{**}\in F_n$.
3. $e_i^*(T_n x_1^{**}) = x_1^{**}(e_i^*)$, $\quad\forall e_i^*\in \bigcup_{2\leqslant i\leqslant n} Z_i^*$.
4. $T_n(e) = e$, $\quad\forall e\in F_n\cap E$.

We put $x_n = T_n x_1^{**}$.

(a) The same computation as before gives $\|x_1^{**}+a(x_1^{**}-x_n)\| \geqslant 1-\varepsilon_1$, for all $n\geqslant 1$.

We now claim that for any $y\in S(F_n)$, $n\geqslant 2$, any scalar a, and any $p\geqslant 0$,

$$\|y+a(x_1^{**}-x_{n+p})\|\geqslant 1-\varepsilon_n.$$

Indeed, there exists $e_i\in Z_n$ such that $\|y-e_i\|\leqslant \varepsilon_n/2$ and $e_i^*(e_i)>1-\varepsilon_n/2$. We get

$$\begin{aligned}\|y+a(x_1^{**}-x_{n+p})\| &\geqslant |e_i^*(y+a(x_1^{**}-x_{n+p}))|\\ &\geqslant |e_i^*(e_i)|-|e_i^*(e_i-y)|-|a|\,|e_i^*(x_1^{**}-x_{n+p})|\\ &\geqslant 1-\varepsilon_n/2-\|e_i-y\|\geqslant 1-\varepsilon_n.\end{aligned}$$

For any scalars $a_1,\ldots,a_{n+p}$ and for any increasing sequence of integers (l_i), we get

$$\begin{aligned}\Big\|\sum_{i=1}^{n+p}a_i(x_1^{**}-x_{l_i})\Big\| &= \Big\|\sum_{i=1}^{n+p-1}a_i(x_1^{**}-x_{l_i})+a_{n+p}(x_1^{**}-x_{l_{n+p}})\Big\|\\ &\geqslant (1-\varepsilon_{l_{n+p-1}})\Big\|\sum_{i=1}^{n+p-1}a_i(x_1^{**}-x_{l_i})\Big\|\\ &\geqslant \prod_{i=n}^{n+p-1}(1-\varepsilon_{l_i})\Big\|\sum_{i=1}^{n}a_i(x_1^{**}-x_{l_i})\Big\|\\ &\geqslant (1-\varepsilon_0)\Big\|\sum_{i=1}^{n}a_i(x_1^{**}-x_{l_i})\Big\|.\end{aligned}$$

This concludes the proof of (a).

(b) Let (l_i) be any increasing sequence of integers and let $a_1,\ldots,a_n$ be any scalars. Then

$$\begin{aligned}\Big\|\sum_{i=1}^{n}a_i x_{l_i}\Big\| &= \Big\|\pi\Big(\sum_{i=1}^{n}a_i x_{l_i}+\sum_{i=n+1}^{n+p}a_i x_1^{**}\Big)\Big\|\\ &\leqslant \|\pi\|\,\Big\|\sum_{i=1}^{n}a_i x_{l_i}+\sum_{i=n+1}^{n+p}a_i x_1^{**}\Big\|\\ &\leqslant \chi(E)^{-1}(1+\varepsilon)\Big\|\Big(\sum_{i=1}^{n+p}a_i\Big)x_1^{**}+\sum_{i=1}^{n}a_i(x_{l_i}-x_1^{**})\Big\|\\ &\leqslant \chi(E)^{-1}(1+\varepsilon)(1-\varepsilon_0)^{-1}\Big\|\Big(\sum_{i=1}^{n+p}a_i\Big)x_1^{**}+\sum_{i=1}^{n+p}a_i(x_{l_i}-x_1^{**})\Big\|\\ &\leqslant \chi(E)^{-1}(1+\varepsilon)(1-\varepsilon_0)^{-1}\Big\|\sum_{i=1}^{n+p}a_i x_{l_i}\Big\|.\end{aligned}$$

This proves that the sequence (x_{l_i}) is basic with

$$K(x_i)\leqslant (1+\varepsilon)(1-\varepsilon_0)^{-1}\chi(E)^{-1}.$$

Moreover, since for all n, $f_1(x_n)=x_1^{**}(f_1)\geqslant 1-\varepsilon_1$, the sequence (x_{l_i}) is nonshrinking.

Hence, in particular, we have proved that there exists a nonshrinking basic sequence (x_n) with basis constant near to $\chi(E)^{-1}$.

