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Subspace mixing properties of operators in R^n with applications to Gluskin spaces

by

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Abstract. There is a constant $c>0$ such that for every $n\in N$ there is an n-dimensional Banach space X_n with the property that whenever G is a compact group of operators acting on X_n then

$$\sup\{\|T\|_{X_n}:\ T\in G\}\geqslant \frac{c(n-t(G))}{n^{1/2}\log^{3/2} n}$$

where $t(G)=\inf\{|\operatorname{tr} T|:\ T\in G\}$.

E. D. Gluskin in [3] introduced a class of random n-dimensional Banach spaces in order to prove that the Banach–Mazur diameter of the set of all n-dimensional Banach spaces is of order n. Later the same author in [4] and independently S. J. Szarek [8] used different variants of spaces defined in [3] to prove the existence of finite-dimensional Banach spaces with the "worst possible" basis constants. Another variant of Gluskin spaces was used by the author in [5] to construct finite-dimensional Banach spaces with the "worst possible" symmetry constants. The importance of the notion of "subspace mixing operators on R^n" in the context of Gluskin spaces was implicit in [4] and "almost explicit" in [5]. The final step in this direction was done by S. J. Szarek [9], who proved that a "vast majority" of Gluskin spaces enjoy the property that every subspace mixing operator on such a space has large norm. The subspace mixing property was used in that paper to prove the existence of finite-dimensional Banach spaces with two essentially different complex structures. Later on, the techniques developed in [9] were used by the same author to construct infinite-dimensional Banach spaces with some pathological properties [10], [11]; however, the credit for the first use of Gluskin spaces to construct pathological infinite-dimensional Banach spaces should be given to J. Bourgain [1].

In this paper we study subspace mixing properties of operators in R^n with special attention turned to operators which belong to a compact group of operators. The main difference in the approach between [9] and this paper lies in the fact that in [9] the author studied the subspace mixing property of an operator T in terms of certain "distances" of T to the line $\{\lambda \mathrm{Id}\}_{\lambda\in R}$ while we study the same property in terms of some other "distances" of T to the operator $n^{-1}(\operatorname{tr} T)\operatorname{Id}$, and the values of the latter "distances" are easier to handle.

Finally, we use the results on subspace mixing properties to prove the existence of finite-dimensional Banach spaces with some pathological properties, or more precisely to establish some pathological properties of a "vast majority" of Gluskin spaces. The main result in this direction can be stated as follows (cf. Th. 5.3 below):

There is a constant $c > 0$ such that for every $n \in N$ there is an n-dimensional Banach space X_n with the property that whenever G is a group of linear operators acting on X_n,

$$\sup\{\|T\|_{X_n}:\ T \in G\} \geqslant \frac{c(n-t(G))}{n^{1/2}\log^{3/2} n}$$

where $t(G) = \inf\{|\operatorname{tr} T|:\ T \in G\}$.

This answers a question posed by S. J. Szarek in [9].

Let us note that the theorem quoted above contains "up to a logarithmic factor" the results proved in [4], [5], [8], [9].

1. Notation and preliminaries. We shall use the standard notation. By $e_1, \ldots, e_n$ we shall denote the standard unit vector basis in $\boldsymbol{R}^n$ and by $\|\cdot\|$ the standard Euclidean norm in $\boldsymbol{R}^n$. S^{n-1} will stand for the unit sphere in $\boldsymbol{R}^n$ while μ_{n-1} will denote the normalized surface Lebesgue measure on S^{n-1}. O_n and h_n will stand for the orthogonal group on $\boldsymbol{R}^n$ and the normalized Haar measure on it respectively. If E is a linear subspace of $\boldsymbol{R}^n$ then by P_E and $E^\perp$ we shall denote the orthogonal projection on E and on the orthogonal complement of E in $\boldsymbol{R}^n$ respectively.

We shall say that a linear operator $T \in L(\boldsymbol{R}^n)$ has the (α, β)*-subspace mixing property* for $\alpha, \beta \geqslant 0$ iff there is a linear subspace $E \subset \boldsymbol{R}^n$ with $\dim E \geqslant \alpha$ such that

$$\|P_{E^\perp} Tx\| \geqslant \beta \|x\| \quad \text{for every } x \in E.$$

The set of all operators in $L(\boldsymbol{R}^n)$ having the (α, β)-subspace mixing property will be denoted by $M_n(\alpha, \beta)$. Obviously for $T \in L(\boldsymbol{R}^n)$:

(i) $T \notin \bigcup_{\alpha,\beta>0} M_n(\alpha, \beta)$ iff $T = \lambda \operatorname{Id}_{\boldsymbol{R}^n}$ for some $\lambda \in \boldsymbol{R}$.

(ii) $T \in M_n(\alpha, \beta)$ iff $\lambda T \in M_n(\alpha, |\lambda|\beta)$ for every $\lambda \in \boldsymbol{R}$.

(iii) $T \in M_n(\alpha, \beta)$ iff $T - \lambda\operatorname{Id}_{\boldsymbol{R}^n} \in M_n(\alpha, \beta)$ for every $\lambda \in \boldsymbol{R}$
iff $T - \lambda\operatorname{Id}_{\boldsymbol{R}^n} \in M_n(\alpha, \beta)$ for some $\lambda \in \boldsymbol{R}$.

We shall write $T \in \widetilde{M}_n(\gamma)$ iff $T \in M_n(\alpha, \beta)$ for some $\alpha, \beta \geqslant 0$ with $\alpha\beta \geqslant \gamma$.

Recall that every operator $T \in L(\boldsymbol{R}^n)$ can be written in the form

$$(*) \qquad T = \sum_{i=1}^{n} \lambda_i(T) \langle \cdot, u_i \rangle v_i$$

where $\lambda_1(T) \geqslant \ldots \geqslant \lambda_n(T) \geqslant 0$ and $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ are orthonormal systems in $\boldsymbol{R}^n$. Any representation of an operator $T \in L(\boldsymbol{R}^n)$ in the form $(*)$ will be called a *polar decomposition* of T. It is well known that while the polar decomposition of an operator T need not be unique the sequence $\{\lambda_i(T)\}_{i=1}^n$ is uniquely determined by T.

