On the differentiation of integrals of functions from $L_p(L)$

by

A. M. Stokolos (Odessa)

Abstract. The following alternative is established for any translation invariant differentiation basis $B$ of rectangles with sides parallel to the coordinate axes: either $B$ differentiates the integral of every summable function, or for every class $L_p(L)$ with $\phi(t) = o(|t|)$ as $t \to \infty$ there is a function whose integral is not differentiated by $B$. A geometric characteristic is introduced which permits to decide which class, $L_p$ or $L_{\log^+} L$, is precisely differentiated by a given basis. Also, a scale of non-translation invariant bases of rectangles with sides parallel to the axes is constructed which differentiate precisely the classes $L_p(L)$ intermediate between $L$ and $L_{\log^+} L$. The results obtained, together with the theorems of Lebesgue and Jensen, Marcinkiewicz and Zygmund, yield a complete description of the behaviour of differentiation of rectangles with sides parallel to the axes. Applications to the theory of multiple Fourier series and extensions from $R^2$ to the multidimensional case are also given.

1. Introduction. A differentiation basis at a point $x \in R^d$ is a collection $B(x)$ of bounded open subsets of $R^d$ containing $x$ such that there is a sequence $(R_k) \subset B(x)$ with $\text{diam } R_k \to 0$ as $k \to \infty$. The family $B = \{R_k \in B(x) \mid x \in R_k\}$ is then called a differentiation basis in $R^d$. A differentiation basis is called translation invariant (briefly: a $T^d$-basis) if it contains all translates of any of its elements.

If a basis $B$ has the property that for each $R_k \in B(x)$ then $R_{k+1} \subseteq R_k$, $R_k$ is then called a Busemann–Feller basis (a BF-basis).

We define the upper and lower derivatives of the integral of a locally integrable function $f$ at a point $x$ with respect to a basis $B$ by

$\tilde{D}^+_B(f, x) = \limsup_{R_k \to 0} \sup_{x \in R_k} \int_{R_k} f(y) dy$

and

$\tilde{D}^-_B(f, x) = \liminf_{R_k \to 0} \inf_{x \in R_k} \int_{R_k} f(y) dy$

We say that a basis $B$ differentiates the integral of $f$ if $\tilde{D}^+_B(f, x) = \tilde{D}^-_B(f, x) = f(x)$ a.e. If $B$ differentiates the integral of every function $f \in \Phi(L)$ (for the definition of the classes $\Phi(L)$ see e.g. [14, p. 650]) then we say that $B$ differentiates $\Phi(L)$; if for every function $g$ on $R^d$ with $g(0) \not= 0$ as $t \to \infty$ there exists an $f \in \Phi(L)$ such that $\tilde{D}^+_B(f, x) = +\infty$ a.e., then we say that $B$ does not differentiate $\Phi(L)$. Finally, $B$ differentiates precisely $\Phi(L)$.
main results in the case of \( R^2 \), which, for the most cases, is typical. \( N \)-dimensional versions are considered in Section 4.

First, we introduce a geometric characteristic of a basis consisting of rectangles. Let \( B \subset B_1 \). For every rectangle \( R \in B \) denote by \( R^* \) the concentric rectangle of minimal measure containing \( R \) with side-lengths of the form \( 2^k, k \in \mathbb{Z} \). Thus to every basis \( B \) we attach, in a natural way, another basis \( B^* = \{ R^* : R \in B \} \), called the basis associated to \( B \).

Further, we will say that two rectangles \( R \) and \( R' \) are \emph{comparable}, and write \( R \prec R' \), if there is a translation placing one of them inside the other. Otherwise we call them \emph{incomparable} and write \( R \not\prec R' \).

We say that a basis \( B \) has property (S) if

\[
(S) \quad \forall \varepsilon > 0 \quad \forall k \in \mathbb{N} \exists \{ R_i \}_{i=1}^k \subset B^* : \quad R_i \not\prec R_j \quad (i \neq j),
\]

\[
\text{diam} R_i < \varepsilon, \quad i = 1, \ldots, k;
\]

i.e. we can find an arbitrary number of arbitrarily small pairwise incomparable rectangles in the associated basis.

Property (S) permits us to formulate a criterion to decide which of the classes \( L \) or \( L \log^+ L \) is precisely differentiated by a given TI-basis:

**Theorem 1.** Let \( B \) be a TI-basis with \( B \subset B_1 \). Then if \( B \) has property (S) then it does not differentiate \( o(L \log^+ L) \), and if \( B \) fails property (S) then it differentiates \( L \).

In connection with this theorem, the problem arises whether there exist at all bases of rectangles with sides parallel to the coordinate axes which differentiate precisely a given class \( L \log^+ L \) intermediate between \( L \) and \( L \log^+ L \). An answer is given by the following theorem.

**Theorem 2.** Let \( \varphi(t) \) be an increasing concave function with \( \varphi(0) = 0 \) and such that \( \varphi(t)/\ln t \) is decreasing for \( t \geq t_0 > 1 \). Then there is a basis \( B \) with \( B \subset B_1 \) such that \( B \in D(L \log^+ L \varphi(t)) \).

\section*{3. Proofs of the main results.} The main tool in the proof of Theorem 1 is the following lemma.

**Lemma 1.** Suppose a basis \( B \) has property (S). Then for arbitrary \( \varepsilon > 0 \) and \( k \in \mathbb{N} \) there are sets \( \Theta = \Theta(\varepsilon, k) \) and \( Y = Y(\varepsilon, k) \) such that

\[
(1) \quad \Theta \subset Y, \quad \text{diam} Y < \varepsilon,
\]

\[
(2) \quad |Y| \geq k2^{-k-1}|\Theta|,
\]

\[
(3) \quad \forall x \in Y \exists R \in B^*(x) : \quad \text{diam} R < \varepsilon, \quad |R \cap \Theta|/|R| \geq 2^{-k}.\]

**Proof.** By property (S) we can find \( k+1 \) pairwise incomparable rectangles \( \{ R_i \}_{i=0}^k \) in \( B^* \) of diameter less than \( \varepsilon \). Let \( |R_0| = 2^{-m}, |R_1| = 2^{-m} \),...
where \( r_n(t) \) are the Rademacher functions.

