of any vector in $l(F_0)$. This establishes Claim 4. Claims 3 and 4 provide the desired contradiction. So $v_1' = 0$. ■

The hypothesis that the subgroups in Theorem 2.1 be nondiscrete cannot be eliminated, as the following example demonstrates.

2.2. EXAMPLE. Define G_1 to be the two-element group and G_2 to be the circle group. Let μ_1 and μ_2 be the Haar measures on G_1 and G_2, respectively Set $G = G_1 \oplus G_2$, $H_1 = G_1 \times \{1\}$, $H_2 = \{0\} \times G_2$. Then $\mu = \mu_1 \otimes \mu_2$ is the Haar measure on G.

Let v be an extreme point of $E(H_1, H_2; \mu)$. The following statements are not hard to prove. There is a Borel set $A \subseteq G_2$ with $\mu_2 A = 1/2$ such that if $F\colon G \to \boldsymbol{R}$ is defined by

$$F(u) = \begin{cases} 2 & \text{if } u \in (\{0\} \times A) \cup (\{1\} \times (G_2 \setminus A)), \\ 0 & \text{if } u \in (\{0\} \times (G_2 \setminus A)) \cup (\{1\} \times A), \end{cases}$$

then $dv = F\,d\mu$. Thus every element of $E(H_1, H_2; \mu)$ is absolutely continuous with respect to μ.

A more intriguing question is whether the subgroups in Theorem 2.1 must be normal. The proof requires this assumption, but in the examples worked out by the author, normality does not seem to be required.

2.3. CONJECTURE. The hypothesis that the H_i be normal subgroups may be dropped from the statement of Theorem 2.1.

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Quadratic operators and invariant subspaces

by

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Abstract. It is shown that every operator has an invariant subspace if and only if every pair of operators satisfying a given pair of quadratic equations has a common invariant subspace. On the other hand, it is also shown that there exists a pair of operators satisfying a cubic and a quadratic equation which generate the algebra of all operators as a strongly closed algebra.

In [2] it is shown that every operator on Hilbert space has a nontrivial invariant subspace if and only if every pair of idempotents has a common nontrivial invariant subspace.

Does the corresponding result hold for pairs of nilpotents of index two? For a nilpotent of index two and an idempotent? More generally, is the existence of common invariant subspaces for pairs of operators satisfying given polynomial equations of degree 2 equivalent to the invariant subspace problem? The purpose of this note is to answer these and certain related questions about generators of the algebra of all bounded linear operators.

We consider bounded linear operators on separable infinite-dimensional complex Hilbert spaces; some remarks on other spaces are given at the end.

The following theorem generalizes [2].

THEOREM 1. *Let p and q be polynomials of degree 2. Then every operator has a nontrivial invariant subspace if and only if every pair $\{A, B\}$ of operators satisfying $p(A) = q(B) = 0$ has a common nontrivial invariant subspace.*

Proof. By dividing by the leading coefficients and completing the square, we can assume that the polynomials have the forms $p(x) = (x-a)^2 - \lambda^2$ and $q(x) = (x-b)^2 - \mu^2$ for complex numbers a, λ, b and μ. Since $A - aI$ and $B - bI$ have a common invariant subspace if and only if A and B do, we can assume that $p(x) = x^2 - \lambda^2$ and $q(x) = x^2 - \mu^2$.

Suppose first that every operator has a nontrivial invariant subspace and $A^2 = \lambda^2 I$, $B^2 = \mu^2 I$. We must show that A and B have a common invariant subspace.