For a subset $A \subset \boldsymbol{R}^n$, $[A]$ will denote the linear hull of A. Sometimes, we shall identify an operator $T \in L(\boldsymbol{R}^n)$ with its matrix representation $(a_{ij})_{i,j=1}^n$ with respect to the basis $\{e_i\}_{i=1}^n$. For $T \in L(\boldsymbol{R}^n)$ we define

$$m(T) = \begin{cases} \lambda_k(T) & \text{for } n = 2k, \\ \lambda_{k+1}(T) & \text{for } n = 2k+1. \end{cases}$$

For $T \in L(\boldsymbol{R}^n)$, $\|T\|_{\mathrm{HS}}$ will denote the Hilbert–Schmidt norm of T and $\operatorname{tr} T$ will denote its trace. We shall make use of the following trivial equalities:

$$\|T\|_{\mathrm{HS}}^2 = \sum_{i=1}^{n} \lambda_i^2(T) = n \int_{S^{n-1}} \|Tx\|^2 \, d\mu_{n-1}(x),$$

$$\operatorname{tr} T = \sum_{i=1}^{n} \lambda_i(T) \langle u_i, v_i \rangle = n \int_{S^{n-1}} \langle Tx, x \rangle \, d\mu_{n-1}(x)$$

for every $T \in L(\boldsymbol{R}^n)$ and every polar decomposition of T.

Finally, we define

$$\delta(T) = \inf\{\|T - \lambda \operatorname{Id}_{\boldsymbol{R}^n}\|_{\mathrm{HS}}:\ \lambda \in \boldsymbol{R}\}$$

for every $T \in L(\boldsymbol{R}^n)$. It can easily be seen that for $T \in L(\boldsymbol{R}^n)$

$$\delta^2(T) = \|T - (n^{-1}\operatorname{tr} T)\operatorname{Id}_{\boldsymbol{R}^n}\|_{\mathrm{HS}}^2 = \|T\|_{\mathrm{HS}}^2 - n^{-1}(\operatorname{tr} T)^2.$$

By $\mathscr{G}_n$ we shall denote the set of all compact groups of operators acting on $\boldsymbol{R}^n$. We define

$$\mathscr{U}\mathscr{G}_n = \bigcup_{G \in \mathscr{G}_n} G, \qquad t(G) = \inf\{|\operatorname{tr} T|:\ T \in G\} \quad \text{for } G \in \mathscr{G}_n.$$

Since $\operatorname{tr}(\operatorname{Id}_{\boldsymbol{R}^n}) = n$ one has $t(G) \leqslant n$ for $G \in \mathscr{G}_n$ and $t(G) = n$ iff $G \subset \{\operatorname{Id}_{\boldsymbol{R}^n}, -\operatorname{Id}_{\boldsymbol{R}^n}\}$. For a compact group G we shall denote by h_G the normalized Haar measure on G.

We shall say that a group $G \in \mathscr{G}_n$ *acts trivially on a subspace* $E \subset \boldsymbol{R}^n$ iff $T|E = \varepsilon \operatorname{Id}_E$, with $\varepsilon \in \{1, -1\}$, for every $T \in G$. Also, we shall say that a group $G \in \mathscr{G}_n$ *acts essentially nontrivially* on $\boldsymbol{R}^n$ iff it does not act trivially on any subspace of $\boldsymbol{R}^n$ with positive dimension.

The letter c with indices or without will always stand for an absolute numerical constant, in general different in different places.

We shall deal only with spaces over the reals; however, all the results after suitable modification remain valid in the complex case.

2. Mixing properties of contractions in R^n. We begin with

PROPOSITION 2.1. *Let* $T \in L(R^n)$. *Then*

$$\int_{S^{n-1}} \|P_{[x]^\perp} T P_{[x]}\|_{HS}^2 \, d\mu_{n-1}(x) = \int_{S^{n-1}} \|P_{[x]^\perp} Tx\|^2 \, d\mu_{n-1}(x)$$

$$= \frac{1}{n(n+2)} \big((n+1)\|T\|_{HS}^2 - (\operatorname{tr} T)^2 - \operatorname{tr} T^2\big) \geqslant \frac{\delta^2(T)}{n+2}.$$

Proof. The first equality, as well as the inequality, is trivial. To prove the second equality set $T = (a_{ij})_{i,j=1}^n$ and observe that for $x = (x_1, \ldots, x_n) \in S^{n-1}$

$$\|P_{[x]^\perp} Tx\|^2 = \|Tx\|^2 - \langle Tx, x\rangle^2 = \|Tx\|^2 - \Big(\sum_{i,j=1}^n a_{ij} x_i x_j\Big)^2.$$

Hence

(1)
$$\int_{S^{n-1}} \|P_{[x]^\perp} Tx\|^2 \, d\mu_{n-1}(x) = \int_{S^{n-1}} \|Tx\|^2 \, d\mu_{n-1}(x) - \int_{S^{n-1}} \Big(\sum_{i,j=1}^n a_{ij} x_i x_j\Big)^2 d\mu_{n-1}(x).$$

Since the first integral on the right-hand side is equal to $n^{-1}\|T\|_{HS}^2$, it remains to evaluate the second one. Expanding the square of the sum, skipping the integrals which are obviously equal to zero and taking into account that

$$\int_{S^{n-1}} x_i^4 \, d\mu_{n-1}(x) = \frac{3}{n(n+2)}, \qquad \int_{S^{n-1}} x_i^2 x_j^2 \, d\mu_{n-1}(x) = \frac{1}{n(n+2)}$$

for $i, j = 1, \ldots, n$ and $i \neq j$, we have

(2)
$$\int_{S^{n-1}} \Big(\sum_{i,j=1}^n a_{ij} x_i x_j\Big)^2 d\mu_{n-1}(x) = \frac{1}{n(n+2)} \Big(3\sum_{i=1}^n a_{ii}^2 + \sum_{i \neq j} a_{ii} a_{jj} + \sum_{i \neq j} a_{ij}^2 + \sum_{i \neq j} a_{ij} a_{ji}\Big)$$

$$= \frac{1}{n(n+2)} \Big(\sum_{i,j=1}^n a_{ij}^2 + \Big(\sum_{i=1}^n a_{ii}\Big)^2 + \sum_{i,j=1}^n a_{ij} a_{ji}\Big)$$

$$= \frac{1}{n(n+2)} \big(\|T\|_{HS}^2 + (\operatorname{tr} T)^2 + \operatorname{tr} T^2\big).$$

Combining (1) and (2) we obtain

$$\int_{S^{n-1}} \|P_{[x]^\perp} Tx\|^2 \, d\mu_{n-1}(x) = \frac{1}{n}\|T\|_{HS}^2 - \frac{1}{n(n+2)} \big(\|T\|_{HS}^2 + (\operatorname{tr} T)^2 + \operatorname{tr} T^2\big)$$

$$= \frac{1}{n(n+2)} \big((n+1)\|T\|_{HS}^2 - (\operatorname{tr} T)^2 - \operatorname{tr} T^2\big),$$

which concludes the proof.

Thus we have

COROLLARY 2.2. $T \in M_n\big(1, (n+2)^{-1/2}\delta(T)\big)$ *for every* $T \in L(R^n)$.