(c) We will suppose $\|T_n\|\leqslant 1$ and $\|T_n^{-1}\|\leqslant 1+\varepsilon_0$. Then

$$\begin{aligned}\Big\|\sum_{i=1}^{n}a_i(x_{l_i}-x_{l_{i+1}})\Big\| &= \|a_1(x_{l_1}-x_{l_2})+\ldots+a_n(x_{l_n}-x_{l_{n+1}})\|\\ &= \|T_{l_{n+1}}[a_1(x_{l_1}-x_{l_2})+\ldots+a_n(x_{l_n}-x_1^{**})]\|\\ &\leqslant \|a_1(x_{l_1}-x_{l_2})+\ldots+a_n(x_{l_n}-x_1^{**})\|.\end{aligned}$$

Since $x_{l_{n-1}}-x_{l_n}=(x_{l_{n-1}}-x_1^{**})-(x_{l_n}-x_1^{**})$ and $(x_{l_i}-x_1^{**})$ is a basic sequence, we have

$$\begin{aligned}&\Big\|\sum_{i=1}^{n}a_i(x_{l_i}-x_{l_{i+1}})\Big\|\\ &\leqslant (1-\varepsilon_0)^{-1}\|a_1(x_{l_1}-x_{l_2})+\ldots+a_{n-1}(x_{l_{n-1}}-x_{l_n})+a_n(x_{l_n}-x_1^{**})\\ &\quad +(a_{n+1}-a_n)(x_{l_{n+1}}-x_1^{**})+(a_{n+2}-a_{n+1})(x_{l_{n+2}}-x_1^{**})+\ldots\\ &\quad \ldots+(a_{n+p}-a_{n+p-1})(x_{l_{n+p}}-x_1^{**})\|\\ &\leqslant (1-\varepsilon_0)^{-1}\|a_1(x_{l_1}-x_{l_2})+\ldots+a_n(x_{l_n}-x_{l_{n+1}})+\ldots+a_{n+p}(x_{l_{n+p}}-x_1^{**})\|\\ &\leqslant (1-\varepsilon_0)^{-1}\|T_{l_{n+p+1}}^{-1}\|\,\|T_{l_{n+p+1}}[a_1(x_{l_1}-x_{l_2})+\ldots+a_{n+p}(x_{l_{n+p}}-x_1^{**})]\|\\ &\leqslant (1-\varepsilon_0)^{-1}(1+\varepsilon_0)\Big\|\sum_{i=1}^{n+p}a_i(x_{l_i}-x_{l_{i+1}})\Big\|,\end{aligned}$$

and so $(x_{l_i}-x_{l_{i+1}})$ is a basic sequence with basis constant near to 1.

We are ready to finish the proof of the second assertion of the theorem. We first assume that E contains an isomorphic copy of l_1. Then E contains almost isometric copies of l_1 [11]. Let (e_i) be the usual basis of l_1. Consider the sequence $(x_n)=(e_1,e_2-e_1,\ldots,e_n-e_{n-1},\ldots)$. Then (x_n) is a non-boundedly complete basis of l_1 with basis constant equal to 1.

Suppose now that E does not contain a copy of l_1. There is no loss of generality in assuming that E is separable. Then our sequence (x_n) has a subsequence (x_{n_i}) which is $\sigma(E^{**},E^*)$-convergent. But (x_{n_i}) does not converge to a point of E, since 0 is the only possible limit point of a basic sequence, but for all n, $f_1(x_n)\geqslant 1-\varepsilon_1$.

We have

$$\sum_{i=1}^{p}(x_{n_i}-x_{n_{i-1}})=-x_{n_0}+x_{n_p}.$$

Therefore $\sup_p \|\sum_{i=1}^{p}(x_{n_i}-x_{n_{i-1}})\|$ is finite, but the series $\sum_{i=1}^{\infty}(x_{n_i}-x_{n_{i-1}})$ does not converge in E. Moreover, $(x_{n_i}-x_{n_{i-1}})$ is basic with basis constant near to 1. ■

Let us mention some applications.

We define $\chi'(E)$ for a Banach space E by

$$\chi'(E) = \inf\{\chi(F)^{-1}:\ F \subseteq E\}.$$

$\chi'(E)$ can be considered as the "shrinking basic sequences index" because of the following corollary:

Corollary 1.2. *Let E be a Banach space. Then*

$$\chi'(E) = \sup\{\lambda:\ \text{every basic sequence } (x_n) \text{ with } K(x_n) < \lambda \text{ is shrinking}\}.$$

Corollary 1.3. *If E is a separable Banach space with $\chi'(E) > 1$, then E^* is separable.*

Proof. Since E is separable, there exists a $\sigma(E^*, E)$-dense sequence (x_n) in $B(E^*)$. So $r[x_n] = 1$ and $[x_n] = E^*$. ■

The above corollary should be compared with the following result: if E is a separable space such that $[e_n]^*$ is separable for every basic sequence (e_n) then E^* is separable [10].

