Subspace mixing properties of operators in $\mathbb{R}^n$ with applications to Gluskin spaces

by

P. MANKIEWICZ (Warszawa)

Abstract. There is a constant $c > 0$ such that for every $n \in \mathbb{N}$ there is an $n$-dimensional Banach space $X_n$ with the property that whenever $G$ is a compact group of operators acting on $X_n$, then

$$\sup \{ \| T \|_{X_n}: T \in G \} \geq \frac{c(n-1)\delta}{n^{1/2} \log n},$$

where $\delta(T) = \inf \{ \| T \|: T \in G \}$.

E. D. Gluskin in [3] introduced a class of random $n$-dimensional Banach spaces in order to prove that the Banach–Mazur diameter of the set of all $n$-dimensional Banach spaces is of order $n$. Later the same author in [4] and independently S. J. Szarek [8] used different variants of spaces defined in [3] to prove the existence of finite-dimensional Banach spaces with the "worst possible" basis constants. Another variant of Gluskin spaces was used by the author in [5] to construct finite-dimensional Banach spaces with the "worst possible" symmetry constants. The importance of the notion of "subspace mixing operators on $\mathbb{R}^n$" in the context of Gluskin spaces was implicit in [4] and "almost explicit" in [5]. The final step in this direction was done by S. J. Szarek [9], who proved that a "vast majority" of Gluskin spaces enjoy the property that every subspace mixing operator on such a space has large norm. The subspace mixing property was used in that paper to prove the existence of finite-dimensional Banach spaces with two essentially different complex structures. Later on, the techniques developed in [9] were used by the same author to construct infinite-dimensional Banach spaces with some pathological properties [10], [11]; however, the credit for the first use of Gluskin spaces to construct pathological infinite-dimensional Banach spaces should be given to J. Bourgain [1].

In this paper we study subspace mixing properties of operators in $\mathbb{R}^n$ with special attention turned to operators which belong to a compact group of operators. The main difference in the approach between [9] and this paper lies in the fact that in [9] the author studied the subspace mixing property of an operator $T$ in terms of certain "distances" of $T$ to the line $\{ t1 \}_{t \in \mathbb{R}}$ while...
we study the same property in terms of some other "distances" of $T$ to the operator $n^{-1}(\operatorname{tr} T)\Id$, and the values of the latter "distances" are easier to handle.

Finally, we use the results on subspace mixing properties to prove the existence of finite-dimensional Banach spaces with some pathological properties, or more precisely to establish some pathological properties of a "vast majority" of Gluskin spaces. The main result in this direction can be stated as follows (cf. Th. 5.3 below):

There is a constant $c > 0$ such that for every $n \in \mathbb{N}$ there is an $n$-dimensional Banach space $X_n$ with the property that whenever $G$ is a group of linear operators acting on $X_n$,  
\[
\sup\|T\|_{X_n} := \sup_{T \in G} \|T\|_{X_n} \geq c\frac{(n - \operatorname{tr}(G))}{n^{1/3} \log^{3/2} n},
\]
where $t(G) = \inf\{\|T\|: T \in G\}$.

This answers a question posed by S. J. Szarek in [9].

Let us note that the theorem quoted above contains "up to a logarithmic factor" the results proved in [4], [5], [8], [9].

1. Notation and preliminaries. We shall use the standard notation. By $e_1, \ldots, e_n$, we shall denote the standard unit vector basis in $R^n$ and by $\|\cdot\|$ the standard Euclidean norm in $R^n$. $S^{n-1}$ will stand for the unit sphere in $R^n$ while $\mu_{n-1}$ will denote the normalized surface Lebesgue measure on $S^{n-1}$. $Q_n$ and $h_n$ will stand for the orthogonal group on $R^n$ and the normalized Haar measure on it respectively. If $E$ is a linear subspace of $R^n$ then by $P_E$ and $E^\perp$ we shall denote the orthogonal projection on $E$ and on the orthogonal complement of $E$ in $R^n$ respectively.

We shall say that a linear operator $T \in L(R^n)$ has the $(\alpha, \beta)$-subspace mixing property for $\alpha, \beta > 0$ if there is a linear subspace $E \subset R^n$ with dim $E \geq \alpha$ such that  
\[
\|P_E T x\| \geq \beta \|x\| \quad \text{for every } x \in E.
\]

The set of all operators in $L(R^n)$ having the $(\alpha, \beta)$-subspace mixing property will be denoted by $M_{\alpha, \beta}(R^n)$. Obviously for $T \in L(R^n)$,

(i) $T \notin \bigcup_{\delta > 0} M_{\alpha, \beta}(R^n)$ if $T = \lambda \Id_{R^n}$ for some $\lambda \in R$.

(ii) $T \in M_{\alpha, \beta}(R^n)$ if $T = \lambda \Id_{R^n}$ for every $\lambda \in R$.

(iii) $T \in M_{\alpha, \beta}(R^n)$ if $T = \lambda \Id_{R^n}$ for every $\lambda \in R$.

We shall write $T \in M_{\alpha}(R^n)$ if $T \in M_{\alpha, \beta}(R^n)$ for some $\alpha, \beta > 0$ with $\alpha \beta = \gamma$.

Recall that every operator $T \in L(R^n)$ can be written in the form  
\[
T = \sum_{\lambda \in \Lambda} \lambda(T) \langle , u_i \rangle v_i
\]
where $\lambda_1(T) \geq \ldots \geq \lambda_n(T) \geq 0$ and $\{u_i\}^{n}_{i=1}$ and $\{v_i\}^{n}_{i=1}$ are orthonormal systems in $R^n$. Any representation of an operator $T \in L(R^n)$ in the form (8) will be called a polar decomposition of $T$. It is well known that while the polar decomposition of an operator $T$ need not be unique the sequence $\{\lambda(T)\}^{n}_{i=1}$ is uniquely determined by $T$.