PROPOSITION 2.3. *For every k-dimensional subspace $E \subset R^n$ and every $T \in L(R^n)$*

$$\int_{O_n} \|P_{U(E^\perp)} T P_{U(E)}\|_{HS}^2 \, dh_n(U) = \frac{k(n-k)}{(n-1)n(n+2)} \big((n+1)\|T\|_{HS}^2 - (\operatorname{tr} T)^2 - \operatorname{tr} T^2\big)$$

$$\geqslant \frac{k(n-k)}{(n-1)(n+2)} \delta^2(T).$$

Proof. Since the integral does not depend on a particular choice of a space E but only on the dimension of E, let $E = [e_1, \ldots, e_k]$. We have

(3)
$$\int_{O_n} \|P_{U(E^\perp)} T P_{U(E)}\|_{HS}^2 \, dh_n(U) = \int_{O_n} \|U P_{E^\perp} U^{-1} T U P_E U^{-1}\|_{HS}^2 \, dh_n(U)$$

$$= \int_{O_n} \|P_{E^\perp} U^{-1} T U P_E\|_{HS}^2 \, dh_n(U).$$

Let $UTU^{-1} = (a_{ij}(U))_{i,j=1}^n$ for $U \in O_n$ and observe that for every $U_1 \in O_n$

$$\int_{O_n} a_{ij}^2(U) \, dh_n(U) = \int_{O_n} a_{ij}^2(UU_1) \, dh_n(U).$$

By a simple "change of coordinates" argument we get

(4)
$$\int_{O_n} a_{ij}^2(U) \, dh_n(U) = \int_{O_n} a_{lm}^2(U) \, dh_n(U)$$

for all $i \neq j$ and $l \neq m$. Let $a(T)$ be the common value of all integrals of the form (4). We have

(5)
$$\int_{O_n} \|P_{E^\perp} U^{-1} T U P_E\|_{HS}^2 \, dh_n(U) = \sum_{j=i}^k \sum_{i=k+1}^n \int_{O_n} a_{ij}^2(U) \, dh_n(U) = k(n-k)\, a(T).$$

On the other hand, Prop. 2.1 means that

(6)
$$(n-1)\, a(T) = \sum_{j=2}^n \int_{O_n} a_{1j}^2(U) \, dh_n(U)$$

$$= \frac{1}{n(n+2)} \big((n+1)\|T\|_{HS}^2 - (\operatorname{tr} T)^2 - \operatorname{tr} T^2\big).$$

Now, combining (3), (5) and (6) completes the proof.

The main result of this section is

THEOREM 2.4. $T \in M_n\big(\tfrac{1}{10}\delta^2(T), \delta(T)/\sqrt{5n}\big)$ *for every* $T \in L(R^n)$ *with* $\|T\| \leqslant 1$.

Proof. We shall assume that n is even. The case of n odd can be treated in a similar way. By Prop. 2.3, there is an $E \subset R^n$, $\dim E = n/2$, such that

(7)
$$\|P_{E^\perp} T P_E\|_{HS}^2 \geqslant \frac{n^2}{4(n-1)(n+2)} \delta^2(T) \geqslant \tfrac{1}{5}\delta^2(T).$$

Let $V = P_{E^\perp} T P_E$ and let

$$V = \sum_{i=1}^{n/2} \lambda_i(V)\langle\cdot, u_i\rangle v_i$$

be a polar decomposition of V. Obviously $\{u_i\}$ and $\{v_i\}$ are orthonormal systems in E and $E^\perp$ respectively and $\lambda_i(V) \leqslant 1$ for $i = 1, \ldots, n/2$. By (7) we have

$$\sum_{i=1}^{n/2} \lambda_i^2(V) \geqslant \tfrac{1}{5}\delta^2(T).$$

This implies that more than $\frac{1}{10}\delta^2(T)$ of the $\lambda_i(V)$'s are greater than $(5n)^{-1/2}\delta(T)$. Set

$$E_0 = [u_i\colon i = 1, \ldots, [\tfrac{1}{10}\delta^2(T)]+1],$$

where $[\frac{1}{10}\delta^2(T)]$ denotes the integer part of $\frac{1}{10}\delta^2(T)$. Then

$$\|P_{E_0^\perp} Tx\| \geqslant \|P_{E^\perp} Tx\| = \|Vx\| \geqslant (5n)^{-1/2}\delta(T)$$

for $x \in E_0$, which yields that $T \in M_n(\frac{1}{10}\delta^2(T), \delta(T)/\sqrt{5n})$ and completes the proof.

Remark 2.5. Observe that the theorem above gives (up to some constants) the best possible "mixing properties" for contractions in $L(\boldsymbol{R}^n)$ with values of δ "proportional" to $\sqrt{n}$, while it is far from the best for contractions with relatively small values of δ.

In the sequel we shall need the following

LEMMA 2.6. *Let $a > b > 0$, $m \in N$ and let $T \in L(\boldsymbol{R}^n)$ be such that there are two orthogonal subspaces $E_1, E_2 \subset \boldsymbol{R}^n$, $\dim E_1 = \dim E_2 = m$, with the properties*:

$$\|Tx\| \geqslant a\|x\| \quad \text{for } x \in E_1, \qquad \|Tx\| \leqslant b\|x\| \quad \text{for } x \in E_2.$$

Then $T \in M_n(\frac{1}{4}m, \frac{1}{4}(a-b))$.

Proof. Without any loss of generality we may assume that $a = 1$. In order to simplify the notation we shall assume that $m = 4k$ for some $k \in N$. Let p be the maximal nonnegative integer such that $T \in M_n(p, \frac{1}{4}(1-b))$. Assume to the contrary that $p < k$, and let $F \subset \boldsymbol{R}^n$ be such that

$$\|P_{F^\perp} Tx\| \geqslant \tfrac{1}{4}(1-b)\|x\| \quad \text{for } x \in F, \quad \dim F = p.$$