As can easily be seen, \( \Theta \) is a union of \( 2^{-2k} 2^{-n_0} 2^{-n_1} \) disjoint rectangles whose projections are dyadic-rational intervals of lengths \( 2^{-n_0} \) and \( 2^{-n_0} \) respectively, and \( Y_k \) is a union of \( 2^{-k} 2^{-n_0} 2^{-n_1} \) disjoint rectangles whose projections are dyadic-rational intervals of lengths \( 2^{-n_0} \) and \( 2^{-n_0} \) respectively. Hence \(|\Theta| = 2^{-2k} |J|, |Y_k| = 2^{-k} |J|, |J| = 2^{-n_0} 2^{-n_0} \).

Further, it is easily seen that

\[
|Y \cap \bigcup_{p=0}^{k-1} Y_p| \leq \frac{1}{2} |Y|, \quad v = 2, \ldots, k.
\]

Therefore

\[
|Y| \geq \frac{1}{2} \sum_{v=0}^{k-1} |Y_v| = \frac{1}{2} (k+1) 2^{-k} |J| = \frac{1}{2} (k+1) 2^{k} |\Theta|,
\]

and so (1) and (2) are proved.

Let now \( v \) be any integer between 0 and \( k \), and let \( R \) be any of the rectangles that form \( Y_v \). It follows from the definitions of \( \Theta \) and \( Y_v \) that \( \Theta \subset Y_v \), \( R \subset \Theta \), and

\[
|\Theta \cap Y_v| = \frac{1}{2} |\Theta|, \quad |Y_v| = \frac{1}{2^k} |J|,
\]

which implies (3) and completes the proof of the lemma.

Remark 1. Write

\[
M^k f(x) = \sup_{R: \mu(R) \leq k} \int_R |f(y)| dy
\]

for the truncated strong maximal operator. Then Lemma 1 essentially means that for arbitrary \( \epsilon > 0 \) and \( k \in \mathbb{N} \) there are sets \( \Theta \) and \( Y \) such that

\[
Y \subset \{ x: M^k \chi_0(x) > 2^{-k} \} \quad \text{and so}
\]

\[
|\{ x: M^k \chi_0(x) > 2^{-k} \}| \approx c \int \frac{\chi_0(x)}{2^{-k} \log + \frac{x}{2^{-k}}} dx,
\]

with \( c \) independent of \( \epsilon, k, \Theta \). This inequality is a converse of the well-known weak type estimate for the strong maximal operator. Such inequalities are of great importance in differentiation theory and constitute the main tool both for proving positive results and for constructing counterexamples. For more details, see [1]-[3], [10].
Putting \( \Phi(t) = t \log t \) for \( t \geq 1 \) we obtain by (5)
\[
\int_{[0,1]^2} \Phi(f(x)) \, dx \leq \sum_{k=1}^\infty \int_{(0,1)^2} \Phi(f_k(x)) \, dx = \sum_{k=1}^\infty \Phi(a_j \cdot 2^k) m_j < \infty,
\]
and so \( f \in L^2(L \cdot \log^+ L)([0,1]^2) \).

On the other hand, by (6) almost every point \( x \) belongs to an infinite sequence of sets \( \tau_j G_0 \). Since \( \tau_j G_0 \) and \( \tau_j E_0 \) satisfy a relation of the type (3), for any \( x \in \tau_j G_0 \) with \( N_j \leq k < N_j \) we can find a rectangle \( R \in B^*(x) \) such that
\[
|R|^{-1} \int_R f(y) \, dy \geq |R|^{-1} \int_R f_k(y) \, dy = \frac{a_j \cdot 2^k |R \cap \tau_j E_0|}{|R|} \geq a_j.
\]

Since \( a_j \uparrow \infty \) as \( j \to \infty \), \( k \to \infty \), we conclude that \( B_B(f, x) = +\infty \) a.e. on \([0,1]^2\), and the obvious relation \( B_B(f, x) \geq \frac{1}{2} B_B(f, x) \) completes the proof of the first part of Theorem 1.

In the proof of the second part we will use the following known facts. Let \( B' \) be a TI-basis generated by translates of rectangles from a monotonic family \( \{R_i\}_{i=1}^\infty \), \( R \subset R_k \) for \( \alpha > \beta \). Then \( B' \) differentiates \( L \), and almost every point is a Lebesgue point with respect to \( B' \). Consequently, the integral of any summable function is differentiated at almost every point by any basis \( B' \) regular with respect to \( B' \) (see [11, Ch. 1, 1.53(d) and 1.53]). But it is not hard to see that if a basis \( B \) fails property (S), then \( B \) decomposes into a finite number of bases generated by monotonic families of rectangles, and into a collection of rectangles with diameters greater than some \( \varepsilon_0 > 0 \). Hence \( B' \) differentiates \( L \), and since \( B \) is regular with respect to \( B' \) (see [11, Ch. 1.53(d)]), it follows that \( B \) also differentiates \( L \) and the proof of Theorem 1 is complete.

We now turn to the proof of Theorem 2. As is known, the differentiation properties of a basis are closely related to its covering properties. In this connection we introduce the following property \( (V_\Psi) \) of weak overlapping, where \( \Psi \) is an increasing function with \( \Psi(0) = 0 \). We say that a basis \( B \) has property \( (V_\Psi) \) if there are constants \( c_j > 0 \) (\( j = 1, 2, 3 \)), \( n_0 \in N \), \( m_0 \in N \) such that for any system \( \{R_i\}_{i=4}^\infty \subset B \) we can find a subsystem \( \{R_i\}_{i=4}^\infty \subset B \) such that for any system \( \{R_i\}_{i=4}^\infty \subset B \) we can find a subsystem
\[
\int \Psi(c_1 \sum_{i=4}^\infty \chi_{R_i}(x) - n_0) \, dx \leq c_2 \sum_{i=4}^\infty |R_i|,
\]
where \( W = \{x : \sum_{i=4}^\infty \chi_{R_i}(x) \geq m_0\} \).


