where $E = \{ t \mid \mathcal{F}g(t) \neq 0 \}$.

Corollary 22. With $A$ and $g$ as in Theorem 21, suppose there exists $H$ in $C^\infty(A)$ such that for some $s < 1$,

$$|\mathcal{F}H(t)| \leq t^s |\mathcal{F}g(t)| \quad \text{a.e.}$$

Then $g$ is not a vector of uniqueness for $\mathcal{F}$.

Corollary 23. If $g$ has a zero of infinite order, then $g$ is not a vector of uniqueness for id/dx on $L^2(\mathbb{R})$.

Corollary 24. If $h$ is not a vector of uniqueness for id/dx on $L^2(\mathbb{R})$ and $|\mathcal{F}g(t)| \geq |\mathcal{F}h(t)|$, for almost all $t$, then $g$ is not a vector of uniqueness.

Corollary 25. If $h$ is a vector of uniqueness for id/dx on $L^2(\mathbb{R})$ and $|\mathcal{F}g(t)| \leq |\mathcal{F}h(t)|$, for almost all $t$, then $g$ is a vector of uniqueness.

Theorem 26. Let $E = \{ t \mid \mathcal{F}g(t) \neq 0 \}$, where $g$ and $A$ are as in Theorem 21. Then $\mathcal{D}(g, A) = \{ f \in L^2(\mathbb{R}) \mid \mathcal{F}f(t) = 0 \text{ whenever } t \notin E \}$, is a vector of uniqueness for $A$.

Proof. Suppose that $n$ is the smallest integer having this property. Hence there exists $\xi_0 \in X$ such that $\mathcal{D}(\xi_0, T\xi_0, \ldots, T^{n-1}\xi_0)$ is linearly independent but $\xi_0, T\xi_0, \ldots, T^n\xi_0$ are not. Then there exists a monic polynomial $P_0$ of degree $n$ such that $P_0(T)\xi_0 = 0$ and if $p$ is another monic polynomial of degree $n$ such that $p(T)\xi_0 = 0$ then $p = P_0$. Let $\eta \in X$ be an arbitrary fixed vector. We now prove that $P_0(T)\eta = 0$. Let $F$ be the linear subspace generated by $\xi_0, T\xi_0, \ldots, T^n\xi_0$. Then $\dim F \leq 2n$. For $\lambda \in C$ we set

$$f_0(\lambda) = \xi_0 + \lambda\eta \in F, \quad f_1(\lambda) = Tf_0(\lambda) \in F, \quad \ldots, \quad f_{n-1}(\lambda) = T^{n-1}f_0(\lambda) \in F, \quad g(\lambda) = T^n f_0(\lambda) \in F.$$
functionals on $F$, denoted by $\varphi_0, \ldots, \varphi_{n-1}$, such that

$$\varphi_i(f)(0) = \delta_{ij} \quad \text{for} \quad 0 \leq i, j \leq n-1.$$  

We define

$$A(\lambda) = \begin{bmatrix} \varphi_0(f_0(\lambda)) & \ldots & \varphi_0(f_{n-1}(\lambda)) \\ \varphi_1(f_0(\lambda)) & \ldots & \varphi_1(f_{n-1}(\lambda)) \\ \vdots & \ddots & \vdots \\ \varphi_{n-1}(f_0(\lambda)) & \ldots & \varphi_{n-1}(f_{n-1}(\lambda)) \end{bmatrix}$$

which is a polynomial of degree $\leq n$, satisfying $A(0) = 1$. Let $E$ be the finite set of its zeros. From the hypothesis we conclude that for $\lambda \notin E$ there exist $a_0(\lambda), \ldots, a_{n-1}(\lambda) \in C$ such that

$$g(\lambda) = a_0(\lambda)f_0(\lambda) + \ldots + a_{n-1}(\lambda)f_{n-1}(\lambda)$$

so we have

$$\varphi_0(g(\lambda)) = a_0(\lambda)\varphi_0(f_0(\lambda)) + \ldots + a_{n-1}(\lambda)\varphi_0(f_{n-1}(\lambda)),$$

$$\varphi_1(g(\lambda)) = a_0(\lambda)\varphi_1(f_0(\lambda)) + \ldots + a_{n-1}(\lambda)\varphi_1(f_{n-1}(\lambda)).$$