Set

$$E_3 = [F \cup TF \cup T^*F \cup T^*TF]^\perp \cap E_1,$$
$$E_4 = [F \cup TF \cup T^*F \cup T^*TF]^\perp \cap E_2$$

and observe that $\dim E_3 \geqslant \dim E_2 - 4\dim F \geqslant 4(k-p) \geqslant 4$. By the same token $\dim E_4 \geqslant 4$. Using the same argument as in the proof of the fact that every $2k$-dimensional ellipsoid has a k-dimensional spherical section we deduce that there are two orthogonal vectors $x_1, x_2 \in E_3 \oplus E_4$ such that $\|Tx_1\| = \|x_1\|$ and $\|Tx_2\| = b\|x_2\|$. Let $E_5 = [x_1, x_2]$ and note that $\|T|E_5\|_{HS}^2 = 1+b^2$, while $|\mathrm{tr}\, P_{E_5} T|E_5| \leqslant 1+b$. Now, if

$$\|P_{E_5^\perp} T|E_5\|_{HS}^2 \geqslant \tfrac{1}{4}(1-b)^2,$$

then there is an $x_0 \in E_5$, $\|x_0\| = 1$, such that

$$\|P_{E_5^\perp} Tx_0\| \geqslant \tfrac{1}{4}(1-b).$$

Since $x_0 \in E_3 \oplus E_4$ it is a matter of a routine calculation to verify that if $F_1 = [F \cup \{x_0\}]$ then

$$\|P_{F_1^\perp} Ty\| = \|P_{F^\perp} Ty\| \geqslant \tfrac{1}{4}(1-b)\|y\| \quad \text{for } y \in F,$$
$$\|P_{F_1^\perp} Tx_0\| = \|P_{[x_0]^\perp} Tx_0\| \geqslant \tfrac{1}{4}(1-b), \quad P_{F_1^\perp} Tx_0 \perp P_{F_1^\perp} TF,$$

and therefore

$$\|P_{F_1^\perp} Tx\| \geqslant \tfrac{1}{4}(1-b)\|x\| \quad \text{for } x \in F_1,$$

a contradiction to the assumption that F has the maximal dimension. On the other hand, if

$$\|P_{E_5^\perp} T|E_5\|_{HS}^2 \leqslant \tfrac{1}{4}(1-b)^2,$$

then treating E_5 as $\boldsymbol{R}^2$ we have

$$\|P_{E_5} T|E_5\|_{HS}^2 - \tfrac{1}{2}(\mathrm{tr}\, P_{E_5} T|E_5)^2 \geqslant \tfrac{1}{4}(1-b)^2.$$

Thus $\delta(P_{E_5} T|E_5) \geqslant \frac{1}{2}(1-b)$ and hence, by Cor. 2.2, there is an $x_0 \in E_5$, $\|x_0\| = 1$, with the property that

$$\|P_{E_5 \cap [x_0]^\perp} Tx_0\| \geqslant \tfrac{1}{4}(1-b),$$

and we get a contradiction just as before, which completes the proof.

3. Mixing properties of operators and compact groups of operators in $\boldsymbol{R}^n$. The theorem below describes the mixing properties of an operator T in terms of $m(T)$ and its trace.

THEOREM 3.1. *There is a constant $c > 0$ such that for every $n \geqslant 4$ and for every $T \in L(\boldsymbol{R}^n)$*

$$T \in \tilde{M}_n(c\,|nm(T) - |\mathrm{tr}\, T||/\log n).$$

Proof. First observe that since

$$|nm(T) - |\mathrm{tr}\, T|| \leqslant nm(T - (n^{-1}\,\mathrm{tr}\, T)\,\mathrm{Id}_{\boldsymbol{R}^n})$$

for $T \in L(\mathbf{R}^n)$ and since $\dot{T}$ has the same mixing properties as $T - \lambda \mathrm{Id}_{\mathbf{R}^n}$ for every $\lambda \in \mathbf{R}$, it is enough to prove the theorem in the case when $\operatorname{tr} T = 0$. In order to simplify the notation in what follows we shall assume that $n = 2^{10} p$, $p = 1, 2, \ldots$

Fix $T \in L(\mathbf{R}^n)$ with $\operatorname{tr} T = 0$ and let

$$T = \sum_{i=1}^{n} \lambda_i(T) \langle \cdot, u_i \rangle v_i$$

be its polar decomposition. We shall consider the following three mutually exclusive cases:

A. Either $\lambda_{n/256}(T) \geqslant \frac{3}{2} m(T)$ or $\lambda_{255n/256} \leqslant \frac{1}{2} m(T)$.

B. A does not hold and $\sum_{i=1}^{n} \lambda_i(T) \leqslant (1+\frac{1}{64})\, nm(T)$.

C. Neither A nor B hold.

Case A. Note that, by Lemma 2.6, we have

$$T \in M_n(2^{-10} n,\ 2^{-3} m(T)) \subset \tilde{M}_n(2^{-13} nm(T))$$

and we are done.

Case B. First observe that

$$\sum_{i=1}^{n/256} \lambda_i(T) \leqslant \sum_{i=1}^{n} \lambda_i(T) - \sum_{i=n/256+1}^{n/2} \lambda_i(T) - \sum_{i=n/2+1}^{255n/256} \lambda_i(T)$$
$$\leqslant (1+\tfrac{1}{64})\, nm(T) - \tfrac{127}{256} nm(T) - \tfrac{127}{512} nm(T) < \tfrac{1}{3} nm(T)$$

and therefore $|\operatorname{tr} T|E_1| < \frac{1}{3} nm(T)$, where $E_1 = [u_1, \ldots, u_{n/256}]$. Thus

(8) $$|\operatorname{tr} T|E_1^{\perp}| < \tfrac{1}{3} nm(T).$$

On the other hand,

$$\|T|E_1^{\perp}\|_{\mathrm{HS}}^2 \geqslant \tfrac{127}{256} nm^2(T) + \tfrac{127}{256}(\tfrac{1}{2} m(T))^2 > \tfrac{1}{2} nm^2(T).$$

Since $\|T|E_1^{\perp}\| < \frac{3}{2} m(T)$ it is easy to see that

(9) $$\|P_{E_1^{\perp}} T|E_1^{\perp}\|_{\mathrm{HS}}^2 > \tfrac{1}{4} nm^2(T).$$

But (8) and (9) imply that

$$\delta^2(\tilde{T}) > \tfrac{5}{81} nm^2(T)$$

where $\tilde{T} = P_{E_1^{\perp}} T|E_1^{\perp}\colon E_1^{\perp} \to E_1^{\perp}$ is considered as an operator acting in $\mathbf{R}^{255n/256}$. Thus, by Theorem 2.4 applied to the operator $(2/3m(T))\,\tilde{T}$ we infer that

$$\tilde{T} \in M_{255n/256}\left(\frac{2n}{9^3}, \frac{m(T)}{9}\right) \subset \tilde{M}_{255n/256}(2^{-13} nm(T)),$$

which obviously implies that $T \in \tilde{M}_n(2^{-13} nm(T))$, concluding the proof in this case.

