sion of the space is a multiple of 4. Slight variants yield the other cases. In particular, B can be defined as

$$\left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \\ & & & -1 \\ \hline & 0 & & 0 \end{array}\right]$$

for any number of 1's. Then A can be defined as above on even-dimensional spaces, and as the direct sum of such an operator and a one-dimensional 0 in the case of odd-dimensional spaces. We omit the details.

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UNIVERSITY OF NEW HAMPSHIRE
Durham, New Hampshire, U.S.A.

UNIVERSITY OF KERMAN
Kerman, Iran

DALHOUSIE UNIVERSITY
Halifax, Nova Scotia, Canada

UNIVERSITY OF TORONTO
Toronto, Ontario, Canada

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Vectors of uniqueness for *id/dx*

by

RALPH DE LAUBENFELS (Athens, O.)

Abstract. A vector x in a Hilbert space is a *vector of uniqueness* for a symmetric operator A if A, with domain restricted to span$\{A^n x \mid n = 0, 1, \ldots\}$, is essentially selfadjoint on the closure of this domain. We characterize vectors of uniqueness for the operator id/dx on $L^2[0, 1]$. Let $\hat{g}(k) \equiv \int_0^1 g(t) e^{-2\pi i k t}\, dt$, $g^{(n)} \equiv$ the nth derivative of g, $E \equiv \{k \mid \hat{g}(k) \neq 0\}$. We show that g fails to be a vector of uniqueness if and only if there exists a nontrivial f such that $f^{(n)}(0) = 0 = f^{(n)}(1)$, for all n, and

$$\sum_{k \in E} \left|\frac{\hat{f}(k)}{\hat{g}(k)}\right|^2 \frac{1}{1+k^2} < \infty.$$

We show that g is a vector of uniqueness if and only if the closure of span$\{g^{(n)} \mid n = 1, 2, \ldots\}$ equals $\{f \in L^2[0, 1] \mid \hat{f}(k) = 0 \text{ when } k \notin E\}$.

We show that g fails to be a vector of uniqueness for id/dx on $L^2(\mathbf{R})$ if and only if there exists a nontrivial f such that $f^{(n)}(0) = 0$, for all n, and

$$\int_E \left|\frac{\mathcal{F}f(t)}{\mathcal{F}g(t)}\right|^2 \frac{dt}{1+t^2} < \infty,$$

where $\mathcal{F}$ is the Fourier transform, and E is the support of $\mathcal{F}g$.

Introduction. Vectors of uniqueness were introduced by Nussbaum [4] (see Definition 2). He showed that a symmetric operator on a Hilbert space is selfadjoint if and only if it has a total set of vectors of uniqueness. In the same paper, and in subsequent papers, the selfadjointness of certain operators is shown by proving that certain classes of vectors are always vectors of uniqueness (see [3]–[5]).

Nussbaum defined vectors of uniqueness in terms of the classical moment problem. He defines x to be a vector of uniqueness for A if the moment sequence $\{\langle A^n x, x\rangle\}_{n=0}^{\infty}$ is determined. We use the equivalent definition given in [6], vol. 2, p. 201 (Definition 2).

It is often advantageous, when considering questions of essential selfadjointness, to focus on vectors of uniqueness. It is precisely in this setting, when the domain of A equals span$\{A^n x \mid n = 0, 1, \ldots\}$, for a fixed x (see Definition 2), that the spectral theorem says that selfadjointness is equivalent to A being unitarily equivalent to multiplication by $f(t) \equiv t$ on $L^2(\mathbf{R}, \mu)$, for some measure μ.

We characterize vectors of uniqueness for id/dx on $L^2[0, 1]$ in terms of Fourier coefficients (Theorem 11). Intuitively, the more slowly the Fourier coefficients go to zero, the more rapidly the derivatives of the function go to infinity, which makes it less likely that the function will be a vector of uniqueness (see Corollaries 15, 16, 24 and 25). For example, functions whose derivatives satisfy a quasi-analytic growth condition are automatically vectors of uniqueness (see [1] and [4]).

Perhaps the most surprising result is that there are so many C^{∞} functions that are not vectors of uniqueness. This includes any function with a zero of infinite order (Corollaries 12, 13, 22 and 23). This is related to quasi-analyticity, since a quasi-analytic class of functions is one that contains no nontrivial functions with a zero of infinite order (see [7], Ch. 19).