(8)
\[
\bigcup_{i=4}^\infty R_i \leq c_3 \sum_{i=4}^\infty |R_i|,
\]

(for the particular case of property \( (V_\varphi) \), see [4, App. II]). Without loss of generality we can assume that \( c_3 > 1 \).

We define the maximal operator corresponding to the basis \( B \) by
\[
M_B f(x) = \sup_{R \in \Phi(L)} |R|^{-1} \int_R |f(y)| \, dy.
\]

**Lemma 2.** Let \( \Phi(t) \), \( \Psi(t) \) be Young conjugate convex functions with \( \Phi(t) \) satisfying the \( \Delta_2 \) condition. Assume that a basis \( B \) has property \( (V_\varphi) \). Then the maximal operator \( M_B \) is of weak type \( L + \Phi(L) \).

\[
|\{x : M_B f(x) > \lambda\}| \leq c_4 \int_{\{M_B \leq \lambda\}} \left( \frac{|f|}{\lambda} + \Phi \left( \frac{|f|}{\lambda} \right) \right) \, dy
\]

for all \( f \in L \cap \Phi(L) \) and \( \lambda > 0 \).

**Proof.** Let
\[
\{x : M_B f(x) > \lambda\} = \bigcup_{k} R_k, \quad R_k \in B, \quad |R_k|^{-1} \int_R |f(y)| \, dy > \lambda.
\]

Take \( \{R_k\} \) satisfying (7), (8). Define
\[
U = \bigcup_{k} R_k, \quad W = \{x : \sum_{i=4}^\infty \chi_{R_i}(x) \geq m_0\}.
\]

Then
\[
\sum_{i=4}^\infty |R_i| \leq \sum_{i=4}^\infty \left( \frac{\int f(y) \, dy}{\lambda} \right) = \sum_{i=4}^\infty \left( \frac{\int \chi_{R_i} f(y) \, dy}{\lambda} \right)
\]

\[
\leq \int_{W} \frac{c_1}{2c^2} \left( \sum_{i=4}^\infty \chi_{R_i}(x) - n_0 \right) \frac{2c_2}{c_1} \left( \frac{|f|}{\lambda} \right) \, dy + m_0 \int_{W} \frac{|f|}{\lambda} \, dy
\]

\[
eq J_1 + J_2 \quad (2c_2 > 1).
\]

By the Young inequality
\[
J_1 \leq \int \Psi \left( \frac{c_1}{2c_2} \sum_{i=4}^\infty \chi_{R_i}(x) - n_0 \right) \, dy + \frac{\Phi}{c_1} \int \frac{2c_2}{c_1} \left( \frac{|f|}{\lambda} \right) \, dy = J_3 + J_4.
\]

Since \( 2c_2 > 1 \) and \( \Psi(t) \) is convex, we obtain by (7)
\[
J_3 \leq \frac{1}{2c_2} \int \Psi \left( \frac{c_1}{2c_2} \sum_{i=4}^\infty \chi_{R_i}(x) - n_0 \right) \, dy \leq \frac{1}{2} \sum \{R_i\},
\]
and since \( \Phi(t) \) satisfies the \( \Delta_2 \) condition,
\[
J_4 \leq c_3 \Phi \left( \frac{|f|}{\lambda} \right) \, dy.
\]


It follows that
\[ \sum_{i} |R_{\alpha_i}| \leq c_{\alpha} \int \left( f(y) + \Phi(f(y)) \right) dy, \]
and so
\[ |x: M_{f}(x) > \lambda| = \left| \bigcup_{\alpha} R_{\alpha} \right| \leq c_{\alpha} \sum_{i} |R_{\alpha_i}| \leq c_{\alpha} \int \left( f(y) + \Phi(f(y)) \right) dy, \]
which completes the proof of Lemma 2.

Using Lemma 2 and a standard technique (see e.g. [2, Ch. II, § 1]) it is easy to obtain the following lemma.

**Lemma 3.** Let \( \Phi(t), \psi(t) \) be Young conjugate convex functions with \( \Phi(t) \) satisfying the \( A_{2} \) condition. Then if a basis \( B \) has property \( (V_{\Phi}) \) then \( B \) differentiates \( \Phi(L) \).

The following lemma gives a method of constructing bases which differentiate precisely \( \Phi(L) \), provided certain covering properties of a simple collection of sets are known. In the present section we restrict our attention to the classes \( L_{\Phi}(L) \) which are close to \( L_{\Phi} \); more precisely, it will be assumed that the inverse function to \( \psi(t) \), denoted by \( \psi(t) \), satisfies the \( A_{2} \) condition:

\[ \exists c_{\psi} > 0, t_{0} > 0 \forall t \geq t_{0}: \exists \psi(t) \leq \psi(c_{\psi} t), \]
(For the general version, see Section 4.)