Case C. First note that

$$\sum_{i=1}^{n/2} \lambda_i(T) > \tfrac{1}{2}(1+\tfrac{1}{64})\, nm(T)$$

and $\lambda_i \geqslant m(T)$ for $i = 1, \ldots, n/2$. Therefore

$$\sum_{i=1}^{n/2} [\lambda_i(T) - m(T)] > \frac{nm(T)}{128}.$$

Hence there is a $j \leqslant \log_2(n/2)$ with the property that

$$\lambda_{2^j}(T) - m(T) > \frac{nm(T)}{2^j\, 128 \log_2(n/2)}$$

which means that

$$\|Tx\| > \left(1 + (2^j\, 128 \log_2(n/2))^{-1} n\right) m(T)$$

for $x \in [u_1, \ldots, u_{2^j}]$. Since $\|Tx\| \leqslant m(T)$ for $x \in [u_{n/2+1}, \ldots, u_n]$, by Lemma 2.6, we infer that

$$T \in M_n(2^{j-2},\ 2^{-j-9} nm(T)/\log_2(n/2)) \subset \tilde{M}_n(2^{-13} nm(T)/\log n),$$

which completes the proof of the theorem.

As an easy consequence of the last theorem we have the following

THEOREM 3.2. *For each* $\alpha \in (0, 1)$ *there is a* $c_\alpha > 0$ *such that for every* $n \geqslant \max\{(4\alpha)^{-1}, (4(1-\alpha))^{-1}\}$ *and for every* $T \in L(\mathbf{R}^n)$

$$T \in \tilde{M}_n(c_\alpha \left|n\lambda_{[\alpha n]}(T) - |\operatorname{tr} T|\right| / \log n).$$

Proof. If $|\lambda_{[\alpha n]}(T) - m(T)| \geqslant \frac{1}{2}\left|n\lambda_{[\alpha n]}(T) - |\operatorname{tr} T|\right| n^{-1}$ then, by Lemma 2.6, for $0 < \alpha \leqslant \frac{1}{2}$

$$T \in M_n(\tfrac{1}{4}[\alpha n],\ \tfrac{1}{8}\left|n\lambda_{[\alpha n]}(T) - |\operatorname{tr} T|\right| n^{-1}) \subset \tilde{M}_n(2^{-6} \alpha \left|n\lambda_{[\alpha n]}(T) - |\operatorname{tr} T|\right|)$$

and we are done. If this is not the case then we have

$$\left|nm(T) - |\operatorname{tr} T|\right| \geqslant \left|n\lambda_{[\alpha n]}(T) - |\operatorname{tr} T|\right| - |n\lambda_{[\alpha n]}(T) - nm(T)|$$
$$> \tfrac{1}{2}\left|n\lambda_{[\alpha n]}(T) - |\operatorname{tr} T|\right|.$$

Hence by the previous theorem

$$T \in \tilde{M}_n(c\left|nm(T) - |\operatorname{tr} T|\right|/\log n) \subset \tilde{M}_n(\tfrac{1}{2} c\left|n\lambda_{[\alpha n]}(T) - |\operatorname{tr} T|\right|/\log n),$$

which concludes the proof in the case $0 \leqslant \alpha \leqslant \frac{1}{2}$. The other case can be obtained in a similar way.

Remark 3.3. It may be of some value to note that the constant c_α in the theorem above may be taken to be $2^{-6}\alpha$ (resp. $2^{-6}(1-\alpha)$) for α sufficiently close to 0 (resp. 1).

Now, we turn our attention to compact groups of operators acting on $\boldsymbol{R}^n$.

THEOREM 3.4. *There is a constant $c>0$ such that for every $n\geqslant 4$ and every $T\in\mathscr{U}\mathscr{G}_n$*

$$\{T, T^{-1}\}\cap\tilde{M}_n(c(n-|\operatorname{tr} T|)/\log n)\neq\emptyset.$$

In particular,

$$G\cap\tilde{M}_n(c(n-t(G))/\log n)\neq\emptyset$$

for every $G\in\mathscr{G}_n$.

Proof. Obviously, it is enough to prove the first part of the theorem. To this end, let $T\in\mathscr{U}\mathscr{G}_n$. Since either $\lambda_{[n/2]}(T)$ or $\lambda_{[n/2]}(T^{-1})$ is not smaller than 1 and since $\operatorname{tr} T=\operatorname{tr} T^{-1}$, replacing perhaps T by T^{-1}, we may assume that $\lambda_{[n/2]}(T)\geqslant 1$. Thus by the previous theorem, since $|\operatorname{tr} T|\leqslant n$, we have

$$T\in\tilde{M}_n(c_{1/2}\,|n\lambda_{[n/2]}(T)-|\operatorname{tr} T||/\log n)\doteq\tilde{M}_n(c_{1/2}(n-|\operatorname{tr} T|)/\log n),$$

which completes the proof.

Remark 3.5. Let us observe that Th. 3.4 answers (at least partially) a problem posed by S. J. Szarek in [9].

In the sequel, we shall need the following notation: if X_n is an n-dimensional linear space and $\langle\ ,\ \rangle$ is a scalar product on X_n then by $\tilde{M}_{n,\langle,\rangle}(\alpha)$, for $\alpha\geqslant 0$, we shall denote the corresponding class of subspace mixing operators in $L(X_n)$ with respect to $\langle\ ,\ \rangle$. Also, by $\mathscr{G}(X_n)$ (resp. $\mathscr{U}\mathscr{G}(X_n)$) we shall denote the set (resp. the union) of all compact groups of operators acting on X_n.

The following result is just a small modification of the theorem above.

THEOREM 3.6. *There is a constant $c>0$ such that for every n-dimensional linear space X_n*

(i) *$T\in\mathscr{U}\mathscr{G}(X_n)$ implies $\{T, T^{-1}\}\cap\tilde{M}_{n,\langle,\rangle}(c(n-|\operatorname{tr} T|)/\log n)\neq\emptyset$ for every scalar product $\langle\ ,\ \rangle$ on X_n. In particular,*

(ii) *$G\cap\tilde{M}_{n,\langle,\rangle}(c(n-t(G))/\log n)\neq\emptyset$ for every $G\in\mathscr{G}(X_n)$ and every scalar product $\langle\ ,\ \rangle$ on X_n.*

4. Compact groups of operators with relatively large values of $t(\cdot)$. In view of the results of the previous section it may be of some interest to give a more detailed description of those groups $G\in\mathscr{G}_n$ for which $t(G)$ is relatively large, i.e. for which $n-t(G)$ is relatively small. To this end let us recall some basic facts about compact groups of operators acting on $\boldsymbol{R}^n$.

Let $G\in\mathscr{G}_n$. Then:

1° There is another scalar product $\langle\ ,\ \rangle_1$ on $\boldsymbol{R}^n$ such that G is a group of isometries of $(\boldsymbol{R}^n, \|\ \|_1)$, where $\|x\|_1=\langle x, x\rangle_1^{1/2}$ for $x\in\boldsymbol{R}^n$.

2° There is a decomposition of $\boldsymbol{R}^n$ into an $\langle\ ,\ \rangle_1$-orthogonal sum of subspaces

$$\boldsymbol{R}^n=E_1\oplus\ldots\oplus E_k \tag{10}$$

with the properties:

(i) $T(E_i)=E_i$ for every $T\in G$ and every $i=1,\ldots,k$.