Proposition 4 and Corollaries 5 and 6 are preliminary results about essential selfadjointness and vectors of uniqueness. Results about id/dx on $L^2[0, 1]$ are contained in Theorems 11 and 17 and Corollaries 12, 13, 15, 16 and 18. Results about id/dx on $L^2(\mathbf{R})$ are contained in Theorems 21 and 26 and Corollaries 22–25.

All operators are linear, on a Hilbert space, with inner product $\langle\ \rangle$.

DEFINITION 1. $D(A) \equiv$ domain of the operator A. $C^{\infty}(A) \equiv \bigcap_{n=1}^{\infty} D(A^n)$.

DEFINITION 2. If x is in $C^{\infty}(A)$, then

$$D(x, A) \equiv \operatorname{span}\{A^n x \mid n = 0, 1, \ldots\}.$$

We say that x is a *vector of uniqueness for A* if the operator $By \equiv Ay$, $D(B) \equiv D(x, A)$, is essentially selfadjoint on $\overline{D(B)}$.

A symmetric operator is selfadjoint if and only if it has a total set of vectors of uniqueness (see [4] and [6], vol. 2, p. 201).

DEFINITION 3. $A^* \equiv$ the adjoint of A, $\bar{A} \equiv$ the closure of A. The *graph-closure* of $D(A)$ is the closure with respect to the *graph norm* $\|x\|_G^2 \equiv \|x\|^2 + \|Ax\|^2$. Note that the graph-closure of $D(A)$ is $D(\bar{A})$.

For completeness, we include a proof of the following well-known proposition.

PROPOSITION 4. *Let* $Af(t) \equiv tf(t)$, $D(A) \equiv \{$*polynomials*$\}$, *in* $L^2(\mathbf{R}, \mu)$. *The following are equivalent*:

(a) *The operator A is essentially selfadjoint on* $\overline{D(A)}$.

(b) *The graph-closure of $D(A)$ equals*

$$\Big\{f \text{ in } L^2(\mathbf{R}, \mu) \Big| \int_{-\infty}^{\infty} t^2 |f(t)|^2 \, d\mu(t) < \infty\Big\}.$$

Proof. Let B be the operator $Bf(t) \equiv tf(t)$ on $L^2(\mathbf{R}, \mu)$,

$$D(B) \equiv \Big\{f \Big| \int_{-\infty}^{\infty} t^2 |f(t)|^2 \, d\mu(t) < \infty\Big\}.$$

(b) $\Rightarrow$ (a). $\bar{A} = B$, which is selfadjoint.

(a) $\Rightarrow$ (b). Since B is an extension of A, and B is closed, B is an extension of $\bar{A}$. Thus $(e^{is\bar{A}}f)(t) = (e^{isB}f)(t) = e^{ist}f(t)$, for all f in $\overline{D(A)}$, where $\{e^{is\bar{A}}\}_{s\in\mathbf{R}}$ is the one-parameter group generated by $\bar{A}$. Suppose g is in $(D(A))^{\perp} = (D(\bar{A}))^{\perp}$, in $L^2(\mathbf{R}, \mu)$. Then, for any real s, $0 = \langle e^{is\bar{A}}1, g\rangle = \int_{-\infty}^{\infty} e^{ist} g(t) \, d\mu(t)$. Thus $g(t)\, d\mu(t) \equiv 0$, so that $g = 0$ in $L^2(\mathbf{R}, \mu)$. Thus $\overline{D(A)} = L^2(\mathbf{R}, \mu)$. Since B is a selfadjoint extension of A on the same space, and A is essentially selfadjoint, $\bar{A} = B$, as desired.

COROLLARY 5. *Let $Af(t) \equiv tf(t)$ on $L^2(\mathbf{R})$, g in $C^{\infty}(A)$, $E \equiv$ support of g. Then g is a vector of uniqueness for A if and only if the graph-closure of $D(g, A)$ equals* $\{f \in L^2(E) \mid \int_E t^2 |f(t)|^2 \, dt < \infty\}$.

Proof. Define the absolutely continuous measure μ by $d\mu(t) \equiv |g(t)|^2 \, dt$, and the unitary operator U: $L^2(E) \to L^2(\mathbf{R}, \mu)$ by $Uh \equiv h/g$. By Proposition 4, since $(UAU^{-1})f(t) = tf(t)$ on $L^2(\mathbf{R}, \mu)$, A is essentially selfadjoint if and only if the graph-closure in $L^2(\mathbf{R}, \mu)$ of $\{$polynomials$\}$ equals

$$\Big\{h \in L^2(\mathbf{R}, \mu) \Big| \int_{-\infty}^{\infty} t^2 |h(t)|^2 |g(t)|^2 \, dt < \infty\Big\}.$$

Since U^{-1} of the latter set is $\{f \in L^2(E) \mid \int_E t^2 |f(t)|^2 \, dt < \infty\}$, the result follows.