For a subset $A \subset R^n$, $[A]$ will denote the linear hull of $A$. Sometimes, we shall identify an operator $T \in L(R^n)$ with its matrix representation $(a_{ij})$ with respect to the basis $\{e_j\}^{n}_{j=1}$. For $T \in L(R^n)$ we define
\[
m(T) = \frac{\lambda_n(T)}{\lambda_{n-2k}} \quad \text{for } n = 2k,
\]
\[
m(T) = \frac{\lambda_{n+1}(T)}{\lambda_{n+2k+1}} \quad \text{for } n = 2k + 1.
\]

For $T \in L(R^n)$, $\|T\|_{\text{HS}}$ will denote the Hilbert-Schmidt norm of $T$ and $\operatorname{tr} T$ will denote its trace. We shall make use of the following trivial equalities:
\[
\|T\|_{\text{HS}} = \sum_{i=1}^{n} \lambda_i(T)^2 = n \int_{S^{n-1}} \|Tx\|^2 d\mu_{n-1}(x),
\]
\[
\operatorname{tr} T = \sum_{i=1}^{n} \lambda_i(T) \langle u_i, v_i \rangle = n \int_{S^{n-1}} \langle Tx, x \rangle d\mu_{n-1}(x)
\]
for every $T \in L(R^n)$ and every polar decomposition of $T$.

Finally, we define
\[
\delta(T) = \inf \{\|T - \lambda \Id_{R^n}\|_{\text{HS}} : \lambda \in R\}
\]
for every $T \in L(R^n)$. It can easily be seen that for $T \in L(R^n)$  
\[
\delta(T) = \|T - \sum_{i\in \Lambda} n^{-1} (\operatorname{tr} T) \Id_{R^n} \|_{\text{HS}} = \|T\|_{\text{HS}} - \frac{n^{-1} (\operatorname{tr} T)^2}{\lambda_{n-1}}
\]
for every $T \in L(R^n)$.

By $\mathcal{G}$ we shall denote the set of all compact groups of operators acting on $R^n$. We define  
\[
\mathcal{G} = \bigcup_{G \in \mathcal{G}} G \quad \text{for } G \in \mathcal{G},
\]
Since $\operatorname{tr} (\Id_{R^n}) = n$ one has $T(G) \leq n$ for $G \in \mathcal{G}$, and $\operatorname{tr}(G) = n$ iff $G \subset \{\Id_{R^n}, -\Id_{R^n}\}$. For a compact group $G$ we shall denote by $h_G$ the normalized Haar measure on $G$.

We shall say that a group $G \in \mathcal{G}$ acts trivially on a subspace $E \subset R^n$ iff $T|E = \lambda \Id_E$ for every $T \in L(E)$, with $\lambda \in \{1, -1\}$, for every $T \in E$. Also, we shall say that a group $G \in \mathcal{G}$ acts essentially nontrivially on $R^n$ iff it does not act trivially on any subspace of $R^n$ with positive dimension.

The letter $c$ with indices or without will always stand for an absolute numerical constant, in general different in different places.

We shall deal only with spaces over the reals; however, all the results after suitable modification remain valid in the complex case.
2. Mixing properties of contractions in \( R^n \). We begin with

**Proposition 2.1.** Let \( T \in L(R^n) \). Then

\[
\frac{1}{n(n+2)} \frac{\langle T, x \rangle^2}{\|T\|_{\infty}^2} = \frac{1}{n(n+2)} \frac{\langle T, x \rangle^2}{\|T\|_{\infty}^2} \equiv \frac{\delta^2(T)}{n+2}.
\]

**Proof.** The first equality, as well as the inequality, is trivial. To prove the second equality set \( T = (a_{ij})_{ij=1}^{n \times n} \) and observe that for \( x = (x_1, \ldots, x_n) \in R^n \)

\[
\|T x\|_{\infty}^2 = \|T x\|_{2}^2 - \langle T x, x \rangle^2 = \|T x\|_{2}^2 - \sum_{i,j=1}^{n} a_{ij} x_i x_j.
\]

Hence

\[
\frac{1}{n(n+2)} \frac{\langle T, x \rangle^2}{\|T\|_{\infty}^2} = \sum_{i,j=1}^{n} \frac{a_{ij} x_i x_j}{\|T\|_{\infty}^2}.
\]

Since the first integral on the right-hand side is equal to \( n^{-1}\|T\|_{\infty}^2 \), it remains to evaluate the second one. Expanding the square of the sum, skipping the integrals which are obviously equal to zero and taking into account that

\[
\sum_{i,j=1}^{n} a_{ij} x_i x_j = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \frac{1}{n(n+2)}
\]

for \( i,j = 1, \ldots, n \) and \( i \neq j \), we have

\[
\sum_{i,j=1}^{n} \frac{a_{ij} x_i x_j}{\|T\|_{\infty}^2} = \frac{1}{n(n+2)} \sum_{i,j=1}^{n} a_{ij} x_i x_j
\]

Combining (1) and (2) we obtain

\[
\frac{1}{n(n+2)} \frac{\langle T, x \rangle^2}{\|T\|_{\infty}^2} = \frac{1}{n(n+2)} \frac{\langle T, x \rangle^2}{\|T\|_{\infty}^2} \equiv \frac{\delta^2(T)}{n+2}.
\]

which concludes the proof.