**Lemma 4.** Let \( \varphi(0) \) be an increasing concave function with \( \varphi(0) = 0 \), and suppose \( \psi(t) \), the inverse function to \( \varphi(t) \), satisfies the \( A_{2} \) condition. Moreover, suppose there are a collection of bounded open sets \( \sigma = \{ R_{\alpha} \} \), \( j = 1, 2, \ldots, i = 1, 2, \ldots, n, \) and constants \( c_{\alpha} > 0, m_{\alpha}, n_{\alpha} \in \mathbb{N} \) such that

\[ \begin{align*}
|R_{\alpha}| & = |R_{\alpha}|, \quad j \geq 1, i = 1, \ldots, n, \\
\forall j \geq 1 \forall \{ R_{\alpha} \} \subset \{ R_{\alpha} \}:
\int_{R_{\alpha}} \psi \left( c_{\alpha} \sum_{x} x_{\alpha}(x) - n_{\alpha} \right) dx & \leq c_{\alpha} \sum_{x} |R_{x}|, \\
\text{where} \quad W_{j} & = \{ x: \sum_{x} x_{\alpha}(x) \geq m_{\alpha} \},
\end{align*} \]

and such that there are measurable sets \( E_{j} \), and numbers \( \lambda_{j}, \beta_{j} \to \infty \) as \( j \to \infty \), satisfying

\[ \begin{align*}
\frac{|R_{\alpha} \cap E_{j}|}{|R_{\alpha}|} & \geq \frac{c_{\alpha}}{\lambda_{j}}, \quad j \geq 1, i = 1, \ldots, n, \\
\bigcup_{j=1}^{n} R_{\alpha} \geq c_{\alpha} \lambda_{j} \varphi(\lambda_{j}) |E_{j}| & \geq 1.
\end{align*} \]

Then there is a basis \( B \) whose every element is a dilation of some member of \( \sigma \) and such that \( B \) differentiates precisely \( L_{\Phi}(L) \).

**Proof.** Since (10)-(13) are dilation invariant, we can assume without loss of generality that

\[ E_{j} \subset \bigcup_{i=1}^{n} R_{i} \subset [0, 1], \quad j \geq 1, \]

and introduce the notation

\[ I = [0, 1], \quad X^{j} = \bigcup_{i=1}^{n} R_{i}, \]

\[ \omega_{j} = \left( \lambda_{j} \varphi(\lambda_{j}) |E_{j}| \right)^{-1}, \quad S_{j} = \sum_{j=1}^{n} \omega_{j}. \]

Clearly,

\[ \sum_{j=1}^{n} \omega_{j} |X^{j}| = \infty. \]

We now start constructing the basis in question. Let \( \{ m_{n} \}_{n=1}^{\infty} \) be an increasing sequence of positive integers tending to infinity, to be defined later. Let \( S_{0} = 0 \) and let \( n \in \mathbb{N}, S_{n-1} + 1 < n < S_{n} \). We divide \( I \) into \( m_{n} \) equal squares \( I_{n}^{j} \):

\[ I = \bigcup_{j=1}^{m_{n}} I_{n}^{j}, \quad |I_{n}^{j}| = m_{n}^{-2}, \quad j = 1, \ldots, m_{n}^{2}. \]

Denote by \( H_{n}^{j} \) dilation with coefficient \( m_{n}^{-1/2} \) taking \( X^{j} \) into \( I_{n}^{j} \) and let

\[ R_{n}^{j} = H_{n}^{j}(R_{n}^{j}), \quad H_{n}^{j} = \bigcup_{j=1}^{m_{n}} H_{n}^{j}(X^{j}), \quad \beta_{n} = |R_{n}^{j}|. \]

In this notation

\[ H_{n}(X^{j}) = \bigcup_{j=1}^{m_{n}} H_{n}^{j}(X^{j}) = \bigcup_{j=1}^{m_{n}} R_{n}^{j}. \]

We now show that the numbers \( m_{n} \) can be chosen so rapidly increasing that \( \limsup H_{n} = 1 \).

At the \( j \)th step of our construction we add to the collection \( B \) the sets \( R_{n-1, j}^{i}, i = 1, \ldots, m_{n-1}^{2}, j = 1, \ldots, m_{n-1} \), where \( S_{n-1} < j \leq S_{n} \). The numbers \( n_{k} \) depend in general on \( j \). Therefore to avoid additional indexation, from now on we write \( n_{k} \) in place of \( n_{k} \).

Let \( m_{1} = 1 \). Then \( \bigcup_{i=1}^{n_{1}} R_{1}^{i} = I \). Choose \( m_{2} \) so large that