COROLLARY 6. *Let $(Af)(k) \equiv kf(k)$ on $l^2(\mathbf{Z})$, g in $C^{\infty}(A)$, $E \equiv \{k \mid g(k) \neq 0\}$. Then g is a vector of uniqueness for A if and only if the graph-closure of $D(g, A)$ equals* $\{f \in l^2(E) \mid \sum k^2 |f(k)|^2 < \infty\}$.

Proof. This is the same as Corollary 5, letting m be the discrete measure supported on E with $m(\{k\}) \equiv |g(k)|^2$.

DEFINITION 7. $g^{(n)} \equiv n$th derivative of g. $\hat{f}(k) \equiv \int_0^1 f(t) e^{-2\pi ikt} \, dt$, the kth Fourier coefficient of f. For E a set of integers,

$$L_E^2[0, 1] \equiv \{f \text{ in } L^2[0, 1] \mid \hat{f}(k) = 0 \text{ when } k \notin E\}.$$

DEFINITION 8. By id/dx on $L^2[0, 1]$ we will mean the operator with domain

$$D(A) = \{\text{absolutely continuous } f \mid f' \text{ is in } L^2[0, 1],\ f(0) = f(1)\}$$
$$= \{f \text{ in } L^2[0, 1] \mid \textstyle\sum k^2 |\hat{f}(k)|^2 < \infty\}.$$

Note that $C^{\infty}(A) = \{$infinitely differentiable $f \mid f^{(n)}(0) = f^{(n)}(1)$, for all $n\}$ (see Definition 1).

LEMMA 9. *Let A be id/dx on $L^2[0,1]$, g in $C^\infty(A)$, $E=\{k\,|\,\hat{g}(k)\neq 0\}$. Then g is a vector of uniqueness for A if and only if the graph-closure of $D(g,A)$ equals $\{f$ in $L^2_E[0,1]\,|\,\sum k^2|\hat{f}(k)|^2<\infty\}$.*

Proof. This follows from Corollary 6, since A is unitarily equivalent, via Fourier series, to the operator considered there.

DEFINITION 10. A function f has a *zero of infinite order* if there exists x such that $f^{(n)}(x)=0$, for all n.

THEOREM 11. *Let A be id/dx on $L^2[0,1]$, g in $C^\infty(A)$, $E\equiv\{k\,|\,\hat{g}(k)\neq 0\}$. Then g fails to be a vector of uniqueness for A if and only if there exists a nontrivial F in $C^\infty(A)\cap L^2_E[0,1]$, with a zero of infinite order, such that*

$$\sum_{k\in E}\frac{1}{1+k^2}\left|\frac{\hat{F}(k)}{\hat{g}(k)}\right|^2<\infty.$$

Proof. Let $\mathcal{D}\equiv\{f\in L^2_E[0,1]\,|\,\sum k^2|\hat{f}(k)|^2<\infty\}$. Define the graph inner product $\langle\ \rangle_G$ on $\mathcal{D}$ by $\langle f,k\rangle_G\equiv\langle f,k\rangle+\langle Af,Ak\rangle$. Note that $\langle f,f\rangle_G=\|f\|_G^2$ (see Definition 3).

By Lemma 9, g fails to be a vector of uniqueness if and only if there exists a nontrivial h in $\mathcal{D}$ such that, for $n=0,1,\ldots$,

$$0=\langle h,A^n g\rangle_G=\langle h,A^n g\rangle+\langle Ah,A^{n+1}g\rangle=\sum_{k=-\infty}^{\infty}(k^n+k^{n+2})\hat{h}(k)\hat{g}(k)$$
$$=i^n\big((h*g)^{(n)}(0)-(h*g)^{(n+2)}(0)\big),$$

where $h*g$ is the convolution of h with g.

If such an h exists, let $F\equiv(h*g)-(h*g)^{(2)}$. Then F has a zero of infinite order, $\hat{F}(k)=0$ when $\hat{g}(k)=0$, and

$$\sum_{\hat{g}(k)\neq 0}\frac{1}{1+k^2}\left|\frac{\hat{F}(k)}{\hat{g}(k)}\right|^2=\sum_{\hat{g}(k)\neq 0}(1+k^2)|\hat{h}(k)|^2=\|h\|_G^2<\infty,$$

as desired.