Thus we have

**Corollary 2.2.** \( T \in M_+(1, (n+2)^{-1/2} \delta^3(T)) \) for every \( T \in L(R^n) \).

**Proposition 2.3.** For every \( k \)-dimensional subspace \( E \subset R^n \) and every \( T \in L(R^n) \)

\[
\left\langle \frac{\|T E\|_{\infty}^2}{\|T\|_{\infty}^2} \right\rangle \frac{\delta^2(T)}{n+2} \geq \frac{k(n-k)}{(n+1)(n+2)} \delta^2(T).
\]

**Proof.** Since the integral does not depend on a particular choice of a space \( E \) but only on the dimension of \( E \), let \( E = [e_1, \ldots, e_k] \). We have

\[
\frac{1}{n(n+2)} \frac{\langle T E, x \rangle^2}{\|T E\|_{\infty}^2} \leq \frac{\delta^2(T)}{n+2}.
\]

Let \( U T U^{-1} = (a_{ij})_{ij=1}^{k \times k} \) for \( U \in O_k \) and observe that for every \( U_i \in O_k \)

\[
\frac{1}{n(n+2)} \frac{\langle T E, x \rangle^2}{\|T E\|_{\infty}^2} \leq \frac{\delta^2(T)}{n+2}.
\]

By a simple "change of coordinates" argument we get

\[
\frac{1}{n(n+2)} \frac{\langle T E, x \rangle^2}{\|T E\|_{\infty}^2} \leq \frac{\delta^2(T)}{n+2}.
\]

for all \( i \neq j \) and \( i \neq m \). Let \( a(T) \) be the common value of all integrals of the form (4). We have

\[
\frac{1}{n(n+2)} \frac{\langle T E, x \rangle^2}{\|T E\|_{\infty}^2} \leq \frac{\delta^2(T)}{n+2}.
\]

Combining (1) and (2) we obtain

\[
\frac{1}{n(n+2)} \frac{\langle T E, x \rangle^2}{\|T E\|_{\infty}^2} \leq \frac{\delta^2(T)}{n+2}.
\]

which concludes the proof.

**Theorem 2.4.** \( T \in M_+(\{\epsilon, \delta\delta^3(T), \delta(T)\sqrt{5}n\}) \) for every \( T \in L(R^n) \) with \( \|T\|_{\infty} \leq 1 \).

**Proof.** We shall assume that \( n \) is even. The case of \( n \) odd can be treated in a similar way. By Prop. 2.3, there is an \( E \subset R^n \), \( \dim E = n/2 \), such that

\[
\frac{1}{n(n+2)} \frac{\langle T E, x \rangle^2}{\|T E\|_{\infty}^2} \geq \frac{\delta^2(T)}{n+2}.
\]

Thus we have

\[
\frac{1}{n(n+2)} \frac{\langle T E, x \rangle^2}{\|T E\|_{\infty}^2} \geq \frac{\delta^2(T)}{n+2}.
\]

which concludes the proof.
Let $V = P_E^\perp TP_E$ and let

$$V = \sum_{i=1}^{\frac{m}{2}} \lambda_i(V) \langle, u_i \rangle u_i$$

be a polar decomposition of $V$. Obviously $[u_i]$ and $[v_i]$ are orthonormal systems in $E$ and $E^\perp$ respectively and $\lambda_i(V) \leq 1$ for $i = 1, \ldots, m/2$. By (7) we have

$$\sum_{i=1}^{\frac{m}{2}} \lambda_i^2(V) \geq \frac{1}{2} \delta^2(T).$$

This implies that more than $\frac{1}{16} \delta^2(T)$ of the $\lambda_i(V)$'s are greater than $(5n)^{-1/2} \delta(T)$. Set

$$E_0 = [u_i; \ i = 1, \ldots, \lfloor \frac{1}{16} \delta^2(T) \rfloor + 1],$$

where $\lfloor \frac{1}{16} \delta^2(T) \rfloor$ denotes the integer part of $\frac{1}{16} \delta^2(T)$. Then

$$\|P_{E_0} T x\| \geq \|P_{E_0} T x\| = \|Vx\| \geq (5n)^{-1/2} \delta(T)$$

for $x \in E_0$, which yields that $T \in M_n(\delta^2(T), \delta(T)\sqrt{5n})$ and completes the proof.

Remark 2.5. Observe that the theorem above gives (up to some constants) the best possible "mixing properties" for contractions in $L(R^n)$ with values of $\delta$ "proportional" to $\sqrt{n}$, while it is far from the best for contractions with relatively small values of $\delta$.

In the sequel we shall need the following

Lemma 2.6. Let $a > 0, m \in N$ and let $T \in L(R^n)$ be such that there are two orthogonal subspaces $E_1, E_2 \subset R^n$, $\dim E_1 = \dim E_2 = m$, with the properties:

$$\|T x\| \geq a \|x\| \quad \text{for} \ x \in E_1,$$

$$\|T x\| \leq b \|x\| \quad \text{for} \ x \in E_2.$$

Then $T \in M_n(a, b, a^2 - b^2)$.

Proof. Without any loss of generality we may assume that $a = 1$. In order to simplify the notation we shall assume that $m = 4k$ for some $k \in N$. Let $p$ be the maximal nonnegative integer such that $T \in M_n(p, 1 - b)$. Assume to the contrary that $p < k$, and let $F \in R^k$ be such that

$$\|P_{E_1} T x\| \geq \frac{1}{2} (1 - b) \|x\| \quad \text{for} \ x \in E_1, \quad \dim F = p.$$

Set

$$E_3 = [F \cup TF \cup T^2 F \cup T^3 F] \cap E_1,$$

$$E_4 = [F \cup TF \cup T^2 F \cup T^3 F] \cap E_2$$

and observe that $\dim E_3 \geq \dim E_2 - 4 \dim F \geq 4 (k - p) \geq 4$. By the same token $\dim E_4 \geq 4$. Using the same argument as in the proof of the fact that every 2k-dimensional ellipsoid has a k-dimensional spherical section we deduce that there are two orthogonal vectors $x_0, y_0 \in E_3 \oplus E_4$ such that $\|T x_0\| = \|x_0\|$ and $\|T y_0\| = 0$. Let $E_k = \{x_0, y_0\}$ and note that $\|T|E_k\|_\infty = 1 + b^2$, while $\|T|E_k\|_\infty < 1 + b$. Now, if $\|P_{E_1} T|E_k\|_\infty \geq \frac{1}{2} (1 - b)^2$, then there is an $x \in E_k$, $\|x\| = 1$, such that $\|P_{E_1} T x\| \geq \frac{1}{2} (1 - b)$.

