$$\int_E \left|\frac{(\mathcal{F}H)(t)}{(\mathcal{F}g)(t)}\right|^2 \frac{dt}{1+t^2} < \infty,$$

where $E \equiv \{t \mid \mathcal{F}g(t) \neq 0\}$.

COROLLARY 22. *With A and g as in Theorem 21, suppose there exists H in $C^\infty(A)$ such that for some $\varepsilon < 1$,*

$$|\mathcal{F}H(t)| \leqslant t^\varepsilon \mathcal{F}g(t)| \text{ a.e.}$$

Then g is not a vector of uniqueness for A.

COROLLARY 23. *If g has a zero of infinite order, then g is not a vector of uniqueness for id/dx on $L^2(\boldsymbol{R})$.*

COROLLARY 24. *If h is not a vector of uniqueness for id/dx on $L^2(\boldsymbol{R})$ and $|\mathcal{F}g(t)| \geqslant |\mathcal{F}h(t)|$, for almost all t, then g is not a vector of uniqueness.*

COROLLARY 25. *If h is a vector of uniqueness for id/dx on $L^2(\boldsymbol{R})$ and $|\mathcal{F}g(t)| \leqslant |\mathcal{F}h(t)|$, for almost all t, then g is a vector of uniqueness.*

THEOREM 26. *Let $E \equiv \{t \mid \mathcal{F}g(t) \neq 0\}$, where g and A are as in Theorem 21. Then $D(g, A) \neq \{f \text{ in } L^2(\boldsymbol{R}) \mid \mathcal{F}f(t) = 0 \text{ when } t \notin E\}$ if and only if there exists a nontrivial F in $C^\infty(A)$, with a zero of infinite order, such that $\mathcal{F}F(t) = 0$ when $\mathcal{F}g(t) = 0$, and*

$$\int_E \left|\frac{\mathcal{F}F(t)}{\mathcal{F}g(t)}\right|^2 dt < \infty.$$

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An improvement of Kaplansky's lemma on locally algebraic operators

by

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Abstract. Let X and Y be two complex vector spaces and let $T_1, \ldots, T_n$ be linear operators from X into Y. Suppose that for every $\xi \in X$ the vectors $T_1\xi, \ldots, T_n\xi$ are linearly dependent. Then, using an analytic argument, we prove that there exists a nontrivial linear combination of these operators having rank $\leqslant n-1$.

Let T be a linear operator on a complex vector space X. Then T is *locally algebraic* if for every $\xi \in X$ there exists a nontrivial polynomial p such that $p(T)\xi = 0$. A standard result of I. Kaplansky ([3], Lemma 14) states that *boundedly locally algebraic* (the degree of p is bounded independently of ξ) implies *algebraic* (for another proof see [5]). This important result has many consequences (see for instance [2]–[4], [6]). In this short paper we present an analytic proof of that result. This argument is very interesting because it implies a surprising extension of Kaplansky's lemma.

THEOREM 1. *Let X be a complex vector space and let T be a linear operator from X into X. Suppose that there exists an integer $n \geqslant 1$ such that $\xi, T\xi, \ldots, T^n\xi$ are linearly dependent for all $\xi \in X$. Then T is algebraic of degree $\leqslant n$.*

Proof. Suppose that n is the smallest integer having this property. Hence there exists $\xi_0 \in X$ such that $\xi_0, T\xi_0, \ldots, T^{n-1}\xi_0$ are linearly independent but $\xi_0, T\xi_0, \ldots, T^n\xi_0$ are not. Then there exists a monic polynomial p_0 of degree n such that $p_0(T)\xi_0 = 0$ and if p is another monic polynomial of degree n such that $p(T)\xi_0 = 0$ then $p = p_0$. Let $\eta \in X$ be an arbitrary fixed vector. We now prove that $p_0(T)\eta = 0$. Let F be the linear subspace generated by $\xi_0, T\xi_0, \ldots, T^n\xi_0, \eta, T\eta, \ldots, T^n\eta$. Then $\dim F \leqslant 2n$. For $\lambda \in \boldsymbol{C}$ we set

$$f_0(\lambda) = \xi_0 + \lambda\eta \in F, \quad f_1(\lambda) = Tf_0(\lambda) \in F, \quad \ldots, \quad f_{n-1}(\lambda) = T^{n-1}f_0(\lambda) \in F,$$
$$g(\lambda) = T^n f_0(\lambda) \in F.$$

