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Asymptotic stability of linear differential equations in Banach spaces

by

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Abstract. Let A be a generator of a strongly continuous bounded semigroup $T(t)$, $t \geqslant 0$. We prove that if the intersection of the spectrum of A and the imaginary axis is at most countable and A^* has no purely imaginary eigenvalues, then the Cauchy problem for the differential equation $\dot{x}(t) = Ax(t)$, $t \geqslant 0$, is asymptotically stable.

We consider the differential equation

$$\dot{x}(t) = Ax(t), \quad t \geqslant 0, \tag{1}$$

in a complex Banach space X, where A is a linear closed operator with a dense domain $D(A) \subset X$. The Cauchy problem for equation (1) is *stable* (i.e., by definition, (1) has a unique bounded solution $x(t)$ which depends continuously on the initial value $x(0) \in D(A)$ w.r.t. the sup-norm topology) if and only if the operator A generates a strongly continuous semigroup $T(t)$, $t \geqslant 0$, which is bounded, i.e.

$$\sup_{t \geqslant 0} \|T(t)\| = M < \infty. \tag{2}$$

This criterion, obtained by S. Krein and P. Sobolevskii, is equivalent to the fact that: (i) the spectrum of A does not meet the half-plane $\operatorname{Re}\lambda > 0$, and (ii) the resolvents $R_\lambda = (A - \lambda I)^{-1}$ satisfy the Miyadera–Feller–Phillips inequality

$$\|R_\lambda^n\| \leqslant \frac{M}{(\operatorname{Re}\lambda)^n}, \quad n = 1, 2, \ldots, \ M = \text{const}. \tag{3}$$

These classical results are presented in detail in the monograph [3]. We notice that, without loss of generality, one can put $M = 1$, which can be obtained by introducing the equivalent norm $\sup_{t \geqslant 0} \|T(t)x\|$. In this case the infinite sequence of inequalities (3) is reduced to one Hille–Yosida inequality

$$\|R_\lambda\| \leqslant \frac{1}{\operatorname{Re}\lambda}, \tag{4}$$

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$T(t)$ turns out to be a semigroup of contractions, and the operator A is dissipative.

The Cauchy problem for equation (1) is said to be *asymptotically stable* (a.s.) if it is stable and

$$\lim_{t\to\infty} T(t)x = 0 \tag{5}$$

for all $x\in X$. This property is of spectral nature. For example, if A is bounded and its spectrum lies in the open left half-plane $\operatorname{Re}\lambda < 0$, then the corresponding Cauchy problem is a.s. This is directly derived from the formula

$$e^{At} = -(2\pi i)^{-1}\int_\Gamma e^{\lambda t} R_\lambda\, d\lambda, \tag{6}$$

where Γ is a closed contour in the half-plane $\operatorname{Re}\lambda < 0$ around the spectrum of A. If the intersection of the spectrum of A with the imaginary axis is not empty, then the a.s. may not hold. In any case, for a.s. it is necessary that the operators A and A^* have no eigenvalues on the imaginary axis. Indeed, if $Ax = i\alpha x$, where $\alpha\in \boldsymbol{R}$, $x\neq 0$, then $T(t)x = e^{i\alpha t}x$ does not tend to zero as $t\to\infty$. Now let $A^*f = i\alpha f$ for some $f\in X^*$, $f\neq 0$. Then for every solution $x(t)$ of equation (1) we have

$$\frac{d}{dt} f\big(x(t)\big) = i\alpha f\big(x(t)\big),$$

hence $f\big(x(t)\big) = f\big(x(0)\big)e^{i\alpha t}$. If $x(t)\to 0$ as $t\to\infty$, then $f\big(x(0)\big) = 0$. Therefore from the a.s. it follows that $f\,|\,D(A) = 0$, i.e. $f = 0$.