Conversely, if such an F exists, we may assume, by translating F if necessary, that $F^{(n)}(0)=0$, for all n. Let

$$h(t)\equiv\sum_{\hat{g}(k)\neq 0}\frac{1}{1+k^2}\cdot\frac{\hat{F}(k)}{\hat{g}(k)}e^{2\pi ikt}.$$

Then h is in $\mathcal{D}$, and $0=F^{(n)}(0)=(h*g)^{(n)}(0)-(h*g)^{(n+2)}(0)$, for all n, so that g fails to be a vector of uniqueness.

COROLLARY 12. *If there exists F in $C^\infty(A)$, with a zero of infinite order, such that, for some $\varepsilon<1$, $|\hat{F}(k)|\leqslant|k^\varepsilon\hat{g}(k)|$, for all k, then g is not a vector of uniqueness for $A\equiv id/dx$ on $L^2[0,1]$.*

COROLLARY 13. *If g has a zero of inifinite order, then g is not a vector of uniqueness for id/dx on $L^2[0,1]$.*

Remark 14. We proved Corollary 13 in [2] by showing that the deficiency indices of id/dx, with domain equal to span $\{g^{(n)}\,|\,n=0,1,\ldots\}$, are nonzero.

COROLLARY 15. *If h is not a vector of uniqueness for id/dx on $L^2[0,1]$, and $|\hat{g}(k)|\geqslant|\hat{h}(k)|$, for all k, then g is not a vector of uniqueness.*

COROLLARY 16. *If h is a vector of uniqueness for id/dx on $L^2[0,1]$, and $|\hat{g}(k)|\leqslant|\hat{h}(k)|$, for all k, then g is a vector of uniqueness.*

THEOREM 17. *Let $E\equiv\{k\,|\,\hat{g}(k)\neq 0\}$, where g is as in Theorem 11. Then $\overline{D(g,A)}\neq L^2_E[0,1]$ if and only if there exists a nontrivial F in $C^\infty(A)\cap L^2_E[0,1]$, with a zero of infinite order, such that*

$$\sum_{k\in E}\left|\frac{\hat{F}(k)}{\hat{g}(k)}\right|^2<\infty.$$

Proof. This is the same as the proof of Theorem 11, using the regular inner product instead of $\langle\ \rangle_G$, with $F\equiv h*g$.

COROLLARY 18. *Let A, g and E be as in Theorem 11. Then g is a vector of uniqueness for A if and only if $\overline{D(Ag,A)}=L^2_E[0,1]$.*

Proof. This follows from Theorems 11 and 17, using the fact that $(g')\hat{}(k)=-ik\hat{g}(k)$.

DEFINITION 19. We will denote by $\mathcal{F}$ the Fourier transform,

$$(\mathcal{F}f)(s)\equiv\int_{-\infty}^{\infty}f(t)e^{-ist}\,dt.$$

LEMMA 20. *Let A be id/dx on $L^2(\boldsymbol{R})$, g in $C^\infty(A)$, and let E be the support of $\mathcal{F}g$. Then g is a vector of uniqueness for A if and only if the graph-closure of $D(g,A)$ equals $\{f\,|\,\mathcal{F}f$ is supported on E, $\int_E t^2|\mathcal{F}f(t)|^2\,dt<\infty\}$.*

Proof. This follows from Corollary 5, as Lemma 9 followed from Corollary 6.

Using Lemma 20 and the Fourier transform in place of Lemma 9 and Fourier series, we get analogous results for id/dx on $L^2(\boldsymbol{R})$.

THEOREM 21. *Let A be id/dx on $L^2(\boldsymbol{R})$, g in $C^\infty(A)$. Then g fails to be a vector of uniqueness for A if and only if there exists a nontrivial H in $C^\infty(A)$, with a zero of infinite order, such that $\mathcal{F}H(s)=0$ when $\mathcal{F}g(s)=0$, and*

$$\int_E \left| \frac{(\mathscr{F}H)(t)}{(\mathscr{F}g)(t)} \right|^2 \frac{dt}{1+t^2} < \infty,$$

where $E \equiv \{t \mid \mathscr{F}g(t) \neq 0\}$.