\[ \left\| \bigcup_{i=1}^{n_{1}} R_{1}^{i} \right\| 
\]
and let \( A_1 = \{ v : I_1 \cap H_1 = \emptyset \} \), \( A_2 = \{ v : I_2 \cap \partial H_1 \neq \emptyset \} \). Obviously,
\[
I \setminus H_1 \subseteq \bigcup_{v \in A_1} I_1 \cup \bigcup_{v \in A_2} I_2,
\]
and therefore
\[
\bigcup_{v \in A_1} I_1 \supseteq (1 - |H_1|) - \bigcup_{v \in A_2} I_2.
\]
But it is easily seen that
\[
\bigcup_{v \in A_2} I_2 \subseteq \bigcup_{v \in A_2} \left( I_2 \cap \left( \bigcup_{i=1}^s \partial R_{s,i}^2 \right) \right) \neq \emptyset,
\]
so that from (16) we obtain
\[
\bigcup_{v \in A_2} I_2 \subseteq \frac{1}{2} (1 - |H_1|)
\]
and hence
\[
\bigcup_{v \in A_1} I_1 \supseteq \frac{1}{2} (1 - |H_1|).
\]
Moreover, it is clear that \(|H_1(X)^q)/|I_2^q| = |H_2|/|I_2^q| = |H_1|/|I_2^q|\). Consequently,
\[
|I \setminus H_1|/|H_1| \leq \left| \bigcup_{v \in A_1} I_1 \right| \subseteq \left| \bigcup_{v \in A_2} I_2 \right| \subseteq \left( 1 - \frac{1}{2} |H_1| \right) \left( 1 - \frac{1}{2} |H_1| \right).
\]
Suppose that we have already chosen \( m_1, \ldots, m_{n-1} \) in such a way that
\[\sum_{v \in A_2} \left( I_2 \cap \bigcup_{i=1}^s \bigcup_{v \in A_2} \partial R_{s,i}^2 \right) \neq \emptyset \]
\[
\min \left( 2^{-q} \beta_{q-1}, 2^{-q} \left| I \setminus H_1 \right| \right), \quad q = 3, \ldots, n-1,
\]
\[
\left| I \setminus H_1 \right| \leq \prod_{v \in A_2} \left( 1 - \frac{1}{2} \left| H_1 \right| \right), \quad 1 \leq p \leq r \leq n-1.
\]
Choose \( m_n \) large enough that
\[
\left| \left( I \setminus H_1 \right) \setminus \left( \bigcup_{v \in A_2} \left( I_2 \cap \bigcup_{i=1}^s \partial R_{s,i}^2 \right) \right) \right| \leq \min \left( 2^{-q} \beta_{q-1}, 2^{-q} \left| I \setminus H_1 \right| \right), \quad q = 3, \ldots, n-1,
\]
and put
\[
A_1^q = \{ v : I_1 \setminus \bigcup_{v \in A_2} H_1 = \emptyset \}, \quad A_2^q = \{ v : I_2 \setminus \bigcup_{v \in A_2} H_1 = \emptyset \}
\]
where \( q \) is a fixed integer between 1 and \( n \). Obviously,
\[
\left( \bigcup_{v \in A_1^q} I_1 \right) \subseteq \left( \bigcup_{v \in A_2^q} I_2 \right) \cup \left( \bigcup_{v \in A_2} I_2 \right)
\]
and therefore
\[
\left| \bigcup_{v \in A_1^q} I_1 \right| \geq \left( 1 - |H_1| \right) - \left| \bigcup_{v \in A_2} I_2 \right|.
\]
It follows from (19) that
\[
\left| \bigcup_{v \in A_1^q} I_1 \right| \leq \frac{1}{2} \left| \bigcup_{v \in A_2} I_2 \right| \leq \frac{1}{2} \left| \bigcup_{v \in A_1^q} I_2 \right|.
\]
Moreover,
\[
\left| \bigcup_{v \in A_2} I_2 \right| \geq \frac{1}{2} \left| \bigcup_{v \in A_1^q} I_2 \right| \geq \frac{1}{2} \left| \bigcup_{v \in A_1^q} I_2 \right|.
\]
Since \(|H_1(X)^q)/|I_2^q| = |H_2|/|I_2^q|\), we therefore obtain
\[
\left| I \setminus H_1 \right| \leq \left| \bigcup_{v \in A_1^q} I_2 \right| \subseteq \left| \bigcup_{v \in A_1^q} I_2 \right| \subseteq \left( 1 - \frac{1}{2} |H_1| \right) \left( 1 - \frac{1}{2} |H_1| \right).
\]
The choice of the numbers \( m_n \) is thus fully described, and clearly
\[
\left| I \setminus H_1 \right| \leq \prod_{v \in A_1^q} \left( 1 - \frac{1}{2} |H_1| \right), \quad \forall q \geq 1.
\]
The infinite product on the right diverges to zero provided \( \sum_{q=1}^{n} |H_1| = \infty \), and this follows easily from (15). Indeed, by dilation, \( |H_2| = |X|^{2^{-q}} \) with \( \frac{s_1}{1} < \frac{s_2}{1} < \ldots < \frac{s_{q-1}}{1} < \frac{s}{1} \), and so
\[
\sum_{q=1}^{n} |H_2| = \sum_{q=1}^{n} \sum_{s=q}^{s_{q-1}} \sum_{r=s_{q-1}+1}^{s_{q}} |H_2| = \sum_{q=1}^{n} \sum_{s=q}^{s_{q-1}} \sum_{r=s_{q-1}+1}^{s_{q}} \omega_k |X^k| = \infty.
\]
But then we have \( \limsup |H_1| = 1 \), and therefore almost every point \( x \in I \) belongs to an infinite sequence \( \{ R_{s}^{(x)} \} \) with diameters tending to zero.

We will show that the collection \( \{ R_{s}^{(x)} \} \) has property \( (V_{ph}) \), and so if we adjoin to it all dilated copies of \( R_{s}^{(x)} \) we obtain a basis \( B \) which differentiates \( L_Q \).

We introduce additional notation to simplify the writing. Instead of \( R_{s}^{(x)} \) we will write \( R_{s} \) where \( x = (k, i, v) \) is a multiindex. In this notation, extracting a subsequence \( \{ R_{s}^{(x)} \} \) from a sequence \( \{ R_{s}^{(x)} \} \) means that we take a subsystem \( \{ R_{s} \} \), \( A' \subseteq A \), of a system \( \{ R_{s} \} \) with some set of indices \( A' \).

Below we use both notations, which should cause no confusion.
Let therefore \( \{R_x\}_{x \in A} \subset B \). Without loss of generality we may assume that \( A \) is countable and write \( \{R_x\}_{x \in A} \) in the form \( \{R^{(k)}_x\}_{k \geq 2} \). Put
\[
Y_x = \bigcup_{k \geq 2} R^{(k)}_x, \quad Y \equiv \bigcup_{x \in A} Y_x.
\]

We will show how to choose a subsystem of \( \{R_x\}_{x \in A} \) with the weak overlapping property (V). Let
\[
\gamma_i = Y_i, \quad Y_i^c = \{ Y_i \setminus Y_i^c : Y_i^{c(k)} = \bigcup_{k \geq 2} \}
\]

Obviously, \( |\bigcup_{i=1}^{\infty} \gamma_i^c| = |\bigcup_{i=1}^{\infty} \gamma_i| \). Furthermore, put
\[
\bar{Y} = \bigcup_{i=1}^{\infty} Y_i, \quad \bar{Y}_i = \{ Y_i \setminus Y_i^c : Y_i \cap \partial \bigcup_{k=1}^{i-1} Y_i \neq \emptyset \}, \quad k \geq 2,
\]

and, finally, let
\[
\bar{Y} = \bigcup_{i=1}^{\infty} \bar{Y}_i, \quad \bar{Y}_i^c = \bigcup_{i=1}^{\infty} Y_i^c, \quad A' = \bigcup_{i=1}^{\infty} A_i.
\]