G. Sklyar and V. Shirman [8] have established that if A is a dissipative operator such that: 1) A is bounded, 2) the spectrum of A has at most countable intersection with the imaginary axis, 3) A^* has no imaginary eigenvalues, then the Cauchy problem for equation (1) is a.s. However, in applications to ordinary/partial differential equations and to optimal control the operator A is usually unbounded. On the other hand, the boundedness of A has been used essentially in the proof in [8]. The aim of this paper is to extend the Sklyar–Shirman criterion to unbounded operators. For this we shall need some simple facts about semigroups of isometries.

Let $U(t)$, $t\geqslant 0$, be a strongly continuous semigroup of isometries (in general, nonsurjective), and let S be its generator.

LEMMA. *If* $\operatorname{Re}\lambda < 0$, *then*

$$\|Sx-\lambda x\| \geqslant |\operatorname{Re}\lambda|\,\|x\|, \tag{7}$$

for all $x\in D(S)$.

This lemma is contained in [3]. Here we give a shorter proof.

Proof. We put $\lambda = -\varrho + i\omega$, $\omega\in\boldsymbol{R}$, $\varrho > 0$, and consider the vector-

valued function $u(t) = e^{-\lambda t}U(t)x$, $t\geqslant 0$. It is clear that

$$\|u(t)\| = e^{\varrho t}\|x\|. \tag{8}$$

On the other hand,

$$u(t) = x + \int_0^t \frac{du(\tau)}{d\tau}\,d\tau = x + \int_0^t e^{-\lambda\tau}U(\tau)(Sx-\lambda x)\,d\tau.$$

Therefore

$$\|u(t)\| \leqslant \|x\| + \frac{e^{\varrho t}-1}{\varrho}\|Sx-\lambda x\|. \tag{9}$$

Comparing (8) and (9), we get (7).

From the lemma it follows that the half-plane $\{\lambda\colon \operatorname{Re}\lambda < 0\}$ lies in a regular component of the operator S (see [1]). It is known that the number $\delta = \dim\{f\colon S^*f = \lambda f\}$ does not depend on λ belonging to the same regular component. Therefore for generators of isometry semigroups one can define the deficiency number δ as the dimension of the eigenspace $\{f\colon S^*f = \lambda f\}$, $\operatorname{Re}\lambda < 0$. It is clear that if $i\boldsymbol{R}\not\subset \operatorname{spec} S$, then $\delta = 0$, and hence $\operatorname{spec} S\subset i\boldsymbol{R}$.

We notice that the equality $\delta = 0$ is equivalent to the extendability of the semigroup $U(t)$ to a strongly continuous group of isometries $U(t)$, $-\infty < t < \infty$ [4]. Indeed, the necessity of the condition $\delta = 0$ is obvious. Conversely, if $\delta = 0$, then, by (7) and the Hille–Yosida theorem, $-S$ is the generator of a strongly continuous semigroup of contractions $V(t)$, $t\geqslant 0$. For every $x\in D(S)$ we have

$$\frac{d}{dt}\{U(t)V(t)x\} = [S, U(t)]V(t)x = 0.$$

Therefore $U(t)V(t) = I$ for all $t\geqslant 0$. (See also [7].)

THEOREM. *Let the operator* A *generate a bounded strongly continuous semigroup* $T(t)$, $t\geqslant 0$. *If the intersection of the spectrum of* A *with the imaginary axis is at most countable and* A^* *has no imaginary eigenvalues, then the Cauchy problem for equation* (1) *is asymptotically stable.*

Proof. We assume without loss of generality that the operator A is dissipative, i.e. $T(t)$ is a semigroup of contractions. Then the functions $\|T(t)x\|$, $t\geqslant 0$, are nonincreasing for each fixed x, and hence the following limit exists:

$$l(x) = \lim_{t\to\infty}\|T(t)x\|, \quad x\in X. \tag{10}$$

$l(x)$ is a seminorm in X, moreover, $l(x)\leqslant\|x\|$. We have to show that $l(x)\equiv 0$. For this, we consider the subspace $L = \ker l$ and suppose, on the contrary, that $L\neq X$. In the quotient space $\tilde X = X/L$ the seminorm l

generates the norm $\tilde{l}$, and the semigroup $T(t)$ acts in $\tilde{X}$ in a natural way, because L is an invariant subspace for all operators $T(t)$, $t \geqslant 0$. Since $l(T(s)x) = l(x)$, $x \in X$, the corresponding operators $\tilde{T}(t)$ in $\tilde{X}$ are isometric. The semigroup of isometries $\tilde{T}(t)$ is strongly continuous, because the seminorm l is dominated by the original norm in X.