(ii) G acts irreducibly on each E_i, $i=1,\ldots,k$, i.e. the group $G_{E_i}=\{T|E_i\colon T\in G\}\subset L(E_i)$ does not admit a nontrivial invariant subspace for $i=1,\ldots,k$.

(iii) If $U\in L(E_i)$ commutes with every element of G_{E_i} then

$$\langle Ux, x\rangle_1=(\dim E_i)^{-1}\operatorname{tr} U$$

for every $x\in E_i$ with $\|x\|_1=1$ and every $i=1,\ldots,k$.

For a fixed group $G\in\mathscr{G}_n$ every decomposition of $\boldsymbol{R}^n$ in the form (10) satisfying 2°(i)–(iii) is said to be a *decomposition* of $\boldsymbol{R}^n$ *into G-irreducible subspaces*. The properties 1°, 2°(i), (ii) and the complex (stronger) version of 2°(iii) can be found for example in [6]. We sketch the proof of the real case of 2°(iii). Let U be an operator in $L(E_i)$ which commutes with G_{E_i}. Then, by the same argument as in the complex case, we infer that $U=\lambda_1\tilde{U}$, where $\lambda_1\in\boldsymbol{R}$ and $\tilde{U}$ is an isometry on $(E_i, \|\ \|_1)$. Set $S=U-\mathrm{Id}_{E_i}$. Since S commutes with G_{E_i}, by the same token we deduce that $S=\lambda_2\tilde{S}$ with $\lambda_2\in\boldsymbol{R}$ and $\tilde{S}$ being an isometry on $(E_i, \|\ \|_1)$. We have for $x\in E_i$, with $\|x\|_1=1$,

$$\lambda_2^2=\langle Sx, Sx\rangle_1=\|Ux\|_1^2-2\langle Ux, x\rangle_1+\|x\|_1=\lambda_1^2+1-2\langle Ux, x\rangle_1.$$

Hence $\langle Ux, x\rangle_1\equiv\text{const}$ for $x\in E_i$ with $\|x\|_1=1$. Now, the exact value of $\langle Ux, x\rangle_1$ follows from the formula

$$\operatorname{tr} U=\dim E_i\int_{S^{E_i}}\langle Ux, x\rangle_1\,d\mu_{S^{E_i}}(x),$$

where $\mu_{S^{E_i}}$ denotes the normalized Lebesgue measure on the unit sphere S^{E_i} of $(E_i, \|\ \|_1)$.

PROPOSITION 4.1. *There is a constant $c>0$ such that for every $n\in\boldsymbol{N}$ and for every $G\in\mathscr{G}_n$ the cardinality of the set of 1-dimensional subspaces in every decomposition of $\boldsymbol{R}^n$ into G-irreducible subspaces is at least $n-c(n-t(G))$.*

Proof. Let $G\in\mathscr{G}_n$. In view of 1° above, without any loss of generality we may and shall assume that $\langle\ ,\ \rangle=\langle\ ,\ \rangle_1$. Let

$$\boldsymbol{R}^n=E_1\oplus\ldots\oplus E_k$$

be a decomposition of $\boldsymbol{R}^n$ into G-irreducible subspaces. Let $n_i = \dim E_i$, $i = 1, \ldots, k$. Obviously, $\sum_{i=1}^k n_i = n$. Assume that $n_1 \geqslant \ldots \geqslant n_k$ and let $j = \max\{i:\ n_i > 1\}$. Note that to prove the proposition it is enough to prove that there is a constant $c > 0$ with the property

$$(*)\qquad \sum_{i=1}^{j} n_i \leqslant c\big(n - t(G)\big).$$

To this end, in order to simplify the notation, we shall assume that all n_i's are even for $i \leqslant j$. Let $m = \sum_{i=1}^j n_i$, and in each E_i, $i \leqslant j$, choose arbitrary two orthogonal subspaces F_i^1 and F_i^2 with $\dim F_i^1 = \dim F_i^2 = n_i/2$. Let

$$F_1 = \bigoplus_{i=1}^{j} F_i^1, \qquad F_2 = \bigoplus_{i=1}^{j} F_i^2.$$

Consider the operators $U_i \in L(E_i)$ defined by

$$(11)\qquad U_i = \int_G T^{-1}|E_i \circ P_{F_i^1} \circ T|E_i\, dh_G(T)$$

for $i = 1, \ldots, j$. Obviously, U_i commutes with every element of G_{E_i}. By 2°(iii), $\langle U_i x, x\rangle = n_i^{-1} \operatorname{tr} U_i$ for every $x \in S^{E_i}$. On the other hand, it follows from (11) that $\operatorname{tr} U_i = \dim F_i^1 = n_i/2$. Hence

$$(12)\qquad \langle U_i x, x\rangle = \tfrac{1}{2}$$

for every $i = 1, \ldots, j$ and every $x \in S^{E_i}$. Set $S^{F_i^2} = S_i$ for $i = 1, \ldots, j$ and observe that by (12)

$$\begin{aligned} m/2 &= \sum_{i=1}^{j} n_i \int_{S_i} \langle U_i x, x\rangle\, d\mu_{S_i}(x) \\ &= \sum_{i=1}^{j} n_i \int_{S_i}\int_G \langle (T^{-1}|E_i \circ P_{F_i^1} \circ T|E_i)x, x\rangle\, dh_G(T)\, d\mu_{S_i}(x) \\ &= \int_G \Big(\sum_{i=1}^{j} n_i \int_{S_i} \|P_{F_i^1} Tx\|^2\, d\mu_{S_i}(x)\Big)\, dh_G(T) \\ &= \int_G \sum_{i=1}^{j} \|P_{F_i^1} T P_{F_i^2}\|_{HS}^2\, dh_G(T) = \int_G \|P_{F_1} T P_{F_2}\|_{HS}^2\, dh_G(T). \end{aligned}$$

Therefore there exists a $T_0 \in G$ such that

$$(13)\qquad \|P_{F_1} T_0 P_{F_2}\|_{HS}^2 \geqslant m/2.$$

Set $\tilde{E} = E_1 \oplus \ldots \oplus E_j$ and let $\tilde{T} = T_0|\tilde{E}$. Obviously $\tilde{T}$ is an isometry in $L(\tilde{E})$. Let $\{u_i:\ i = 1, \ldots, m/2\}$ be an orthonormal basis in F_1 and let $\{u_i:\ i = m/2+1, \ldots, m\}$ be an orthonormal basis in F_2. Since $\tilde{E}$ is the orthogonal sum of F_1 and F_2 we infer that $\{u_i:\ i = 1, \ldots, m\}$ is an orthogonal basis in $\tilde{E}$. Let $(\alpha_{ij})_{i,j=1}^m$ be the matrix representation of $\tilde{T}$ with respect to this basis. Since $\|\tilde{T}\|_{HS}^2 = m$, by (13), we have

$$\begin{aligned} |\operatorname{tr} \tilde{T}| &= \Big|\sum_{i=1}^{m} \alpha_{ii}\Big| \leqslant \sum_{i=1}^{m} |\alpha_{ii}| \leqslant \sqrt{m}\Big(\sum_{i=1}^{m} \alpha_{ii}^2\Big)^{1/2} \leqslant \sqrt{m}(m - m/2)^{1/2} \\ &= m/\sqrt{2}. \end{aligned}$$

Thus

$$t(G) \leqslant |\operatorname{tr} T_0| \leqslant |\operatorname{tr} \tilde{T}| + n - m$$

which yields (*) and concludes the proof.