By Cramer’s formulas the $\alpha_i$ coincide on $C \setminus E$ with rational functions. Relation (2) can be written as

$$p_2(T)f_0(\lambda) = 0 \quad \text{for} \quad \lambda \notin E,$$

$$p_3(T)f_0(\lambda) = T^*a_{n-1}(\lambda)T^{n-1} - a_0(\lambda)I.$$  

Denote by $\beta_1(\lambda), \ldots, \beta_n(\lambda)$ the roots of the polynomial $p_3$. We have

$$(T - \beta_1(\lambda)I) \ldots (T - \beta_n(\lambda)I)f_0(\lambda) = 0 \quad \text{for} \quad \lambda \notin E$$

and obviously $(T - \beta_1(\lambda)I) \ldots (T - \beta_n(\lambda)I)f_0(\lambda) \neq 0$ for $\lambda \notin E$, by the definition of $E$. So (5) implies that $\beta_i(\lambda)$ is in the spectrum of $T$. A similar argument implies that $\beta_1(\lambda), \ldots, \beta_n(\lambda)$ are also in the spectrum of $T$. Consequently $|\beta_i(\lambda)| = \|T\|$ for $i = 1, \ldots, n$ and $\lambda \notin E$, where $\|T\|$ is a norm on the invariant subspace $F$. So the symmetric functions $a_0(\lambda), \ldots, a_{n-1}(\lambda)$ are also bounded on $C \setminus E$. Because the $\alpha_i$ coincide with rational functions on $C \setminus E$ we conclude from Liouville’s Theorem that there are constant numbers $\gamma_0, \ldots, \gamma_{n-1} \in C$ such that $a_i(\lambda) = \gamma_i$ for $\lambda \notin E$. Let $p(\lambda) = z^n - \gamma_0z^{n-1} - \ldots - \gamma_n$. Then $p(T)f_0(\lambda) = 0$ on $C \setminus E$, but also on $C$ by continuity of $\lambda$. In particular, $p(T)f_0(\lambda) = 0$ on $C \setminus E$, and so on $C$. Let $\lambda \in C$ be a complex number. Then $p(\lambda) = 0$. Consequently $p_0(T)f_0(\lambda) = 0$ for all $\eta \in X$. Hence $p_0(T) = 0$, so $T$ is algebraic of degree $\leq n$. 

A slight modification of the argument now gives

**Theorem 2.** Let $X$ and $Y$ be two complex vector space vectors, and let $T_1, \ldots, T_n$ be linear operators from $X$ into $Y$. Suppose that for every $\xi \in X$ the vectors $T_1\xi, \ldots, T_n\xi$ are linearly dependent. Then there exist $\lambda_1, \ldots, \lambda_n \in C$, not all zero, such that $Q = \lambda_1T_1 + \ldots + \lambda_nT_n$ has finite rank $\leq n-1$. Moreover, if $X = Y$ and the $T_i$ commute, then $Q^2 = 0$. 

**Proof.** If for all $\xi \in X$, the vectors $T_1\xi, \ldots, T_{n-1}\xi$ are linearly dependent, it is enough to prove the result with $T_1, \ldots, T_n$. So suppose that there exists $\xi \in X$ such that $T_1\xi, \ldots, T_{n-1}\xi$ are linearly independent and $T_n\xi$ are not. Then there exist $\lambda_1, \ldots, \lambda_n \in C$ such that

$$(\lambda_n + a_{n-1}T_{n-1} + \ldots + a_1T_1)\xi = 0.$$  

Let $\eta \in X$ be an arbitrary fixed vector and let $F$ be the linear subspace of $Y$ generated by $T_1\eta, \ldots, T_{n-1}\eta, T_n\eta$. Then $F \leq 2(n-1)$. For $\lambda \in C$ we set

$$f_0(\lambda) = \xi + \lambda\eta, \quad f_1(\lambda) = T_1f_0(\lambda) \in F, \quad \ldots, \quad f_{n-1}(\lambda) = T_{n-1}f_0(\lambda) \in F.$$  

$$g(\lambda) = T_nf_0(\lambda) \in F.$$  

Because $f_1(0), \ldots, f_{n-1}(0)$ are linearly independent in $F$ there exist $n-1$ linear functionals on $F$, denoted by $\varphi_1, \ldots, \varphi_{n-1}$, such that

$$\varphi_i(f_j(0)) = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq n-1.$$  