Because $f_0(0), \ldots, f_{n-1}(0)$ are linearly independent in F there exist n linear

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functionals on F, denoted by $\varphi_0, \ldots, \varphi_{n-1}$, such that

(1) $$\varphi_i(f_j(0)) = \delta_{ij} \quad \text{for } 0 \leqslant i, j \leqslant n-1.$$

We define

$$\Delta(\lambda) = \begin{vmatrix} \varphi_0(f_0(\lambda)) & \ldots & \varphi_0(f_{n-1}(\lambda)) \\ \ldots & \ldots & \ldots \\ \varphi_{n-1}(f_0(\lambda)) & \ldots & \varphi_{n-1}(f_{n-1}(\lambda)) \end{vmatrix}$$

which is a polynomial of degree $\leqslant n$, satisfying $\Delta(0) = 1$. Let E be the finite set of its zeros. From the hypothesis we conclude that for $\lambda \notin E$ there exist $\alpha_0(\lambda), \ldots, \alpha_{n-1}(\lambda) \in C$ such that

(2) $$g(\lambda) = \alpha_0(\lambda) f_0(\lambda) + \ldots + \alpha_{n-1}(\lambda) f_{n-1}(\lambda)$$

so we have

(3) $$\begin{gathered} \varphi_0(g(\lambda)) = \alpha_0(\lambda)\varphi_0(f_0(\lambda)) + \ldots + \alpha_{n-1}(\lambda)\varphi_0(f_{n-1}(\lambda)), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_{n-1}(g(\lambda)) = \alpha_0(\lambda)\varphi_{n-1}(f_0(\lambda)) + \ldots + \alpha_{n-1}(\lambda)\varphi_{n-1}(f_{n-1}(\lambda)). \end{gathered}$$

By Cramer's formulas the α_i coincide on $C\setminus E$ with rational functions. Relation (2) can be written as

(4) $$\begin{gathered} p_\lambda(T) f_0(\lambda) = 0 \quad \text{for } \lambda \notin E, \text{ with} \\ p_\lambda(T) = T^n - \alpha_{n-1}(\lambda) T^{n-1} - \ldots - \alpha_0(\lambda) 1. \end{gathered}$$

Denote by $\beta_1(\lambda), \ldots, \beta_n(\lambda)$ the roots of the polynomial p_λ. We have

(5) $$(T - \beta_1(\lambda)1)\ldots(T - \beta_n(\lambda)1) f_0(\lambda) = 0 \quad \text{for } \lambda \notin E$$

and obviously $(T - \beta_2(\lambda)1)\ldots(T - \beta_n(\lambda)1) f_0(\lambda) \neq 0$ for $\lambda \notin E$, by the definition of E. So (5) implies that $\beta_1(\lambda)$ is in the spectrum of T. A similar argument implies that $\beta_2(\lambda), \ldots, \beta_n(\lambda)$ are also in the spectrum of T. Consequently $|\beta_i(\lambda)| \leqslant \|T\|$ for $i = 1, \ldots, n$ and $\lambda \notin E$, where $\|\ \|$ is a norm on the invariant subspace F. So the symmetric functions $\alpha_0(\lambda), \ldots, \alpha_{n-1}(\lambda)$ are also bounded on $C\setminus E$. Because the α_i coincide with rational functions on $C\setminus E$ we conclude from Liouville's Theorem that there are constant numbers $\gamma_0, \ldots, \gamma_{n-1} \in C$ such that $\alpha_i(\lambda) = \gamma_i$ for $\lambda \notin E$. Let $p(z) = z^n - \gamma_{n-1} z^{n-1} - \ldots - \gamma_0$. Then $p(T) f_0(\lambda) = 0$ on $C\setminus E$, but also on C, by continuity in λ. In particular, $p(T)\xi_0 = 0$, so $p = p_0$. Consequently $p_0(T)\eta = 0$ for all $\eta \in X$. Hence $p_0(T) = 0$, so T is algebraic of degree $\leqslant n$. ∎