COROLLARY 22. *With* A *and* g *as in Theorem* 21, *suppose there exists* H *in* $C^\infty(A)$ *such that for some* $\varepsilon < 1$,

$$|\mathscr{F}H(t)| \leqslant t^\varepsilon \mathscr{F}g(t)| \text{ a.e.}$$

Then g *is not a vector of uniqueness for* A.

COROLLARY 23. *If* g *has a zero of infinite order, then* g *is not a vector of uniqueness for* id/dx *on* $L^2(\boldsymbol{R})$.

COROLLARY 24. *If* h *is not a vector of uniqueness for* id/dx *on* $L^2(\boldsymbol{R})$ *and* $|\mathscr{F}g(t)| \geqslant |\mathscr{F}h(t)|$, *for almost all* t, *then* g *is not a vector of uniqueness.*

COROLLARY 25. *If* h *is a vector of uniqueness for* id/dx *on* $L^2(\boldsymbol{R})$ *and* $|\mathscr{F}g(t)| \leqslant |\mathscr{F}h(t)|$, *for almost all* t, *then* g *is a vector of uniqueness.*

THEOREM 26. *Let* $E \equiv \{t \mid \mathscr{F}g(t) \neq 0\}$, *where* g *and* A *are as in Theorem* 21. *Then* $D(g, A) \neq \{f \text{ in } L^2(\boldsymbol{R}) \mid \mathscr{F}f(t) = 0 \text{ when } t \notin E\}$ *if and only if there exists a nontrivial* F *in* $C^\infty(A)$, *with a zero of infinite order, such that* $\mathscr{F}F(t) = 0$ *when* $\mathscr{F}g(t) = 0$, *and*

$$\int_E \left| \frac{\mathscr{F}F(t)}{\mathscr{F}g(t)} \right|^2 dt < \infty.$$

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DEPARTMENT OF MATHEMATICS
OHIO UNIVERSITY
Athens, Ohio 45701, U.S.A.

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An improvement of Kaplansky's lemma on locally algebraic operators

by

BERNARD AUPETIT* (Québec)

Abstract. Let X and Y be two complex vector spaces and let $T_1, \ldots, T_n$ be linear operators from X into Y. Suppose that for every $\xi \in X$ the vectors $T_1\xi, \ldots, T_n\xi$ are linearly dependent. Then, using an analytic argument, we prove that there exists a nontrivial linear combination of these operators having rank $\leqslant n-1$.

Let T be a linear operator on a complex vector space X. Then T is *locally algebraic* if for every $\xi \in X$ there exists a nontrivial polynomial p such that $p(T)\xi = 0$. A standard result of I. Kaplansky ([3], Lemma 14) states that *boundedly locally algebraic* (the degree of p is bounded independently of ξ) implies *algebraic* (for another proof see [5]). This important result has many consequences (see for instance [2]–[4], [6]). In this short paper we present an analytic proof of that result. This argument is very interesting because it implies a surprising extension of Kaplansky's lemma.

THEOREM 1. *Let* X *be a complex vector space and let* T *be a linear operator from* X *into* X. *Suppose that there exists an integer* $n \geqslant 1$ *such that* $\xi, T\xi, \ldots, T^n\xi$ *are linearly dependent for all* $\xi \in X$. *Then* T *is algebraic of degree* $\leqslant n$.

Proof. Suppose that n is the smallest integer having this property. Hence there exists $\xi_0 \in X$ such that $\xi_0, T\xi_0, \ldots, T^{n-1}\xi_0$ are linearly independent but $\xi_0, T\xi_0, \ldots, T^n\xi_0$ are not. Then there exists a monic polynomial p_0 of degree n such that $p_0(T)\xi_0 = 0$ and if p is another monic polynomial of degree n such that $p(T)\xi_0 = 0$ then $p = p_0$. Let $\eta \in X$ be an arbitrary fixed vector. We now prove that $p_0(T)\eta = 0$. Let F be the linear subspace generated by $\xi_0, T\xi_0, \ldots, T^n\xi_0, \eta, T\eta, \ldots, T^n\eta$. Then $\dim F \leqslant 2n$. For $\lambda \in \boldsymbol{C}$ we set

$$f_0(\lambda) = \xi_0 + \lambda\eta \in F, \quad f_1(\lambda) = Tf_0(\lambda) \in F, \quad \ldots, \quad f_{n-1}(\lambda) = T^{n-1}f_0(\lambda) \in F,$$
$$g(\lambda) = T^n f_0(\lambda) \in F.$$

Because $f_0(0), \ldots, f_{n-1}(0)$ are linearly independent in F there exist n linear

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