Define
\[
u(\bar{Y}) = \{ Y_i \setminus Y_i^c : Y_i \cap \partial \bigcup_{k=1}^{i-1} Y_i \neq \emptyset \}, \quad i \geq 2.
\]

It follows from (17) that \( |\nu(\bar{Y})| \leq 2^{-i}\sum_{i=1}^{\infty} |\gamma_i^c| \), and clearly
\[
\sum_{i=1}^{\infty} |\nu(\bar{Y})| \leq \sum_{i=1}^{\infty} |\bar{Y}_i^c| \leq |\bar{Y}_i|.
\]

But
\[
|\nu(\bar{Y})| \leq \sum_{i=2}^{\infty} |\nu(\bar{Y})| \leq \sum_{i=2}^{\infty} 2^{-i} |\bar{Y}_i^c| \leq |\bar{Y}_i|.
\]

Therefore
\[
\sum_{i=1}^{\infty} |\nu(\bar{Y})| \leq |\bar{Y}_i| - |\bar{Y}_1^c| \leq |\bar{Y}_1|.
\]

that is,
\[
2|\bar{Y}_1^c| \geq |\bar{Y}| = |\bigcup_{x \in A} Y_x|.
\]

This means that \( |\bigcup_{x \in A} R_x| \leq 2 |\bigcup_{x \in A} R_x| \), and an inequality of the type (8) is proved. Further, it is easily seen that the sets \( \bar{Y}_x \) are pairwise disjoint and on each of them we have a \( (V_\phi) \) estimate by condition (11) of the lemma. Hence
\[
\int \psi(\sum_{x \in A} X_{w_0}(x))dx = \sum_{i,j \geq 1} \int \psi(\sum_{x \in A} X_{w_0}(x))dx \leq c_0 \sum_{i,j \geq 1} |\bigcup_{x \in A} R_x| = c_0 \sum_{x \in A} |R_x|,
\]

where \( W = \{ x : \sum_{x \in A} X_{w_0}(x) \geq m_0 \} \), \( U = \bigcup_{x \in A} R_x \).

We have thus established estimates (7) and (8), i.e. property \( (V_\phi) \). Moreover, since \( \psi \) satisfies the \( A_\phi \) condition, its conjugate function is equivalent to \( \psi \) (see \( \psi \) (6, Th. 6.1)]), and we conclude by Lemma 3 that \( B \) differentiates \( L_0(L) \).

We now show that \( B \) does not differentiate \( o(L_0(L)) \). Let \( g(t) \downarrow 0 \) as \( t \to \infty \). Then there are numbers \( w_i \in N \), \( 0 \leq w_i \leq \lambda_i \), such that
\[
\sum_{i=1}^{\infty} w_i |X|^{\infty} = \infty,
\]
\[
\sum_{i=1}^{\infty} w_i |X|^{\infty} \sqrt{\psi(\lambda_i)} < \infty,
\]

where the \( \lambda_i \) are taken from condition (12) of the lemma. Write \( \lambda_i = \gamma_i \psi(\lambda_i) \psi(\lambda_i) \), where \( \gamma_i \) slowly enough that
\[
\gamma_i = o\left( \psi(\lambda_i)^{-1/2} \right) \quad \text{as} \quad k \to \infty.
\]

In every \( \lambda_i \) with \( S_{i-1} < \lambda_i \leq S_i \), we place \( H(\alpha_i) \), a delayed copy of \( E_\lambda \) (the dilate \( H \) has been defined at the beginning of the proof). Put
\[
Y_i = \begin{cases} \bigcup_{k=1}^{\infty} H(\alpha_i), & S_{i-1} < \lambda_i \leq S_i, \\ \emptyset, & S_i - w_i < \lambda_i \leq S_{i-1}, \end{cases}
\]

and define the functions
\[
\lambda(x) = \gamma_i \lambda_i \sqrt{\psi(\lambda_i)}, \quad S_{i-1} < \lambda_i \leq S_i, \quad f(x) = \sup_{\lambda} f(x).
\]

Write \( F(t) = g(t) \psi(t), \quad t > 0 \). Obviously,
\[
\int f(x)dx \leq \sum_{k=1}^{\infty} \int f(x)dx = \sum_{k=1}^{\infty} \sum_{S_{i-1} < \lambda_i \leq S_i} F(\lambda_i) |Q_i| \leq \sum_{k=1}^{\infty} g(\gamma_i) \lambda_i \sqrt{\psi(\lambda_i)} \psi(\lambda_i) |Q_i|.
\]

Write \( F(t) = g(t) \psi(t), \quad t > 0 \). Obviously,
\[
\int f(x)dx \leq \sum_{k=1}^{\infty} \int f(x)dx = \sum_{k=1}^{\infty} \sum_{S_{i-1} < \lambda_i \leq S_i} F(\lambda_i) |Q_i| \leq \sum_{k=1}^{\infty} g(\gamma_i) \lambda_i \sqrt{\psi(\lambda_i)} \psi(\lambda_i) |Q_i|.
\]
By (21), (22), (13) we obtain
\[
\int f(x) dx \leq c_{12} \sum_k w_k |X_k^2| \sqrt{g(x_k)} < \infty.
\]
Thus \( f \in \mathcal{E}(L) \mathcal{E}(L)[0,1]^2 \).

We now show that \( E_B(f, x) = +\infty \) a.e. on \([0,1]^2\). By repeating the considerations used in the construction of the basis it is not difficult to show that almost every \( x \in I \) belongs to an infinite sequence \( \{R_n^*, x_n^*\} \).