We define

$$A(\lambda) = \begin{bmatrix} \varphi_1(f_1(\lambda)) & \ldots & \varphi_1(f_{n-1}(\lambda)) \\ \varphi_2(f_1(\lambda)) & \ldots & \varphi_2(f_{n-1}(\lambda)) \\ \vdots & \ddots & \vdots \\ \varphi_{n-1}(f_1(\lambda)) & \ldots & \varphi_{n-1}(f_{n-1}(\lambda)) \end{bmatrix}$$

which is a polynomial of degree $\leq n-1$, satisfying $A(0) = 1$, and

$$A(\lambda) = \begin{bmatrix} \varphi_1(f_1(\lambda)) & \ldots & \varphi_1(f_{n-1}(\lambda)) \\ \varphi_2(f_1(\lambda)) & \ldots & \varphi_2(f_{n-1}(\lambda)) \\ \vdots & \ddots & \vdots \\ \varphi_{n-1}(f_1(\lambda)) & \ldots & \varphi_{n-1}(f_{n-1}(\lambda)) \end{bmatrix}$$

which is also a polynomial of degree $\leq n-1$, satisfying $A(0) = \alpha_n$, and $A(0) = 0$. If $F$ denotes the set of zeros of $A$ then, arguing as in the proof of Theorem 1, we conclude that

$$A(\lambda)T_n - A_{n-1}(\lambda)T_{n-1} - \ldots - A_1(\lambda)T_1f_0(\lambda) = 0$$

on $C \setminus E$, and so, by continuity, on all $C$. Let $\alpha_1 = 1$ and let $\beta_1, \ldots, \beta_n$ be the coefficients of $\lambda$ respectively in $-A_1(\lambda), \ldots, -A_{n-1}(\lambda), A_1(\lambda)$. Setting $Q = \alpha_1T_1 + \ldots + \alpha_nT_n$ (which does not depend on $\eta$) and looking at the coefficients of degree 0 and 1 in $\lambda$, from (9) we obtain

$$Qf_0 = 0, \quad Q\eta + R\xi = 0.$$  

6. **End.**
Consequently $Q\eta$ is in the linear subspace generated by $T_1 \xi_0, \ldots, T_{n-1} \xi_0$. So $Q$ has a finite rank $\leq n-1$. If moreover the $T_i$ commute, then $Q$ and $R$ commute, so $Q^2 \eta = -QR\xi_0 = -RQ\xi_0 = 0$. Hence $Q^2 = 0$.

Remark. Let $P$ and $Q$ be two different projections having the same range of dimension $1$, defined on a complex vector space $X$. For every $\xi \in X$, the vectors $P\xi$ and $Q\xi$ are dependent and obviously there are linear combinations of $P$ and $Q$ having rank one. But $\alpha P + \beta Q \neq 0$ for any $\alpha, \beta \in C$. So in general it is impossible to have $Q = 0$ in Theorem 2.

References


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Extension of $C^\infty$ functions from sets with polynomial cusps

by

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Abstract. We give a simple construction of a continuous linear operator extending $C^\infty$ functions from compact subsets of $R^n$ with polynomial cusps including fat subanalytic sets.

1. Introduction. Whitney's extension theorem [15] yields a continuous linear operator extending $C^k$ functions ($k$ finite) defined on closed subsets $X$ of $R^n$. For $C^\infty$ functions such an operator does not in general exist (see e.g. [12, p. 79]). However, Mityagin [4] and Seeley [7] proved the existence of an extension operator if $X$ is a half-space of $R^n$. Stein [9] showed that such an operator exists if $X$ is the closure of a Lipschitz domain in $R^n$ of class Lip1. Stein's result was then extended by Bierstone [1] to the case of a domain with boundary which is Lipschitz of any order. By the main result of Bierstone [1] involving Hironaka's desingularization theorem, an extension operator exists if $X$ is a fat (i.e. int $X \neq X$) closed subanalytic subset of $R^n$. If $X$ is Nash subanalytic (not necessarily fat) the existence problem was solved by Bierstone and Schwarz [3]. Recently Wacht [14] has constructed an extension operator for fat closed subanalytic sets in $R^n$ without making use of the Hironaka desingularization theorem. For closed subsets of $R^n$ admitting some polynomial cusps, the existence of an extension operator was shown by Tidten [10].

In this paper we construct an extension operator for the family of compact uniformly polynomially cuspidal (briefly, UPC) subsets of $R^n$ (see Theorem 4.1). The UPC sets were introduced in [6] as follows.

Definition 1.1. A subset $X$ of $R^n$ is said to be UPC if there exist positive constants $M$ and $m$, and a positive integer $d$ such that for each point $x$ in $X$, one may choose a polynomial map $h_x: R \rightarrow R^d$ of degree at most $d$ satisfying the following conditions:

(i) $h_x(0,0) \in X$ and $h_x(0) = x$;

(ii) $\text{dist}(h(a), R^n-X) \geq M^m$ for all $x \in X$ and $t \in (0,1)$.

Every bounded convex domain in $R^n$ and every bounded Lipschitz domain are UPC. More generally, every subset of $R^n$ with a parallelepiped