Let $\mathscr{R}$ denote the closure of the range of $A + \lambda I$. Since

$$(AB+BA)(A+\lambda I) = ABA + B(\lambda^2) + \lambda(AB+BA)$$

and

$$(A+\lambda I)B(A+\lambda I) = ABA+\lambda BA+\lambda AB+\lambda^2 B$$

are equal, $\mathscr{R}$ is invariant under $AB+BA$, and the restriction of $AB+BA$ to $\mathscr{R}$ is equal to the restriction of $(A+\lambda I)B$. If the dimension of $\mathscr{R}$ is greater than 1, let $\mathscr{M}$ be any nontrivial invariant subspace of the restriction of $AB+BA$ to $\mathscr{R}$; if the dimension of $\mathscr{R}$ is 1, let $\mathscr{M} = \mathscr{R}$. If the dimension of $\mathscr{R}$ is 0 the result is trivially true.

Let $\mathscr{N} = \mathscr{M}\vee(B\mathscr{M})$. We claim that $\mathscr{N}$ is a nontrivial common invariant subspace of A and B. Since $B^2 = \mu^2 I$, $\mathscr{N}$ is obviously invariant under B. Also, $(A-\lambda I)\mathscr{M} = \{0\}$, so $A\mathscr{M}\subset\mathscr{M}$. Now the invariance of $\mathscr{M}$ under $AB+BA$ gives $AB\mathscr{M}\subset\mathscr{M}\vee B\mathscr{M} = \mathscr{N}$, so $\mathscr{N}$ is a common invariant subspace.

Since $\mathscr{M}\neq\{0\}$, $\mathscr{N}\neq\{0\}$. It only remains to be shown that $\mathscr{N}\neq\mathscr{H}$. In the case where $\mathscr{M} = \mathscr{R}$, $\mathscr{N}$ has dimension at most 2 and thus is not $\mathscr{H}$. In the case where $\mathscr{M}$ is a proper subspace of $\mathscr{R}$, we show that $\mathscr{N}$ is proper by showing that $(A+\lambda I)\mathscr{N}$ is not dense in $\mathscr{R}$. To see this recall that $A\mathscr{M}\subset\mathscr{M}$ and

$$(A+\lambda I)B\mathscr{M} = (AB+BA)\mathscr{M}\subset\mathscr{M},$$

so

$$(A+\lambda I)\mathscr{N}\subset(A+\lambda I)\mathscr{M}\vee(A+\lambda I)B\mathscr{M}\subset\mathscr{M}.$$

Hence $\mathscr{N}$ is proper.

To prove the converse, fix $p(x) = x^2-\lambda$ and $q(x) = x^2-\mu$ and suppose that every pair $\{A, B\}$ satisfying $p(A) = q(B) = 0$ has a common nontrivial invariant subspace. Let T be any operator. The definitions of the corresponding operators A and B depend upon whether λ or $\mu = 0$.

Let λ_1 be any square root of λ and μ_1 any square root of μ.

Case (i): If $\lambda = \mu = 0$ (i.e., we are assuming that all pairs of nilpotents of index 2 have common invariant subspaces), define

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \quad\text{and}\quad B = \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}.$$

Case (ii): If $\lambda = 0$ and $\mu\neq 0$, let

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \quad\text{and}\quad B = \mu_1\begin{bmatrix} 2T-I & 2T \\ 2I-2T & I-2T \end{bmatrix}.$$

Case (iii): If $\lambda\neq 0$ and $\mu\neq 0$, define

$$A = \lambda_1\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad\text{and}\quad B = \mu_1\begin{bmatrix} 2T-I & 2T \\ 2I-2T & I-2T \end{bmatrix}.$$

It is clear in all three cases that $p(A) = q(B) = 0$.

Now assume that $\mathscr{N}$ is a nontrivial common invariant subspace of $\{A, B\}$. In each case, $T\oplus T$ is a linear combination of $AB+BA$ and $I\oplus I$ (compute $AB+BA$), so $\mathscr{N}$ is invariant under $T\oplus T$. Let $\mathscr{N}_1 = \{x\colon x\oplus 0\in\mathscr{N}\}$ and $\mathscr{N}_2 = \{y\colon 0\oplus y\in\mathscr{N}\}$. Then the $\mathscr{N}_i$ are closed subspaces and are invariant under T.