Remark. Let E be a Banach space which is an M-ideal of its bidual. Then $\chi(E) = 1/2$ [7]. This class is stable under subspaces. Hence $\chi'(E) = 2$.

We now give an isomorphic version of the following result of G. Godefroy and P. Saphar [7]. Let E be a Banach space such that E^* contains no proper 1-norming subspace and let (T_n) be a sequence of contractions on E such that for every $x \in E$, $\lim_n \|T_n x - x\| = 0$. Then we have $\lim_n \|T_n^* x^* - x^*\| = 0$ for every $x^* \in E^*$.

We will replace the metric assumption (the T_n are contractions) by an algebraic one (the T_n commute).

Our next result is

Corollary 1.4. *Let E be a separable Banach space such that ($\exists \alpha > 1$, $\forall (x_n)$ basic sequence: $K(x_n) < \alpha \Rightarrow (x_n)$ shrinking). Then $\forall X \subseteq E$, $\forall T_n: X \to X$ with*

$$(*) \qquad \begin{gathered} \operatorname{rank}(T_n) < \infty, \qquad \sup_n \|T_n\| < \alpha, \\ T_n T_k = T_k T_n, \quad \forall k, n, \quad \|T_n x - x\| \to 0, \quad \forall x \in X, \end{gathered}$$

we have $\|T_n^* x^* - x^*\| \to 0$, $\forall x^* \in X^*$.

Proof. Let $(T_n)_{n\geq 1}$ be a sequence of operators on X satisfying $(*)$. Then $\lim_n \|T_n^* x^* - x^*\| = 0$ for every x^* in the norm-closed subspace Γ of X^* generated by $\bigcup_{n=1}^{\infty} T_n^*(X^*)$ [16]. But it is easy to check that $r(\Gamma) \geqslant (\sup\|T_n\|)^{-1}$ and thus $r(\Gamma) > \alpha^{-1}$, and this implies that $\Gamma = X^*$. ■

Let us note that under the same assumptions on E, the projections associated with a basic sequence satisfy the conclusion of Corollary 1.4. We get here the result ($\lim_n \|T_n^* x^* - x^*\| = 0$ for every $x^* \in X^*$) for *any* sequence T_n satisfying $(*)$.

2. A smoothness condition. Examples. Let us introduce a condition of uniform smoothness which is close to the one given in [4], [5]. We use the same terminology. Let E be a Banach space. $\mathscr{D}(E)$ is the set of points of the unit sphere where the norm is Fréchet-smooth; for every $x \in \mathscr{D}(E)$, we denote by f_x the differential of this norm at x.

Definition 2.1. We say that E is *almost uniformly smooth* (a.u.s.) if there exists a family $(A_\varepsilon)_{0<\varepsilon<1}$ of subsets of $\mathscr{D}(E)$ such that:

(a) $\forall \varepsilon \in]0, 1[$, $\exists \delta(\varepsilon) > 0$:

$$y \in B(E^*),\ x \in A_\varepsilon,\ y(x) \geqslant 1-\delta(\varepsilon) \Rightarrow \|y - f_x\| \leqslant \varepsilon.$$

(b) $\forall \varepsilon$: $B(E^*) = \overline{\operatorname{conv}}\{f_x:\ x \in A_\varepsilon\} + \varepsilon B(E^*)$.

Examples. $c_0(\Gamma)$ is a.u.s. for any Γ. Every superreflexive Banach space is a.u.s. for every equivalent norm [5]. A nontrivial class of new examples is provided by our next result:

Proposition 2.2. *If X, Y are superreflexive Banach spaces then $K(X, Y)$ is a.u.s.*

Proof. The proof follows the method of [4], [7]; we only give the main ideas. The following facts are known. Let $x \in S(X)$ (resp. $y \in S(Y)$) and assume that x (resp. y) is strongly exposed in $B(X)$ (resp. $B(Y)$) by $f_x \in X^*$ (resp. $f_y \in Y^*$). Then $x \otimes y$ is strongly exposed in $B(X \otimes Y)$ by $f_x \otimes f_y$ [7]. Moreover, if there is a uniformity in the strong exposition of x and y then there is also a uniformity in the strong exposition of $x \otimes y$ [4]. Since a superreflexive Banach space is a.u.s., the unit ball of X and the unit ball of Y^* are of type (b) from Definition 2.1. Using the technique of [7], we conclude that the unit ball of $X^* \otimes_\varepsilon Y$ is also of type (b). ■