Now, we are ready to derive the main result of this section.

THEOREM 4.2. *There is a constant $c > 0$ such that for every $n \in N$ and for every group $G \in \mathscr{G}_n$ there is a subspace $E_G \subset \boldsymbol{R}^n$ with $\dim E_G \geqslant n - c\big(n - t(G)\big)$ on which G acts trivially.*

Proof. Fix $G \in \mathscr{G}_n$. Replacing perhaps G by the group $-G \cup G$ we may assume that $-\mathrm{Id}_{\boldsymbol{R}^n} \in G$. As in the proof of the previous proposition, without any loss of generality we may assume that G is a group of isometries of $\boldsymbol{R}^n$. Since the trace is a continuous functional on $L(\boldsymbol{R}^n)$ equipped with the Hilbert–Schmidt norm and its norm is equal to $\sqrt{n}$ and since

$$G \subset S_{n^2} \stackrel{\mathrm{df}}{=} \{T \in L(\boldsymbol{R}^n):\ \|T\|_{HS} = \sqrt{n}\}$$

we infer that there is a constant $c_1 \in (0, 1)$ with the property that if $T \in S_{n^2}$ and $\operatorname{tr} T \geqslant c_1 n$ then $\|T - \mathrm{Id}_{\boldsymbol{R}^n}\|_{HS} \leqslant \frac{1}{4}\sqrt{n}$.

Now, observe that if $t(G) < c_1 n$ then the theorem holds trivially with $c = 2/(1 - c_1)$ and therefore it is enough to prove it for groups satisfying $t(G) \geqslant c_1 n$. Let G be such a group and define

$$G_0 = \{T \in G:\ \operatorname{tr} T \geqslant c_1 n\} = \{T \in G:\ \operatorname{tr} T > 0\}.$$

We claim that G_0 is a subgroup of G and $G = -G_0 \cup G_0$. Indeed, the second property of G_0 is trivial, and to see that G_0 is a subgroup, note that for $T_1, T_2 \in G_0$ we have $\|\mathrm{Id}_{\boldsymbol{R}^n} - T_1 T_2\|_{HS} \leqslant \frac{1}{2}\sqrt{n}$, hence $\operatorname{tr}(\mathrm{Id}_{\boldsymbol{R}^n} - T_1 T_2) = n - \operatorname{tr} T_1 T_2 \leqslant \frac{1}{4}n$, which implies $\operatorname{tr} T_1 T_2 > 0$ and therefore $T_1 T_2 \in G_0$. In particular, we have $t(G_0) = t(G) \geqslant c_1 n$.

Now, let $\boldsymbol{R}^n = E_1 \oplus \ldots \oplus E_m$ be a decomposition of $\boldsymbol{R}^n$ into G-irreducible subspaces. Let $\{E_i\}_{i=1}^k$ be the set of all 1-dimensional subspaces in this decomposition. By Prop. 4.1, we have $k \geqslant n - c_2\big(n - t(G)\big)$, where c_2 is some numerical constant. Let $u_i \in E_i$, $\|u_i\| = 1$ for $i = 1, \ldots, k$, and define

$$A = \{i:\ Tu_i = u_i \text{ for every } T \in G_0\}, \qquad B = \{1, \ldots, k\} \setminus A.$$

Set $F = [u_i\colon i \in B]$ and observe that if $i \in B$ then

$$\int_{G_0} \langle Tu_i, u_i \rangle \, dh_{G_0}(T) = 0.$$

Hence

$$\int_{G_0} \operatorname{tr}(T|F)\, dh_{G_0} = \sum_{i \in B} \int_{G_0} \langle Tu_i, u_i \rangle \, dh_{G_0}(T) = 0.$$

Thus there is a $T_0 \in G_0$ with $\operatorname{tr}(T_0|F) \leqslant 0$. We have

$$t(G) \leqslant \operatorname{tr} T_0 \leqslant \operatorname{tr}(T_0|F) + \dim F^{\perp} \leqslant n - \dim F,$$

which implies that

$$\# B = \dim F \leqslant n - t(G).$$

This yields

$$\# A = k - \# B \geqslant n - c_2(n - t(G)) - (n - t(G)) = n - (c_2 + 1)(n - t(G)).$$

Now, setting $c = c_2 + 1$ and $E_G = [u_i\colon i \in A]$ concludes the proof.

Remark 4.3. The theorem above states that if $n - t(G)$ is relatively small then G acts trivially on a large-dimensional subspace.

5. Application to the pathological properties of Gluskin spaces. We begin with the crucial result due to S. J. Szarek [9].

THEOREM 5.1 (S. J. Szarek). *There is a constant $c > 0$ such that for each $n \geqslant 4$ there is a norm $|||\cdot|||_n$ on $\mathbf{R}^n$ with the property*

$$|||T|||_n \geqslant \frac{c\alpha}{\sqrt{n \log n}}$$

for every operator $T \in \tilde{M}_n(\alpha)$.

Remark 5.2. In fact, Szarek proved slightly more, namely, that Th. 5.1 holds for a "vast majority" of norms defined by E. D. Gluskin in [4].

Combining Th. 5.1 with Th. 3.4 we immediately get an answer to a problem posed by S. J. Szarek in [9].

THEOREM 5.3. *There is a constant $c > 0$ such that for every $n \geqslant 4$ there is an n-dimensional Banach space X_n with the property that for every $T \in \mathscr{U}\mathscr{G}(X_n)$*

$$\max\{\|T\|_{X_n}, \|T^{-1}\|_{X_n}\} \geqslant \frac{c(n - |\operatorname{tr} T|)}{n^{1/2} \log^{3/2} n}.$$

In particular,

$$\sup\{\|T\|_{X_n}\colon T \in G\} \geqslant \frac{c(n - t(G))}{n^{1/2} \log^{3/2} n}$$

for every $G \in \mathscr{G}(X_n)$.