If T has compression spectrum, then T has a nontrivial invariant subspace, so we can assume that the range of $T-\gamma I$ is dense for all γ. Under this assumption we will show that at least one $\mathscr{N}_i$ is nontrivial.

Since $\mathscr{N}\supset\mathscr{N}_1\oplus\mathscr{N}_2$, at least one $\mathscr{N}_i$ is not $\mathscr{H}$. If $\mathscr{N}_1 = \mathscr{H}$ then $\mathscr{N}_2 = \mathscr{H}$; this is clear in case (i) (since $\mathscr{N}_2$ then contains the range of T) and follows in the other cases from the fact that

$$\begin{bmatrix} 2T-I & 2T \\ 2I-2T & I-2T \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 2Tx-x \\ (2I-2T)x \end{bmatrix}$$

implies that $\mathscr{N}_2$ contains the range of $2I-2T$. Hence $\mathscr{N}_1\neq\mathscr{H}$. Similarly, if $\mathscr{N}_2 = \mathscr{H}$, then $\mathscr{N}_1 = \mathscr{H}$; this is obvious in cases (i) and (ii) (apply A) and follows in case (iii) from

$$\begin{bmatrix} 2T-I & 2T \\ 2I-2T & I-2T \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 2Ty \\ y-2Ty \end{bmatrix}.$$

Therefore neither $\mathscr{N}_1$ nor $\mathscr{N}_2$ is $\mathscr{H}$.

If $\mathscr{N}_1$ and $\mathscr{N}_2$ were both $\{0\}$, then $\mathscr{N}$ would contain a vector $x\oplus y$ with $x\neq 0$ and $y\neq 0$. Applying A in cases (i) and (ii) and $A+\lambda_1 I$ in case (iii) shows that $\mathscr{N}_1\neq\{0\}$, which is a contradiction.

We have shown, then, that at least one of $\mathscr{N}_1$ and $\mathscr{N}_2$ is a nontrivial invariant subspace of T.

Chandler Davis [1] has shown that there exist three Hermitian idempotents that generate $\boldsymbol{B}(\mathscr{H})$ as a strongly closed algebra; Davis [1] also pointed out that a pair of such operators cannot generate. Can a pair of quadratic operators generate $\boldsymbol{B}(\mathscr{H})$?

THEOREM 2. *If A and B are quadratic operators, then the weakly closed algebra generated by $\{I, A, B\}$ is not $\boldsymbol{B}(\mathscr{H})$.*

Proof. If the operator C satisfies $C^2+aC+b = 0$, α is any solution of $\alpha^2-a\alpha+b = 0$ and $\beta = a-2\alpha$, then

$$(C+\alpha I)^2 = -\beta(C+\alpha I).$$

Thus translating the operators and dividing by appropriate square roots will reduce this result to the following three cases.

Case (i): $A^2 = A$, $B^2 = B$.
Case (ii): $A^2 = 0$, $B^2 = 0$.
Case (iii): $A^2 = 0$, $B^2 = B$.

In cases (i) and (ii) a simple computation shows that $(A-B)^2$ commutes with both A and B, so the algebra generated by A and B has a nontrivial commutant unless $(A-B)^2$ is a multiple of the identity. If $(A-B)^2$ is a multiple of the identity, then the algebra generated by $\{A, B, I\}$ is at most 5-dimensional.

In case (iii) the operator $AB+BA-A$ commutes with both A and B. If $AB+BA-A$ is a multiple of I, then the algebra generated by $\{A, B, I\}$ is at most 5-dimensional.

Do similar results hold for cubic operators? The following provides a strong counterexample.