Since $x_0 \in E_3 \oplus E_4$ it is a matter of a routine calculation to verify that if $F_1 = [F \cup \{x_0\}]$ then

$$\|P_{E_1} T y\| = \|P_{E_1} T y\| \geq \frac{1}{2} (1 - b) \|y\| \quad \text{for} \ y \in F,$$

$$\|P_{E_1} T x\| = \|P_{E_1} T x\| \geq \frac{1}{2} (1 - b), \quad P_{E_1} T x_0 \perp P_{E_1} T F,$$

and therefore

$$\|P_{E_1} T x\| \geq \frac{1}{2} (1 - b) \|x\| \quad \text{for} \ x \in F_1,$$

a contradiction to the assumption that $F$ has the maximal dimension. On the other hand, if $\|P_{E_1} T|E_k\|_\infty \leq \frac{1}{2} (1 - b)^2$, then treating $E_3$ as $R^m$ we have

$$\|P_{E_0} T|E_k\| \leq \frac{1}{2} (1 - b)^2.$$

Thus $\|P_{E_0} T|E_k\| \geq \frac{1}{2} (1 - b)$ and hence, by Cor. 2.2, there is an $x_0 \in E_3$, $\|x_0\| = 1$, with the property that

$$\|P_{E_0 \perp x_0} T x_0\| \geq \frac{1}{2} (1 - b),$$

and we get a contradiction just as before, which completes the proof.

3. Mixing properties of operators and compact groups of operators in $R^n$.

The theorem below describes the mixing properties of an operator $T$ in terms of $m(T)$ and its trace.

Theorem 3.1. There is a constant $c > 0$ such that for every $n \geq 4$ and for every $T \in L(R^n)$

$$T \in M_n(c \|nm(T) - \|tr T\|/\log n),$$

Proof. First observe that since

$$\|nm(T) - \|tr T\| \leq nm(T - (n^{-1}) tr T) Id_n,$$
for $T \in L(R^n)$ and since $T$ has the same mixing properties as $T - \lambda I_{dp}$ for every $\lambda \in R$, it is enough to prove the theorem in the case when $\text{tr} T = 0$. In order to simplify the notation in what follows we shall assume that $n = 2^{10} p$, $p = 1, 2, \ldots$

Fix $T \in L(R^n)$ with $\text{tr} T = 0$ and let

$$T = \sum_{i=1}^{n} \lambda_i (T) \mathbf{e}_i \mathbf{e}_i^T$$

be its polar decomposition. We shall consider the following three mutually exclusive cases:

A. Either $\lambda_{2^{55} 2^{26}}(T) \geq \frac{1}{2} m(T)$ or $\lambda_{2^{55} 2^{26}} \leq \frac{1}{4} m(T)$.

B. $A$ does not hold and $\sum_{i=1}^{n} \lambda_i (T) \leq (1 + \frac{1}{4}) nm(T)$.

C. Neither $A$ nor $B$ hold.

Case A. Note that, by Lemma 2.6, we have

$$T \in M_a(2^{-10} n, 2^{-2} m(T)) = \tilde{M}_a(2^{-10} nm(T))$$

and we are done.

Case B. First observe that

$$\sum_{i=1}^{n} \lambda_i (T) \leq \sum_{i=1}^{n} \lambda_i (T) - \sum_{i=1}^{n} \lambda_i (T) \leq (1 + \frac{1}{2}) nm(T) - \frac{1}{4} nm(T) < \frac{1}{2} nm(T)$$

and therefore $|tr T| |E_1| < \frac{1}{2} nm(T)$, where $E_1 = [u_1, \ldots, u_{2^{55}}]$. Thus

$$|tr T| |E_1| < \frac{1}{2} nm(T)$$

which obviously implies that $T \in \tilde{M}_a(2^{-10} nm(T))$, concluding the proof in this case.

Case C. First note that

$$\sum_{i=1}^{n} \lambda_i (T) > \frac{1}{2} (1 + \frac{1}{2}) nm(T)$$

and $\lambda_i \geq m(T)$ for $i = 1, \ldots, n/2$. Therefore

$$\sum_{i=1}^{n} \lambda_i (T) - m(T) > \frac{nm(T)}{128}$$

Hence there is a $j \leq \log_2(n/2)$ with the property that

$$\lambda_j(T) - m(T) > \frac{nm(T)}{128 \log_2(n/2)}$$

which means that

$$||T x|| > (1 + 2^{10} \log_2(n/2))^{-1} n m(T)$$

for $x \in [u_1, \ldots, u_j]$. Since $||T x|| \leq m(T)$ for $x \in [u_{j+1}, \ldots, u_n]$, by Lemma 2.6, we infer that

$$T \in M_a(2^{-10} m(T) \log_2(n/2) < \tilde{M}_a(2^{-10} m(T)/\log n)$$

which completes the proof of the theorem.

As an easy consequence of the last theorem we have the following

**Theorem 3.2.** For each $a \in (0, 1)$ there is a $c_a > 0$ such that for every $n \geq \max \{4a^{-1}, 4(1-a)^{-1}\}$ and for every $T \in L(R^n)$

$$T \in \tilde{M}_a(c_a |\lambda_1(T) - |tr T|/\log n)$$

**Proof.** If $|\lambda_1(T) - |tr T| > \frac{1}{2} |\lambda_1(T) - |tr T||^{-1}$ then, by Lemma 2.6, for $0 < a \leq \frac{1}{2}$

$$T \in M_a(\frac{1}{2} \|\lambda(T) - |tr T|\|^{-1} < \tilde{M}_a(2^{-6} \|\lambda_1(T) - |tr T||^{-1})$$

and we are done. If this is not the case then we have

$$nm(T) - |tr T| \geq |\lambda_1(T) - |tr T|| - |\lambda_1(T) - nm(T)|$$

and we infer that

$$nm(T) - |tr T| \geq |\lambda_1(T) - |tr T||$$

Hence by the previous theorem

$$To \tilde{M}_a(c_a |nm(T) - |tr T||/\log n) < \tilde{M}_a(c_a |nm(T) - |tr T||/\log n)$$

which concludes the proof in the case $0 \leq a \leq \frac{1}{2}$. The other case can be obtained in a similar way.