If \( S_{n-1} < I_n < S_{n-1} + w_n \), then we obtain by (12) (writing \( R = R_n^*, x_n^* \) for simplicity)
\[
|R|^{-1} \int f(y) dy \geq |R|^{-1} \int f_n(x) dy \geq \frac{\lambda_k}{|R|} \geq c_{13} \lambda_k.
\]
Since \( \lambda_k \to \infty \) as \( k \to \infty \), it follows that \( \hat{D}_B(f_n, x_n) = +\infty \) a.e. on \([0,1]^2\), i.e. \( B \) does not differentiate \( \mathcal{E}(L) \mathcal{E}(L)[0,1]^2 \). Decomposing \( R^2 \) into a union of unit squares we obtain a basis in \( R^2 \). Thus the proof of Lemma 4 is complete.

Using Lemma 4 it is not difficult to prove Theorem 2. First, we need some additional information about the function \( \psi(t) \). We show that
\[
\phi(ab) \leq \phi(a) + \phi(b), \quad \forall a, b \geq t_0.
\]
Indeed, we obviously have \( \phi(ab)/\ln(ab) \leq \phi(a)/\ln a \) and \( \phi(ab)/\ln(ab) \leq \phi(b)/\ln b \), i.e. \( \ln a/\ln(ab) \leq \phi(a)/\phi(ab) \) and \( \ln b/\ln(ab) \leq \phi(b)/\phi(ab) \). Adding the last two inequalities gives (23). For the inverse function \( \psi(t) \) we then obtain
\[
\psi(a) \psi(b) \leq \psi(a+b), \quad \forall a, b \geq t_0.
\]
Further, let \( c_{14} = \psi(t_0) \). By the concavity of \( \psi(t) \) we have
\[
\psi(t) \leq c_{14} t, \quad \forall t \geq t_0.
\]
Hence
\[
\tau \psi(t) = \psi(\tau \psi(t)) \leq \psi(\psi(t) + t) = \psi((c_{14} + 1)t), \quad \forall t \geq t_0,
\]
i.e. \( \psi(t) \) satisfies the \( \lambda_3 \) condition.
\[
\tau \psi(t) \leq \psi(c_{14} t), \quad \forall t \geq t_0.
\]

We introduce the function
\[
F(t) = \begin{cases} t^{-2} \psi(t), & t > 0, \\ 0, & t \leq 0. \end{cases}
\]
By (26), \( t^2 \psi(t) \leq \psi(c_{14} t)/c_{14} \), and so
\[
\psi(t) \leq c_{14} F(c_{14} t), \quad \forall t \geq t_0.
\]
By the definition of $F(t)$ we obtain

$$J = \int \frac{1}{t} \sum_{j=1}^{n} x_{y_{j}}(x) \, dx,$$

$$U = \bigcup_{m=0}^{n} \bigcup_{k=0}^{n} E_{m,k}.$$

Clearly, $|E_{m,k}| \leqslant \Psi(k) \Psi(m + k)$, and it can easily be seen that

$$\sum_{j=1}^{n} x_{y_{j}}(x) = k + 1, \quad \forall x \in E_{m,k}.$$

Hence

$$J \leq \sum_{m=0}^{n} \sum_{k=0}^{n-1} \int \frac{\sum_{j=1}^{n} x_{y_{j}}(x)}{t} \, dx \leq \sum_{m=0}^{n} \sum_{k=0}^{n-1} \frac{\Psi(k)}{\Psi(m + k)} \frac{\Psi(m + k)}{\Psi(m + k)}.$$

But $k \geq m \geq 0$, $i_{m,k} \geq x_{m} \geq 0$, and so $\Psi(k) \Psi(m + k) \leq \Psi(i_{m,k})$, and since it is easily seen that $i_{m,k} \leq i_{m,k}$, we finally obtain

$$J \leq \frac{1}{\Psi(m + k)} \sum_{m=0}^{n} \sum_{k=0}^{n-1} \frac{\Psi(m + k)}{\Psi(m + k)} \leq \frac{2n}{\Psi(m + k)}.$$

On the other hand, $\sum_{m=0}^{n} |R_{m}^{n}| = n! \Psi(m + k)$. By (27) and (28) we obtain for $m \geq 3n$

$$\forall |R_{m}^{n}| \leq \frac{n! \Psi(m + k)}{\Psi(m + k)} \leq \frac{2n}{\Psi(m + k)}.$$

and (11) is proved. All conditions of Lemma 4 are therefore satisfied, and the proof of Theorem 2 is completed by using the conclusion of that lemma.

Remark 2. It follows from Lemma 2 that the maximal operator corresponding to the constructed basis has weak type $L + \varphi(L)$.

4. N-dimensional analogues and some generalizations. All definitions introduced in Sections 1–3 carry over without change to the case of several variables. Theorems 1 and 2 also remain valid, with $B_{1}$ understood as $B_{1}(R^{n})$. The proof can be reduced to a two-dimensional argument by considering projections on two-dimensional coordinate hyperplanes.

The important and essentially new element in our proof of Theorem 2 is Lemma 4. The same method yields the following more general result which is of independent interest:

**Lemma A.** Let $\Phi(t)$ and $\Phi^{*}(t)$ be Young conjugate convex functions with $\Phi(t)$ satisfying the $A_{2}$ condition. Let $\sigma = [R_{k}], \ k = 1, \ldots, m_{k}, n_{k}, N$, and $[E_{m,k}]_{m=1}^{n_{k}}$ be collections of bounded open sets in $R^{n}$ and $A_{m,k}$ a sequence of numbers with $A_{m,k} \rightarrow \infty$ as $n \rightarrow \infty$ such that there are constants $c_{i} > 0, i = 0, \ldots, 6$, satisfying the following conditions for all $N$:

(i) $[R_{k}] = [R_{k}], \ k, j = 1, \ldots, m_{k}$,

(ii) $\forall [R_{k}]_{m=1}^{n_{k}} = [R_{k}]_{m=1}^{n_{k}}$:

$$\int \frac{\Phi^{*}(c_{i} \sum_{j=1}^{n_{k}} x_{y_{j}}(x))}{\Lambda} \, dx \leq c_{d} \sum_{i=1}^{n_{k}} |R_{k}^{n}|,$$

where $\Lambda = \{x: \sum_{j=1}^{n_{k}} x_{y_{j}}(x) \geq c_{d}\}$.