THEOREM 3. *There exist Hermitian operators A and B such that $A^2 = A$, $B(B-I)(B+I) = 0$ and $\{A, B\}$ generates $\boldsymbol{B}(\mathscr{H})$ as a strongly closed algebra.*

Proof. Let $\mathscr{H}_0$ be a separable infinite-dimensional space and let

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_0 \oplus \mathscr{H}_0 \oplus \mathscr{H}_0.$$

Let B denote the operator with matrix

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with respect to $\mathscr{H}_0 \oplus \mathscr{H}_0 \oplus \mathscr{H}_0 \oplus \mathscr{H}_0$.

Now let K be any operator on $\mathscr{H}_0 \oplus \mathscr{H}_0$ that is compact, injective, positive, less than I, of uniform multiplicity 1, and has no nontrivial invariant subspaces in common with $\left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$. (The existence of such a K follows from the proof of Corollary 1 of [3]). Let A denote the operator with matrix

$$\begin{bmatrix} K & \sqrt{K(1-K)} \\ \sqrt{K(1-K)} & 1-K \end{bmatrix}$$

with respect to $(\mathscr{H}_0 \oplus \mathscr{H}_0) \oplus (\mathscr{H}_0 \oplus \mathscr{H}_0)$. Clearly $A^2 = A$.

Recall that a set $\mathscr{S}$ generates $\boldsymbol{B}(\mathscr{H})$ as a strongly closed algebra if $\mathscr{S} \cup \{I\}$ generates $\boldsymbol{B}(\mathscr{H})$ as a strongly closed algebra (for if A is the algebra generated by $\mathscr{S}$ then $A \vee \{I\}$ is the algebra generated by $\mathscr{S} \cup \{I\}$, A is an ideal in $A \vee \{I\}$, so if $A \vee \{I\} = \boldsymbol{B}(\mathscr{H})$ then A contains all finite rank operators). Thus it suffices to show that $\{A, B, I\}$ generates $\boldsymbol{B}(\mathscr{H})$.

Let $\boldsymbol{B}$ denote the strongly closed algebra generated by $\{A, B, I\}$. Since B^2 is the projection of $\mathscr{H}$ onto $\mathscr{H}_0 \oplus \mathscr{H}_0$, $\left[\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \boldsymbol{B}$. The algebra generated by K and $\left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ is an irreducible von Neumann algebra, hence is $\boldsymbol{B}(\mathscr{H}_0 \oplus \mathscr{H}_0)$, so $\boldsymbol{B}$ contains every operator of the form $\left[\begin{smallmatrix} C & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for $C \in \boldsymbol{B}(\mathscr{H}_0 \oplus \mathscr{H}_0)$. Thus $\boldsymbol{B}$ contains all operators of the form

$$\begin{bmatrix} 0 & C\sqrt{K(1-K)} \\ 0 & 0 \end{bmatrix}.$$

For P any injective operator on a space $\mathscr{K}$, $\{CP\colon C \in \boldsymbol{B}(\mathscr{K})\}$ is strongly dense in $\boldsymbol{B}(\mathscr{K})$ (in fact, strictly dense: if $\{x_1, \ldots, x_n\}$ is linearly independent, then so is $\{Px_1, \ldots, Px_n\}$, so for each $\{y_1, \ldots, y_n\}$ there exists a C such that $CPx_i = y_i$ for all i). Therefore $\boldsymbol{B}$ contains every operator of the form $\left[\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right]$. Since $\boldsymbol{B}$ is selfadjoint, $\boldsymbol{B}$ also contains all operators of the form $\left[\begin{smallmatrix} 0 & 0 \\ D & 0 \end{smallmatrix}\right]$. Also

$$\begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & DC \end{bmatrix}.$$

So $\boldsymbol{B} = \boldsymbol{B}(\mathscr{H})$.

This theorem includes the result of Davis [1].