A slight modification of the argument now gives

THEOREM 2. *Let X and Y be two complex vector spaces and let $T_1, \ldots, T_n$ be linear operators from X into Y. Suppose that for every $\xi \in X$ the*

vectors $T_1\xi, \ldots, T_n\xi$ are linearly dependent. Then there exist $\alpha_1, \ldots, \alpha_n \in C$, not all zero, such that $Q = \alpha_1 T_1 + \ldots + \alpha_n T_n$ has finite rank $\leqslant n-1$. Moreover, if $X = Y$ and the T_i commute, then $Q^2 = 0$.

Proof. If for all $\xi \in X$, the vectors $T_1\xi, \ldots, T_{n-1}\xi$ are linearly dependent, it is enough to prove the result with $T_1, \ldots, T_{n-1}$. So suppose that there exists $\xi_0 \in X$ such that $T_1\xi_0, \ldots, T_{n-1}\xi_0$ are linearly independent and $T_1\xi_0, \ldots, T_n\xi_0$ are not. Then there exist $\alpha_1, \ldots, \alpha_{n-1} \in C$ such that

(6) $$(T_n + \alpha_{n-1} T_{n-1} + \ldots + \alpha_1 T_1)\xi_0 = 0.$$

Let $\eta \in X$ be an arbitrary fixed vector and let F be the linear subspace of Y generated by $T_1\xi_0, \ldots, T_n\xi_0, T_1\eta, \ldots, T_n\eta$. Then $\dim F \leqslant 2(n-1)$. For $\lambda \in C$ we set

(7) $$\begin{gathered} f_0(\lambda) = \xi_0 + \lambda\eta, \quad f_1(\lambda) = T_1 f_0(\lambda) \in F, \quad \ldots, \quad f_{n-1}(\lambda) = T_{n-1} f_0(\lambda) \in F, \\ g(\lambda) = T_n f_0(\lambda) \in F. \end{gathered}$$

Because $f_1(0), \ldots, f_{n-1}(0)$ are linearly independent in F there exist $n-1$ linear functionals on F, denoted by $\varphi_1, \ldots, \varphi_{n-1}$, such that

(8) $$\varphi_i(f_j(0)) = \delta_{ij} \quad \text{for } 1 \leqslant i, j \leqslant n-1.$$

We define

$$\Delta(\lambda) = \begin{vmatrix} \varphi_1(f_1(\lambda)) & \ldots & \varphi_1(f_{n-1}(\lambda)) \\ \ldots & \ldots & \ldots \\ \varphi_{n-1}(f_1(\lambda)) & \ldots & \varphi_{n-1}(f_{n-1}(\lambda)) \end{vmatrix}$$

which is a polynomial of degree $\leqslant n-1$, satisfying $\Delta(0) = 1$, and

$$\Delta_i(\lambda) = \begin{vmatrix} \varphi_1(f_1(\lambda)) & \ldots & \varphi_1(g(\lambda)) & \ldots & \varphi_1(f_{n-1}(\lambda)) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varphi_{n-1}(f_1(\lambda)) & \ldots & \underset{i\text{th column}}{\varphi_{n-1}(g(\lambda))} & \ldots & \varphi_{n-1}(f_{n-1}(\lambda)) \end{vmatrix}$$

which is also a polynomial of degree $\leqslant n-1$, satisfying $-\Delta_i(0) = \alpha_i$, by (6) and (8). If E denotes the set of zeros of Δ then, arguing as in the proof of Theorem 1, we conclude that