As an application of the result above we shall deduce slightly weaker versions (they differ by a logarithmic factor) of results due to E. D. Gluskin [4] and S. J. Szarek [8].

COROLLARY 5.4. *There is a constant $c > 0$ such that for each $n \geqslant 4$ there is an n-dimensional Banach space X_n with the property*

$$\|P\|_{X_n} \geqslant \frac{ck}{n^{1/2} \log^{3/2} n}$$

for every rank k projection P, with $k \leqslant n/2$.

Proof. For every such projection P it is enough to consider the group G_P consisting of $\mathrm{Id}_{X_n} - 2P$ and Id_{X_n}, and apply Th. 5.3 (note that $\operatorname{tr}(\mathrm{Id}_{X_n} - 2P) = n - 2k$).

As another application of Th. 5.3 let us mention the following weaker version ("up to a logarithmic factor") of a result proved by S. J. Szarek in [9]:

COROLLARY 5.5. *There is a constant $c > 0$ such that for every $n \geqslant 2$ there is a $2n$-dimensional Banach space X_{2n} with the property that for every complex n-dimensional Banach space $Y_n^{\mathbf{C}}$ the Banach–Mazur distance over the reals $d_{\mathbf{R}}(X_{2n}, Y_n^{\mathbf{C}})$ is at least $cn^{1/2} \log^{-3/2} n$.*

Proof. Let X_{2n} be a Banach space satisfying Th. 5.3 and let $T\colon X_{2n} \to Y_n$ be an "$\mathbf{R}$-linear" operator realizing the distance between X_{2n} and $Y_n^{\mathbf{C}}$. Define

$$A = T^{-1} \circ i\, \mathrm{Id}_{Y_n^{\mathbf{C}}} \circ T.$$

Since $A^2 = -\mathrm{Id}_{X_{2n}}$ it can easily be seen that $\operatorname{tr} A = 0$. Now, it is enough to apply Th. 5.3 for the group $\{A, -A, -\mathrm{Id}, \mathrm{Id}\}$ and to note that on the other hand $\|A\|_{X_{2n}} \leqslant d_{\mathbf{R}}(X_{2n}, Y_n^{\mathbf{C}})$.

The next result is a one more application of Th. 5.3 concerning a "vast majority" of Gluskin spaces.

THEOREM 5.6. *There is a constant $c > 0$ such that for every $n \geqslant 4$ there is an n-dimensional Banach space X_n with the property: if*

$$G \in \mathscr{G}(X_n) \quad \text{and} \quad \sup\{\|T\|_{X_n}\colon T \in G\} = A$$

then G acts trivially on some subspace of X_n of dimension $> n - cAn^{1/2} \log^{3/2} n$.

Proof. The theorem follows directly from Th. 5.3 and Th. 4.2.

Remark 5.7. (i) Note that Th. 5.6 is nontrivial only if $A < c^{-1} n^{1/2} \log^{-3/2} n$.

(ii) On the other hand, Th. 5.6 states that the situation in Corollary 5.4 is typical (G_P acts trivially on $\ker P$, while $\dim \ker P = n - k$).

Now, we shall deal with the problem of representing an operator T

acting on a finite-dimensional Banach space as a sum of some number of "small rank" operators with "small" norms.

THEOREM 5.8. *There is a constant $c>0$ such that for every $n \geqslant 4$ there is an n-dimensional Banach space X_n with the property that for every $T \in L(X_n)$, if $T=\sum_{i=1}^{k} T_i$ with* rank $T_i \leqslant n/2$ *for* $i=1, \ldots, k$ *then*

$$\sup\{\|T_i\|_{X_n}: i=1, \ldots, k\} \geqslant \frac{c\,|\operatorname{tr} T|}{k n^{1/2} \log^{3/2} n}.$$

In particular, if $\mathrm{Id}_{X_n}=\sum_{i=1}^{k} T_i$ *with* rank $T_i \leqslant n/2$ *for* $i=1, \ldots, k$ *then*

$$\sup\{\|T_i\|_{X_n}: i=1, \ldots, k\} \geqslant \frac{c n^{1/2}}{k \log^{3/2} n}.$$

Proof. First observe that for at least one of the T_i's we have $|\operatorname{tr} T_i| \geqslant k^{-1}|\operatorname{tr} T|$ while $\lambda_{[3n/4]}(T_i)=0$, and next apply Th. 3.1 and Th. 5.1.

Remark 5.9. A standard argument shows that Th. 5.8 implies the following result due to S. J. Szarek [11]:

There is a constant $c>0$ such that for every $n \in N$ there is an n-dimensional Banach space X_n with the property that whenever X_n is C_1-isomorphic to a C_2-complemented subspace of an m-dimensional Banach space Y_m then the basis constant of Y_m is greater than

$$c C_1^{-1} C_2^{-1} m^{-1} n^{3/2} \log^{-3/2} n.$$

This shows that the estimates obtained from the corresponding positive result due to A. Pełczyński [7] are "up to a logarithmic factor" the best possible.

For a finite-dimensional Banach space X denote by $\mathcal{N}\mathcal{G}(X)$ the set of those compact groups of operators on X which act essentially nontrivially on X. Define

$$\mathrm{ws}(X, G)=\sup\{\|T\|_X: T \in G\}$$

for every $G \in \mathcal{N}\mathcal{G}(X)$ and the *weak symmetry constant* $\mathrm{ws}(X)$ of X by

$$\mathrm{ws}(X)=\inf\{\mathrm{ws}(X, G): G \in \mathcal{N}\mathcal{G}(X)\}.$$

Obviously, for every finite-dimensional Banach space X:

(i) $\mathrm{ws}(X)=\inf\{d(X, Y): \dim Y=\dim X$ and $\mathrm{ws}(Y)=1\}$,

(ii) $\mathrm{ws}(X) \leqslant \mathrm{s}(X)$ where $\mathrm{s}(X)$ denotes the symmetry constant of X defined by D. J. H. Garling and Y. Gordon in [2] (cf. also [5]) and

(iii) $\mathrm{ws}(X) \leqslant \sqrt{\dim X}$.

Th. 5.6 yields the following result which generalizes "up to a logarithmic" factor the main result of [5].

THEOREM 5.10. *There is a constant $c>0$ such that for each $n \geqslant 4$ there is an n-dimensional Banach space X_n with*

$$\mathrm{ws}(X_n) \geqslant c n^{1/2} \log^{-3/2} n.$$

Remark 5.11. One can construct for each $n \in N$ an n-dimensional Banach space X_n with $\mathrm{ws}(X_n)=1$ and $\mathrm{s}(X_n) \geqslant c\sqrt{n}$ for some numerical constant $c>0$.

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