Of course, the proposition is not true in general for reflexive Banach spaces. If $X = \bigoplus_{l_2} l_\infty^n$ and $Y = \boldsymbol{R}$ then $K(X, Y) = \bigoplus_{l_2} l_1^n$ and $K(X, Y)$ is not a.u.s. [4]. J. Partington has shown that every Banach space may be equivalently renormed to have property β ([13], see also [15]). For a reflexive Banach space it is easy to see that property β implies almost uniform smoothness. Therefore if X and Y are reflexive Banach spaces then $K(X, Y)$ admits an equivalent almost uniformly smooth norm.

We use the technique of [4] in the proof of the main result of this chapter.

THEOREM 2.3. *If E is a.u.s. then $\chi(E)$ is strictly less than 1.*

Proof. Suppose $\chi(E)=1$. There are a sequence (y_n) in $S(E^{**}\setminus E)$ and a sequence of projections (π_n) such that π_n is the projection of $E\oplus \mathbf{R}y_n$ onto E with kernel $\mathbf{R}y_n$ and $1\leqslant \|\pi_n\|\leqslant 1+1/n^2$. Therefore

$$(1)\qquad \|e\|\leqslant (1+1/n^2)\|e+y_n\| \quad \text{for any } e\in E.$$

By using a technique similar to that in [4] we show that $\ker y_n\cap B(E^*)$ is "almost" $\sigma(E^*, E)$-dense in $B(E^*)$. More precisely, there is an integer N such that for every $n\geqslant N$ and every $h\in S(E)$, one has

$$\ker y_n\cap B(E^*)\not\subset \{x\in B(E^*)\colon |h(x)|\leqslant 1-1/n\}.$$

The idea is the following. Suppose this is not true. Let N be an integer such that there are $n\geqslant N$ and $h\in S(E)$ satisfying

$$\ker y_n\cap B(E^*)\subset \{x\in B(E^*)\colon |h(x)|\leqslant 1-1/n\}.$$

Then for $u=(n+1/n)h\in E$ it can be shown as in [4] that $\|u\|>(1+1/n^2)\|u-y_n\|$, a contradiction with (1).

Therefore

$$\exists N,\ \forall n\geqslant N,\ \forall x\in A_\varepsilon,\ \exists y_{(x)}^{(n)}\in\ker y_n\cap B(E^*)\colon\quad y_{(x)}^{(n)}(x)>1-1/n.$$

Let $M\geqslant \max\{N, 1/\delta(\varepsilon)\}$; then for $n\geqslant M$ we get $\|y_{(x)}^{(n)}-f_x\|\leqslant\varepsilon$. Now,

$$|y_n(f_x)|\leqslant |y_n(f_x-y_{(x)}^{(n)})|+|y_n(y_{(x)}^{(n)})|.$$

Since $y_{(x)}^{(n)}\in\ker y_n$, one has

$$\forall\varepsilon>0, \exists M,\ \forall n\geqslant M\colon\quad \sup_{x\in A_\varepsilon}|y_n(f_x)|\leqslant\varepsilon.$$

By Definition 2.1(b), this implies $\|y_n\|\to 0$ as $n\to\infty$, a contradiction. ∎

COROLLARY 2.4. *If X and Y are superreflexive Banach spaces then $\chi(K(X, Y))$ is strictly less than 1.*

We now mention two questions.

QUESTION 2.5. Let X, Y be superreflexive Banach spaces. Does there exist $\alpha>1$ such that if (x_n) is a basic sequence in $K(X, Y)$ with $K(x_n)<\alpha$ then (x_n) is shrinking? Of course, Theorem 2.3 does not imply the existence of α since the property $\chi(E)<1$ is not hereditary (L. V. Gladun and A. M. Plichko have constructed [6] an example of a Banach space X which contains a norm-one complemented hyperplane Y with $\chi(Y)=1$, $\chi(X)\leqslant 2/3$).

The second question concerns a renorming problem:

QUESTION 2.6. Is it possible to renorm a Banach space E with separable dual so that $\chi(E)<1$? (see [2]). The answer is affirmative whenever E is a quasi-reflexive Banach space: if E is a quasi-reflexive Banach space then there exists on E an equivalent norm such that $\chi(E)=1/2$ [8].

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