(9) $$(\Delta(\lambda) T_n - \Delta_{n-1}(\lambda) T_{n-1} - \ldots - \Delta_1(\lambda) T_1) f_0(\lambda) = 0$$

on $C\setminus E$, and so, by continuity, on all C. Let $\alpha_n = 1$ and let $\beta_1, \ldots, \beta_n$ be the coefficients of λ respectively in $-\Delta_1(\lambda), \ldots, -\Delta_{n-1}(\lambda), \Delta(\lambda)$. Setting $Q = \alpha_1 T_1 + \ldots + \alpha_n T_n$ (which does not depend on η!), $R = \beta_1 T_1 + \ldots + \beta_n T_n$ (which depends on η!) and looking at the coefficients of degree 0 and 1 in λ, from (9) we obtain

$$Q\xi_0 = 0, \quad Q\eta + R\xi_0 = 0.$$

Consequently $Q\eta$ is in the linear subspace generated by $T_1\xi_0, \ldots, T_{n-1}\xi_0$. So Q has a finite rank $\leqslant n-1$. If moreover the T_i commute, then Q and R commute, so $Q^2\eta = -QR\xi_0 = -RQ\xi_0 = 0$. Hence $Q^2 = 0$. ■

Remark. Let P and Q be two different projections having the same range of dimension 1, defined on a complex vector space X. For every $\xi \in X$ the vectors $P\xi$ and $Q\xi$ are dependent and obviously there are linear combinations of P and Q having rank one. But $\alpha P + \beta Q \neq 0$ for any $\alpha, \beta \in \boldsymbol{C}$. So in general it is impossible to have $Q = 0$ in Theorem 2.

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Extension of C^∞ functions from sets with polynomial cusps

by

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Abstract. We give a simple construction of a continuous linear operator extending C^∞ functions from compact subsets of $\boldsymbol{R}^n$ with polynomial cusps including fat subanalytic sets.

1. Introduction. Whitney's extension theorem [15] yields a continuous linear operator extending C^k functions (k finite) defined on closed subsets X of $\boldsymbol{R}^n$. For C^∞ functions such an operator does not in general exist (see e.g. [12, p. 79]). However, Mityagin [4] and Seeley [7] proved the existence of an extension operator if X is a half-space of $\boldsymbol{R}^n$. Stein [9] showed that such an operator exists if X is the closure of a Lipschitz domain in $\boldsymbol{R}^n$ of class Lip 1. Stein's result was then extended by Bierstone [1] to the case of a domain with boundary which is Lipschitz of any order. By the main result of Bierstone [1] involving Hironaka's desingularization theorem, an extension operator exists if X is a *fat* (i.e. $\overline{\operatorname{int} X} \supset X$) closed subanalytic subset of $\boldsymbol{R}^n$. If X is Nash subanalytic (not necessarily fat) the existence problem was solved by Bierstone and Schwarz [3]. Recently Wachta [14] has constructed an extension operator for fat closed subanalytic sets in $\boldsymbol{R}^n$ without making use of the Hironaka desingularization theorem. For closed subsets of $\boldsymbol{R}^n$ admitting some polynomial cusps, the existence of an extension operator was shown by Tidten [10].

In this paper we construct an extension operator for the family of compact *uniformly polynomially cuspidal* (briefly, UPC) subsets of $\boldsymbol{R}^n$ (see Theorem 4.1). The UPC sets were introduced in [6] as follows.

Definition 1.1. A subset X of $\boldsymbol{R}^n$ is said to be UPC if there exist positive constants M and m, and a positive integer d such that for each point x in $\bar{X}$, one may choose a polynomial map $h_x\colon \boldsymbol{R} \to \boldsymbol{R}^n$ of degree at most d satisfying the following conditions:

(i) $h_x((0, 1]) \subset X$ and $h_x(0) = x$;
(ii) $\operatorname{dist}(h_x(t), \boldsymbol{R}^n - X) \geqslant Mt^m$ for all x in X and $t \in (0, 1]$.

Every bounded convex domain in $\boldsymbol{R}^n$ and every bounded Lipschitz domain are UPC. More generally, every subset of $\boldsymbol{R}^n$ with a parallelepiped