(iii) $[R_{k}] \cap E_{m,k} \geq \frac{c_{j}}{A_{m,k}}$, $k = 1, \ldots, m_{k}$,

(iv) $\left| \bigcap_{k=1}^{n_{k}} R_{k} \right| \geq c_{d} \Phi(A_{m,k})|E_{m,k}|$.

Then there is a basis $B(R^{n})$ whose every element is a dilation of some member of $\sigma$ and such that $B(R^{n})$ differentiates precisely $\Phi(t)$.

**Lemma A** gives a method of constructing bases in $R^{n}$ consisting of elements of a given type and differentiating precisely the classes $\Phi(t)$.

The results obtained have applications in the theory of multiple Fourier series. Let $[m_{k}], [n_{k}]$ be two sequences of positive integers tending to infinity, and let $\sigma_{m_{k}}^{n_{k}}(f, x)$ be the (C, 1) means of the Fourier series of a function $f(x)$ on the rectangle $[0, n_{k}^{1}] \times [0, m_{k}^{1}]$. From the method of proof of Theorem 2.14 in [15, Ch. XVII] it follows that if $f$ is not differentiated by the TI-basis consisting of rectangles of the form $[0, n_{k}^{1}] \times [0, m_{k}^{1}]$ then

$$\lim_{n_{k} \rightarrow \infty} \sigma_{m_{k}}^{n_{k}}(f, x) = \infty \quad a.e. \text{ on } [0, 2\pi]^{2}.$$

On the other hand, if $f \in L^{2}(0, 2\pi)^{2}$ then

$$\lim_{n_{k} \rightarrow \infty} \sigma_{m_{k}}^{n_{k}}(f, x) = f(x) \quad a.e. \text{ on } [0, 2\pi]^{2}.$$

(see [5]). Proceeding by analogy with the proof of Theorem 3.1 in [15, Ch. XVII] it is not difficult to show that if $[m_{k}^{1}] \times [0, m_{k}^{1}]$ is a monotonic family of rectangles then for all $f \in L^{2}(0, 2\pi)^{2}$

$$\lim_{n_{k} \rightarrow \infty} \sigma_{m_{k}}^{n_{k}}(f, x) = f(x) \quad a.e. \text{ on } [0, 2\pi]^{2}.$$

Combining the above and the proof of Theorem 1 we obtain the following alternative for any sequences $[m_{k}], [n_{k}]$ of positive integers tending
to infinity: either \( \sigma_m \to f(x) \) converges to \( f(x) \) a.e. for all \( f \in L(0, 2\pi) \), or for any function \( g(t) \) with \( \{ g(t) \} \to 0 \) as \( t \to \infty \) there is an \( f \in g(L) \log^+ L(0, 2\pi) \) such that

\[
\limsup_{k \to \infty} \sigma_m \to f(x) = +\infty \quad \text{a.e. on } [0, 2\pi].
\]

Analogous results hold for \((C, \alpha, \beta)\) summability \((0 < \alpha \leq 1, 0 < \beta \leq 1)\).

References


DEPARTMENT OF MATHEMATICS AND MECHANICS
OSKRA STATE UNIVERSITY
Poes Ypsilene 3, 270000 Odessa, U.S.S.R.

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On subspaces of \( H^1 \) isomorphic to \( H^1 \)

by

PAUL F. X. MÜLLER (Linz)

Abstract. We show that any subspace of \( H^1 \) which is isomorphic to \( H^1 \) contains a complemented copy of \( H^1 \). \( H^1 \) is proved to be primary.

Introduction. This work is best regarded as an appendix to the book Symmetric Structures in Banach Spaces by W. Johnson, B. Maurey, G. Schechtman and L. Tzafriri (JMSF), where the result analogous to our Theorem 1 is proved for \( L^p \) spaces \((1 < p < \infty)\).

We use their notation and follow their arguments rather closely.

I feel obliged to indicate at which point the treatment of \( H^1 \) spaces requires different tools than that for \( L^p \) spaces \((1 < p < \infty)\):

In trying to find complemented subspaces in the range of embeddings on \( L^p \), JMSF rely on the following martingale inequality due to E. M. Stein: Given an increasing sequence of \( \sigma \)-fields \(\mathcal{F}_{\xi n} \) in \([0, 1]\) with corresponding conditional expectations \( E_{\mathcal{F}_{\xi n}} \), for any \( 1 < p < \infty \) there exists \( C_p \in \mathbb{R}^+ \) such that for any sequence of measurable functions \( f_{\xi n} \) the following holds:

\[
\left\| \sum_{n=1}^{\infty} \left| E_{\mathcal{F}_{\xi n}} f_{\xi n} \right|^{p/2} \right\|_{\mathcal{F}_{\xi n}} \leq C_p \left\| \sum_{n=1}^{\infty} f_{\xi n} \right\|_{\mathcal{F}_{\xi n}}.
\]

There exist examples (cf. [S3], p. 105) showing that this inequality does not hold for \( p = 1 \) or \( p = \infty \).

Here we modify the selection process of [JMSF] in such a way that projections can be constructed which are bounded on \( H^1 \). At this point the third component of the vector measure used below becomes crucial.

Definitions and notation. Recall that \( H^1 \) is the closed linear span of the \( L^\infty \)-normalized Haar system

\[
\{ h_n : (n) \in \mathcal{A} \}
\]

where \( \mathcal{A} = \{(n) : n \in \mathbb{N}, 0 \leq i \leq 2^n - 1 \} \)

under the norm

\[
\| f \|_{H^1} = |S(f)|, \quad S(f) = \left( \sum a_n^i h_n^i \right)^{1/2},
\]

with \( f = \sum a_n h_n \).