**Remark 3.3.** It may be of some value to note that the constant $c_a$ in
the theorem above may be taken to be $2^{-n}a$ (resp. $2^{-n}(1-a)$) for $a$ sufficiently close to 0 (resp. 1).

Now, we turn our attention to compact groups of operators acting on $R^n$.

**Theorem 3.4.** There is a constant $c > 0$ such that for every $n \geq 4$ and every $T \in \mathfrak{G}_n$,

$[T, T^{-1}] \cap \mathfrak{M}_n(c(n-\text{tr} T)/\log n) \neq \emptyset$.

In particular,

$G \cap \mathfrak{M}_n(c(n-\tau(G))/\log n) \neq \emptyset$.

for every $G \in \mathfrak{G}_n$.

**Proof.** Obviously, it is enough to prove the first part of the theorem.

To this end, let $T \in \mathfrak{G}_n$. Since either $\lambda_{w21}(T)$ or $\lambda_{w21}(T^{-1})$ is not smaller than 1 and since $\text{tr} T = \text{tr} T^{-1}$, replacing perhaps $T$ by $T^{-1}$, we may assume that $\lambda_{w21}(T) \geq 1$. Thus by the previous theorem, since $\text{tr} T \leq n$, we have

$T \in \mathfrak{M}_n(c/\lambda_{w21}(T)-\text{tr} T)/\log n) \leq \mathfrak{M}_n(c(n-\text{tr} T)/\log n)\),

which completes the proof.

**Remark 3.5.** Let us observe that Th. 3.4 answers (at least partially) a problem posed by S. J. Szarek in [9].

In the sequel, we shall need the following notation: if $X_n$ is an n-dimensional linear space and $\langle \cdot, \cdot \rangle$ is a scalar product on $X_n$ then by $M_{a1,1}(X_n)$, for $a \geq 0$, we shall denote the corresponding class of subspace mixing operators in $L(X_n)$ with respect to $\langle \cdot, \cdot \rangle$. Also, by $\mathfrak{G}(X_n)$ (resp. $\mathfrak{K}(X_n)$) we shall denote the set (resp. the union) of all compact groups of operators acting on $X_n$.

The following result is just a small modification of the theorem above.

**Theorem 3.6.** There is a constant $c > 0$ such that for every n-dimensional linear space $X_n$,

(i) $T \in \mathfrak{K}(X_n)$ implies $\langle T, T^{-1} \rangle \cap \mathfrak{M}_{a1,1}(c(n-\text{tr} T)/\log n) \neq \emptyset$ for every scalar product $\langle \cdot, \cdot \rangle$ on $X_n$. In particular,

(ii) $\mathfrak{G} \cap \mathfrak{M}_{a1,1}(c(n-\tau(G))/\log n) \neq \emptyset$ for every $G \in \mathfrak{K}(X_n)$ and every scalar product $\langle \cdot, \cdot \rangle$ on $X_n$.

4. Compact groups of operators with relatively large values of $\tau(G)$. In view of the results of the previous section it may be of some interest to give a more detailed description of those groups $G \in \mathfrak{G}$ for which $\tau(G)$ is relatively large, i.e. for which $n-\tau(G)$ is relatively small. To this end let us recall some basic facts about compact groups of operators acting on $R^n$.

Let $G \in \mathfrak{G}_n$. Then:

1° There is another scalar product $\langle \cdot, \cdot \rangle_1$ on $R^n$ such that $G$ is a group of isometries of $(R^n, \|\cdot\|_1)$, where $\|x\|_1 = |\langle x, x_1 \rangle|^2$ for $x \in R^n$.

2° There is a decomposition of $R^n$ into an $\langle \cdot, \cdot \rangle_1$-orthogonal sum of subspaces $R^n = E_1 \oplus \cdots \oplus E_k$

with the properties:

(i) $T(E_i) = E_i$ for every $T \in G$ and every $i = 1, \ldots, k$.

(ii) $G$ acts irreducibly on each $E_i$, $i = 1, \ldots, k$, i.e. the group $G_{E_i} = \{T \in G : T \subseteq E_i\}$ does not admit a nontrivial invariant subspace for $i = 1, \ldots, k$. 

(iii) If $U \subseteq L(E_i)$ commutes with every element of $G_{E_i}$ then

$\langle UX, \gamma \rangle = (\dim E_i)^{-1} \langle U, \gamma \rangle \text{ tr } U$

for every $x \in E_i$ and $\|\gamma\|_1 = 1$ and every $i = 1, \ldots, k$.

For a fixed group $G \in \mathfrak{G}_n$, every decomposition of $R^n$ in the form (10) satisfying 2°-iii is said to be a decomposition of $R^n$ into $G$-irreducible subspaces. The properties 1°, 2°, and the complex (stronger) version of 2°-iii can be found for example in [6]. We sketch the proof of the real case of 2°-iii. Let $U$ be an operator in $L(E_i)$ which commutes with $G_{E_i}$. Then, by the same argument as in the complex case, we infer that $U = \lambda_1 \tilde{U}$, where $\lambda_1 \in \mathbb{R}$ and $\tilde{U}$ is an isometry on $(E_i, \|\cdot\|_1)$. Set $S = U - \text{Id}_{E_i}$. Since $S$ commutes with $G_{E_i}$, by the same token we deduce that $S = \lambda_2 \tilde{S}$ with $\lambda_2 \in \mathbb{R}$ and $\tilde{S}$ being an isometry on $(E_i, \|\cdot\|_1)$. We have for $x \in E_i$, with $\|\gamma\|_1 = 1$,

$\lambda_2^2 = 1 - 2 \langle UX, \gamma \rangle + \|\gamma\|_1 = \lambda_1^2 + 1 - 2 \langle UX, \gamma \rangle_1$.