Now we take the completion E of $\tilde{X}$ w.r.t. the norm $\tilde{l}$. We get a Banach space E and a strongly continuous semigroup $U(t)$ of isometries in it, the extension by continuity of the semigroup $\tilde{T}(t)$. Let S be the generator of $U(t)$. We show that $\operatorname{spec} S \subset \operatorname{spec} A$. Let $\lambda \notin \operatorname{spec} A$, $R_\lambda = (A - \lambda I)^{-1}$. Since

(11) $$l(R_\lambda x) = \lim_{t\to\infty} \|R_\lambda T(t) x\| \leqslant \|R_\lambda\|\, l(x),$$

the resolvent R_λ has a natural extension to a bounded operator $\hat{R}_\lambda$ in E. If $\operatorname{Re}\lambda > 0$, then

$$R_\lambda x = -\int_0^\infty (T(t)x) e^{-\lambda t}\, dt, \qquad x \in X.$$

This implies

$$\hat{R}_\lambda \hat{x} = -\int_0^\infty (U(t)\hat{x}) e^{-\lambda t}\, dt, \qquad \hat{x} \in E.$$

Therefore $\hat{R}_\lambda$ coincides with the resolvent $R_\lambda(S)$ for all λ, $\operatorname{Re}\lambda > 0$. Now by the Hilbert identity

$$\hat{R}_\mu - \hat{R}_\lambda = (\mu - \lambda)\hat{R}_\lambda \hat{R}_\mu, \qquad \lambda, \mu \notin \operatorname{spec} A,$$

hence

$$\hat{R}_\mu - R_\lambda(S) = (\mu - \lambda) R_\lambda(S) \hat{R}_\mu, \qquad \operatorname{Re}\lambda > 0,\ \mu \notin \operatorname{spec} A.$$

Therefore $\operatorname{Im}\hat{R}_\mu \subset D(S)$ and

$$(S - \lambda I)\hat{R}_\mu = I + (\mu - \lambda)\hat{R}_\mu,$$

which implies $(S - \mu I)\hat{R}_\mu = I$. Analogously, we get $\hat{R}_\mu (S - \mu I) = I | D(S)$. Thus $\mu \notin \operatorname{spec} S$ (and $\hat{R}_\mu = R_\mu(S)$).

From the inclusion $\operatorname{spec} S \subset \operatorname{spec} A$ it follows that the intersection $\operatorname{spec} S \cap i\boldsymbol{R}$ is at most countable. But in this case $\operatorname{spec} S \subset i\boldsymbol{R}$, and therefore $\operatorname{spec} S$ is at most countable. Moreover, $\operatorname{spec} S \neq \emptyset$, since S is a generator of a group of isometries (see [5]). Thus $\operatorname{spec} S$ is a nonempty at most countable closed subset of $i\boldsymbol{R}$. Therefore it contains an isolated point $i\omega \in i\boldsymbol{R}$. To this point corresponds the Riesz projection $P \neq 0$, commuting with S and with all $U(t)$. The subspace $\Omega = \operatorname{Im} P$ is invariant for S and for all $U(t)$, and the corresponding semigroup of isometries $U_\omega(t) = U(t)|\Omega$ is generated by the

bounded operator $S_\omega = S|\Omega$, $\operatorname{spec} S_\omega = \{i\omega\}$. It is well known that isometries with one-point spectrum are scalars (see e.g. [6]). From this and from the spectral mapping theorem, valid for uniformly continuous semigroups, it follows that $S_\omega = i\omega$. Therefore Ω is the eigenspace of S with eigenvalue $i\omega$. Then every linear functional $h \in \operatorname{Im} P^*$, $h \neq 0$, is an eigenfunctional for S^*, with the same eigenvalue. Now we extend the functional h to the whole space X using the sequence of homomorphisms $X \to \tilde{X} \to E$. We get a nonzero functional $f \in X^*$ which is an eigenfunctional for A^* with eigenvalue $i\omega$; a contradiction.