Hence $\langle UX, \gamma \rangle = \text{ const}$ for $x \in E_i$ with $\|\gamma\|_1 = 1$. Now, the exact value of $\langle UX, \gamma \rangle_1$ follows from the formula

$\text{tr } U = \dim E_i \langle UX, \gamma \rangle_1 \mu_{G_{E_i}}(\gamma)$

where $\mu_{G_{E_i}}$ denotes the normalized Lebesgue measure on the unit sphere $S^{n-1}$ of $(E_i, \|\cdot\|_1)$.

**Proposition 4.1.** There is a constant $c > 0$ such that for every $n \in \mathbb{N}$ and for every $G \in \mathfrak{G}_n$ the cardinality of the set of 1-dimensional subspaces in every decomposition of $R^n$ into $G$-irreducible subspaces is at least $n-c(n-\tau(G))$.

**Proof.** Let $G \in \mathfrak{G}_n$. In view of 1° above, without any loss of generality we may and shall assume that $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1$. Let

$R^n = E_1 \oplus \cdots \oplus E_k$
be a decomposition of $R^k$ into $G$-irreducible subspaces. Let $n_i = \dim E_i$, $i = 1, \ldots, k$. Obviously, $\sum_{i=1}^{k} n_i = n$. Assume that $n_1 \geq \cdots \geq n_k$ and let $j = \max\{i: n_i > 1\}$. Note that to prove the proposition it is enough to prove that there is a constant $c > 0$ with the property

\[ j \leq n_j \leq c(n - r(G)). \]

To this end, in order to simplify the notation, we shall assume that all $n_i$'s are even for $i < j$. Let $m = \sum_{i=1}^{j-1} n_i$, and in each $E_i$, $i \leq j$, choose arbitrary two orthogonal subspaces $F_i^0$ and $F_i^1$ with $\dim F_i^0 = \dim F_i^1 = n_i/2$. Let

\[ F_i = \bigoplus_{i=1}^{j-1} F_i^0, \quad F_j = \bigoplus_{i=1}^{j-1} F_i^1. \]

Consider the operators $U_i \in L(E_i)$ defined by

\[ U_i = \int T^{-1} \{ E_j \circ P_j \circ T \} E_i \, d\alpha(T) \]

for $i = 1, \ldots, j$. Obviously, $U_i$ commutes with every element of $G_{E_i}$. By 2(iii),

\[ \langle U_i, x, x \rangle = n_i^{-1} \tr U_i \text{ for every } x \in S_{E_i}. \]

On the other hand, it follows from (11) that $\tr U_i = \dim F_i = n_i/2$. Hence

\[ \langle U_i, x, x \rangle = \frac{1}{2} \]

for every $i = 1, \ldots, j$ and every $x \in S_{E_i}$. Set $S_{F_j}^2 = S_j$ for $i = 1, \ldots, j$ and observe that by (12)

\[ m/2 = \sum_{i=1}^{j} n_i \left\{ \langle U_i, x, x \rangle \right\} d\mu_{\text{hyp}}(x) = \sum_{i=1}^{j} n_i \left\{ \langle T^{-1} \{ E_j \circ P_j \circ T \} E_i, x, x \rangle \right\} d\mu_{\text{hyp}}(x) = \frac{1}{2} \sum_{i=1}^{j} \| P_j \|_{\text{hyp}} \| P_j \|_{\text{hyp}} d\mu_{\text{hyp}}(x) d\mu_{\text{hyp}}(T) \]

and

\[ = \frac{1}{2} \sum_{i=1}^{j} \| P_j \|_{\text{hyp}} \| P_j \|_{\text{hyp}} d\mu_{\text{hyp}}(T) = \frac{1}{2} \| P_j \|_{\text{hyp}} \| P_j \|_{\text{hyp}} d\mu_{\text{hyp}}(T). \]

Therefore there exists a $T_0 \in G$ such that

\[ \| P_j \|_{\text{hyp}} \| T_0 P_j \|_{\text{hyp}} \| T_0 P_j \|_{\text{hyp}} \| T_0 P_j \|_{\text{hyp}} \geq m/2. \]

Set $E = E_1 \oplus \cdots \oplus E_j$ and let $\tilde{T} = T_0|E$. Obviously $\tilde{T}$ is an isometry in $L(E)$. Let $\{u_i: i = 1, \ldots, m/2\}$ be an orthonormal basis in $F_1$, and let $\{u_i: i = m/2 + 1, \ldots, m\}$ be an orthonormal basis in $F_2$. Since $\tilde{E}$ is the orthogonal sum of $F_1$ and $F_2$ we infer that $\{u_i: i = 1, \ldots, m\}$ is an orthogonal basis in $\tilde{E}$. Let $(\tilde{u}_i)_{i=1}^{m}$ be the matrix representation of $\tilde{T}$ with respect to this basis.

Since $\| \tilde{T} \|_{\text{hyp}} = m$, by (13) we have

\[ \| \tilde{T} \|_{\text{hyp}} \leq \sqrt{m}, \]

which yields (1) and concludes the proof.

Now we are ready to derive the main result of this section.

Theorem 42. There is a constant $c > 0$ such that for every $n \in N$ and for

\[ \text{every group } G \in \mathcal{G}, \text{there is a subspace } E_n < R^k \text{ with } \dim E_n \geq n - c(n - r(G)) \]

on which $G$ acts trivially.

Proof. Fix $G \in \mathcal{G}$. Replacing perhaps $G$ by the group $G_0 \cap G$, we may assume that $-\text{Id}_{E_n \oplus E_n} \in G$. As in the proof of the previous proposition, without any loss of generality we may assume that $G$ is a group of isometries of $R^k$. Since the trace is a continuous functional on $L(R^k)$ equipped with the Hilbert–Schmidt norm and its norm is equal to $\sqrt{n}$ and since

\[ G \subset S_n \iff \{ T \in L(R^k): \| T \|_{\text{hyp}} = \sqrt{n} \} \]

we infer that there is a constant $c_1 \in (0, 1)$ with the property that if $T \in S_n$ and $\| T^{-1} \text{Id}_{E_n \oplus E_n} \|_{\text{hyp}} \leq \frac{1}{2} \sqrt{n}$.

Now, observe that if $T < c_1 n$, then the theorem holds trivially with $c = 2(1 - c_1)$ and therefore it is enough to prove it for groups satisfying $T(G) \geq c_1 n$. Let $G$ be such a group and define

\[ G_0 = \{ T \in G: \| T \| < c_1 n \} = \{ T \in G: \| T \| > 0 \}. \]

We claim that $G_0$ is a subgroup of $G$, and $G = -G_0 \cup G_0$. Indeed, the second property of $G_0$ is trivial, and to see that $G_0$ is a subgroup, note that for $T_1, T_2 \in G_0$ we have

\[ \| T_1^{-1} T_2 \|_{\text{hyp}} \leq \frac{1}{2} \sqrt{n}, \] hence $\| \text{tr} (\text{Id}_{E_n \oplus E_n} - T_1 T_2) \| = n - \| T_1 T_2 \| < \frac{1}{2} n$, which implies $\| T_1 T_2 \| > 0$ and therefore $T_1 T_2 \in G_0$. In particular, we have $t(G_0) = t(G) \geq c_1 n$.

Now, let $E_n = E_1 \oplus \cdots \oplus E_n$ be a decomposition of $R^k$ into $G$-irreducible subspaces. Let $\{E_i\}_{i=1}^{k}$ be the set of all 1-dimensional subspaces in this decomposition. By Prop. 4.1, we have $k \geq n - c_2 (n - r(G))$, where $c_2$ is some numerical constant. Let $u_0 \in E_i$, $\| u_0 \| = 1$ for $i = 1, \ldots, k$, and define

\[ A = \{ i: T u_0 = u_0 \text{ for every } T \in G_0 \}, \quad B = \{ 1, \ldots, k \} \setminus A. \]
Set $F = \{ u_i : i \in B \}$ and observe that if $i \in B$ then
\[
\left\{ \left\langle Tu_i, u_i \right\rangle \right\}_{d_0} (T) = 0.
\]
Hence
\[
\left\{ \text{tr}(T|F) \right\}_{d_0} = \sum_{i \in B} \left\{ \left\langle Tu_i, u_i \right\rangle \right\}_{d_0} (T) = 0.
\]
Thus there is a $T_0 \in G_0$ with $\text{tr}(T_0|F) = 0$. We have
\[
\rho(G) \leq \text{tr} T_0 \leq \left( \text{tr}(T_0|F) + \dim F \right) \leq n - \dim F,
\]
which implies that
\[
\# B = \dim F \leq n - \rho(G).
\]
This yields
\[
\# A = k - \# B \geq n - c_2(n - \rho(G)) = n - (c_2 + 1)(n - \rho(G)).
\]
Now, setting $c = c_2 + 1$ and $E_0 = \{ u_i : i \in A \}$ concludes the proof.

**Remark 4.3.** The theorem above states that if $n - \rho(G)$ is relatively small then $G$ acts trivially on a large-dimensional subspace.

**5. Application to the pathological properties of Gloskin spaces.** We begin with the crucial result due to S. J. Szarek.

**Theorem 5.1 (S. J. Szarek).** There is a constant $c > 0$ such that for each $n \geq 4$ there is a norm $\| \cdot \|_n$ on $\mathbb{R}^n$ with the property
\[
\| T \|_{n} \geq \frac{cn}{\sqrt{n \log n}}
\]
for every operator $T \in \mathcal{M}_n(\mathbb{R})$.

**Remark 5.2.** In fact, Szarek proved slightly more, namely, that Th. 5.1 holds for a "vast majority" of norms defined by E. D. Gloskin in [3].

Combining Th. 5.1 with Th. 3.4 we immediately get an answer to a problem posed by S. J. Szarek in [3].

**Theorem 5.3.** There is a constant $c > 0$ such that for every $n \geq 4$ there is an $n$-dimensional Banach space $X_n$ with the property that for every $T \in \mathcal{G}(X_n)$
\[
\max \left\{ \| T \|_{X_n}, \| T^{-1} \|_{X_n} \right\} \geq \frac{c(n - \text{tr} T)}{n^{1/2} \log^{3/2} n},
\]
In particular,
\[
\sup \left\{ \| T \|_{X_n} : T \in G \right\} \geq \frac{c(n - \rho(G))}{n^{1/2} \log^{3/2} n}
\]
for every $G \in \mathcal{G}(X_n)$.

As an application of the result above we shall deduce slightly weaker versions (they differ by a logarithmic factor) of results due to E. D. Gloskin [4] and S. J. Szarek [3].

**Corollary 5.4.** There is a constant $c > 0$ such that for each $n \geq 4$ there is an $n$-dimensional Banach space $X_n$ with the property
\[
\| P \|_{X_n} \geq \frac{ck}{n^{1/2} \log^{3/2} n}
\]
for every rank $k$ projection $P$, with $k \leq n/2$.

Proof. For every such projection $P$ it is enough to consider the group $G_0$ consisting of $\text{Id}_{X_n} - 2P$ and $\text{Id}_{X_n}$, and apply Th. 5.3 (note that $\text{tr}(\text{Id}_{X_n} - 2P) = n - 2k$).

As another application of Th. 5.3 let us mention the following weaker version ("up to a logarithmic factor") of a result proved by S. J. Szarek in [4]:

**Corollary 5.5.** There is a constant $c > 0$ such that for every $n \geq 2$ there is a 2n-dimensional Banach space $X_{2n}$ with the property that for every complex $n$-dimensional Banach space $Y_n$ the Banach–Mazur distance over the reals $d_{BM}(X_{2n}, Y_n)$ is at least $cn^{1/2} \log^{-3/2} n$.