Remark. Under the conditions of the theorem the operator A also does not have imaginary eigenvalues, since the Cauchy problem is a.s. Therefore, if the Banach space X is reflexive, then in the formulation of the theorem we can require the absence of imaginary eigenvalues of A instead of A^*.

Corollary. *If the spectrum of the generator A of a bounded strongly continuous group does not intersect the imaginary axis, then the Cauchy problem for equation* (1) *is a.s.*

Remark. The assumptions of the main theorem do not imply the stronger conclusion

(12) $$\int_0^\infty \|T(t)x\|\, dt < \infty$$

even in the case of a bounded generator (see [2]).

On the other hand, we note that the system may be a.s. even if $\operatorname{spec} A \cap i\boldsymbol{R}$ is uncountable (cf. [8]).

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C_p-Estimates for certain kernels on local fields

by

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Abstract. We give necessary and sufficient conditions for certain operators defined on $L^2(\boldsymbol{K})$ ($\boldsymbol{K}$ a local field) to belong to Schatten–von Neumann ideals. The operators considered are defined by a type of integral kernels.

1. The purpose of this paper is to extend to the case of a local field $\boldsymbol{K}$ the results proved in [9] for the real case. They concern necessary and sufficient conditions for certain kernels to give rise to operators (on $L^2(\boldsymbol{K})$) belonging to Schatten–von Neumann classes (for the theory of Schatten–von Neumann classes, see, for instance, [1]). Though the main ideas are the same as in [9], their actual application needs several adaptations to the new context.

In order to present the results, we have first to establish the notation and to remind some facts from the theory of local fields; the basic reference for this topic is [7].

Let $\boldsymbol{K}$ be a local field, that is, a locally compact, nondiscrete, totally disconnected field with the valuation $|\ |$. We write $\mathfrak{O} = \{x \in \boldsymbol{K},\ |x| \leqslant 1\}$, $\mathfrak{O}^* = \{x \in \boldsymbol{K},\ |x| = 1\}$, $\mathfrak{P} = \{x \in \boldsymbol{K},\ |x| < 1\}$. It is known that there exists $\mathfrak{p} \in \mathfrak{P}$ such that $\mathfrak{P} = \mathfrak{p}\mathfrak{O}$ (this $\mathfrak{p}$ will be fixed in the sequel). The residue space $\mathfrak{O}/\mathfrak{P}$ is a finite field; let Q be a complete set of representatives for it. If $\operatorname{card} Q = q$, then the image of $\boldsymbol{K}^*$ in $(0, \infty)$ under the valuation $|\ |$ is the multiplicative subgroup of $(0, \infty)$ generated by q; also $|\mathfrak{p}| = q^{-1}$. We have $\mathfrak{P}^k = \{x \in \boldsymbol{K},\ |x| \leqslant q^{-k}\}$, and we will write $S_k = \{x \in \boldsymbol{K},\ |x| = q^{-k}\}$; Φ_k will be the characteristic function of $\mathfrak{P}^k$. $\mathscr{S} = \mathscr{S}(\boldsymbol{K})$ will denote the space of finite linear combinations of characteristic functions of balls.

The Fourier transform on $\boldsymbol{K}$ is defined as follows: let χ be a fixed character on $\boldsymbol{K}$ that is trivial on $\mathfrak{O}$ but is nontrivial on $\mathfrak{P}^{-1}$. Then, for $f \in L^1(\boldsymbol{K})$,

$$\hat{f}(x) = \int_{\boldsymbol{K}} f(\xi)\overline{\chi(x\xi)}\, d\xi.$$

The standard properties of the Fourier transform can be found in [7, Chap. II].