Proof. Let $X_{2n}$ be a Banach space satisfying Th. 5.3 and let $T : X_{2n} \to Y_n$ be an "$R$-linear" operator realizing the distance between $X_{2n}$ and $Y_n$.

Define
\[
A = T^{-1} \circ \text{Id}_{X_n} \circ T.
\]
Since $A^2 = -\text{Id}_{X_n}$, it can easily be seen that $\text{tr} A = 0$. Now, it is enough to apply Th. 5.3 for the group $\{ A, -A, -\text{Id}, \text{Id} \}$ and to note that on the other hand $\| A \|_{X_{2n}} \leq d_{BM}(X_{2n}, Y_n)$.

The next result is a more application of Th. 5.3 concerning a "vast majority" of Gloskin spaces.

**Theorem 5.6.** There is a constant $c > 0$ such that for every $n \geq 4$ there is an $n$-dimensional Banach space $X_n$ with the property if
\[
G \in \mathcal{G}(X_n) \quad \text{and} \quad \sup \left\{ \| T \|_{X_n} : T \in G \right\} = A
\]
then $G$ acts trivially on some subspace of $X_n$ of dimension $n - cAn^{1/2} \log^{3/2} n$.

Proof. The theorem follows directly from Th. 5.3 and Th. 4.2.

**Remark 5.7.** (i) Note that Th. 5.6 is nontrivial only if $A < c^{-1} n^{1/2} \log^{-3/2} n$.

(ii) On the other hand, Th. 5.6 states that the situation in Corollary 5.4 is typical ($G_0$ acts trivially on $\ker P$, while $\dim \ker P = n - k$).

Now, we shall deal with the problem of representing an operator $T$...
acting on a finite-dimensional Banach space as a sum of some number of "small rank" operators with "small" norms.

Theorem 5.8. There is a constant \( c > 0 \) such that for every \( n \geq 4 \) there is an \( n \)-dimensional Banach space \( X_n \) with the property that for every \( T \in L(X_n) \) if \( T = \sum_{i=1}^k T_i \) with rank \( T_i \leq n/2 \) for \( i = 1, \ldots, k \), then

\[
\sup \left\{ \|T_i\|_{X_n} : i = 1, \ldots, k \right\} \geq c \frac{\sqrt{\text{tr}(T)}}{kn^{1/2} \log^{1/2} n}.
\]

In particular, if \( \text{Id}_{X_n} = \sum_{i=1}^k T_i \) with rank \( T_i \leq n/2 \) for \( i = 1, \ldots, k \), then

\[
\sup \left\{ \|T_i\|_{X_n} : i = 1, \ldots, k \right\} \geq c \frac{n^{1/2}}{k \log^{1/2} n}.
\]

Proof. First observe that for at least one of the \( T_i \)’s we have \( \|T_i\| \geq k^{-1} \text{tr}(T) \) while \( \lambda_{2(n+1)}(T_i) = 0 \), and next apply Th. 3.1 and Th. 5.1.

Remark 5.9. A standard argument shows that Th. 5.8 implies the following result due to S. J. Szarek [11]:

There is a constant \( c > 0 \) such that for every \( n \in \mathbb{N} \) there is an \( n \)-dimensional Banach space \( X_n \) with the property that whenever \( X_n \) is \( C_1 \)-isomorphic to a \( C_2 \)-complemented subspace of an \( m \)-dimensional Banach space \( Y_m \) then the basis constant of \( Y_m \) is greater than

\[
c C_1^{-1} C_2^{-1} m^{-1} n^{1/2} \log^{-1/2} n.
\]

This shows that the estimates obtained from the corresponding positive result due to A. Pelczyński [7] are "up to a logarithmic factor" the best possible.

For a finite-dimensional Banach space \( X \) denote by \( \mathcal{N}(X) \) the set of those compact groups of operators on \( X \) which act essentially nontrivially on \( X \). Define

\[
\text{ws}(X, G) = \sup \left\{ \|T_i\|_G : T \mapsto G \right\}
\]

for every \( G \in \mathcal{N}(X) \) and the weak symmetry constant \( \text{ws}(X) \) of \( X \) by

\[
\text{ws}(X) = \inf \{ \text{ws}(X, G) : G \in \mathcal{N}(X) \}.
\]

Obviously, for every finite-dimensional Banach space \( X \):

(i) \( \text{ws}(X) = \inf \|d(X, Y) : \dim Y = \dim X \) and \( \text{ws}(Y) = 1 \),

(ii) \( \text{ws}(X) \leq \text{ws}(X) \) where \( \text{ws}(X) \) denotes the symmetry constant of \( X \) defined by D. J. H. Garling and Y. Gordon in [2] (cf. also [5]) and

(iii) \( \text{ws}(X) \leq \sqrt{\dim X} \).

Th. 5.6 yields the following result which generalizes "up to a logarithmic" factor the main result of [5].

Subspace mixing properties

Theorem 5.10. There is a constant \( c > 0 \) such that for each \( n \geq 4 \) there is an \( n \)-dimensional Banach space \( X_n \) with

\[
\text{ws}(X_n) \geq c n^{1/2} \log^{-1/2} n.
\]

Remark 5.11. One can construct for each \( n \in \mathbb{N} \) an \( n \)-dimensional Banach space \( X_n \) with \( \text{ws}(X_n) = 1 \) and \( \text{ws}(X_n) \geq c \sqrt{n} \) for some numerical constant \( c > 0 \).

Acknowledgements. I would like to thank Staszez Szarek for interesting discussions concerning Gluskin spaces and Nassif Ghousoub for a fascinating private lecture on some deep historical roots of an actual problem. Also, I would like to thank Tadek Figiel for plenty of remarks concerning the manuscript of this paper.

References

[1] J. Bourgain, A complex Banach space such that \( X \) and \( \overline{X} \) are not isomorphic, preprint.


Institute of Mathematics, Polish Academy of Sciences

Received March 25, 1986

Revised version July 30, 1986