COROLLARY ([1]). *There exist three Hermitian projections such that the strongly closed algebra they generate is $\boldsymbol{B}(\mathscr{H})$. In fact, two of them can be taken to be mutually orthogonal.*

Proof. In the notation of the proof of Theorem 3 above, let $P_1 = A$, let P_2 denote the projection of $\mathscr{H}$ onto $\mathscr{H}_0 \oplus \{0\} \oplus \{0\} \oplus \{0\}$ and let P_3 denote the projection of $\mathscr{H}$ onto $\{0\} \oplus \mathscr{H}_0 \oplus \{0\} \oplus \{0\}$. Clearly the algebra generated by $\{P_2, P_3\}$ contains B, so the algebra generated by $\{P_1, P_2, P_3\}$ is $\boldsymbol{B}(\mathscr{H})$.

There are several remarks to be made about the situation on other spaces. First of all, on nonseparable Hilbert spaces every countable collection of operators has a common nontrivial invariant subspace, and thus cannot generate $\boldsymbol{B}(\mathscr{H})$.

Theorem 1 can be varied to apply to operators on Banach spaces, but the quadratic operators and the given operator may operate on different spaces. Read [4] has shown that there is an operator on l^1 with no nontrivial invariant subspaces. Since $l^1 \oplus l^1$ is isomorphic to l^1, this shows (using one of the variants of the idempotent case of Theorem 1 given in [2]) that there exists a pair $l^1 = \mathscr{M} \oplus \mathscr{N} = \mathscr{K} \oplus \mathscr{L}$ of direct-sum decompositions of l^1 with no nontrivial equal subdecompositions. The corresponding question for l^2 is equivalent to the invariant subspace problem (see [2]).

With a little care, Theorem 3 can be proved on finite-dimensional spaces too.

THEOREM 4. *On every finite-dimensional complex space there exist Hermitian operators A and B satisfying $A^2 = A$ and $B(B+1)(B-1) = 0$ such that every linear transformation is a (noncommutative) polynomial in A and B.*

Proof. The proof of Theorem 3 applies without change if the dimension of the space is a multiple of 4. Slight variants yield the other cases. In particular, B can be defined as

$$\left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & & & -1 \\ \hline & 0 & & 0 \end{array}\right]$$

for any number of 1's. Then A can be defined as above on even-dimensional spaces, and as the direct sum of such an operator and a one-dimensional 0 in the case of odd-dimensional spaces. We omit the details.

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Vectors of uniqueness for *id*/*dx*

by

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Abstract. A vector x in a Hilbert space is a *vector of uniqueness* for a symmetric operator A if A, with domain restricted to span $\{A^n x \mid n = 0, 1, \ldots\}$, is essentially selfadjoint on the closure of this domain. We characterize vectors of uniqueness for the operator id/dx on $L^2[0, 1]$. Let $\hat{g}(k) \equiv \int_0^1 g(t) e^{-2\pi ikt}\, dt$, $g^{(n)} \equiv$ the nth derivative of g, $E \equiv \{k \mid \hat{g}(k) \neq 0\}$. We show that g fails to be a vector of uniqueness if and only if there exists a nontrivial f such that $f^{(n)}(0) = 0 = f^{(n)}(1)$, for all n, and

$$\sum_{k \in E} \left|\frac{\hat{f}(k)}{\hat{g}(k)}\right|^2 \frac{1}{1+k^2} < \infty.$$

We show that g is a vector of uniqueness if and only if the closure of span $\{g^{(n)} \mid n = 1, 2, \ldots\}$ equals $\{f \in L^2[0, 1] \mid \hat{f}(k) = 0 \text{ when } k \notin E\}$.

We show that g fails to be a vector of uniqueness for id/dx on $L^2(\boldsymbol{R})$ if and only if there exists a nontrivial f such that $f^{(n)}(0) = 0$, for all n, and

$$\int_E \left|\frac{\mathscr{F}f(t)}{\mathscr{F}g(t)}\right|^2 \frac{dt}{1+t^2} < \infty,$$

where $\mathscr{F}$ is the Fourier transform, and E is the support of $\mathscr{F}g$.

Introduction. Vectors of uniqueness were introduced by Nussbaum [4] (see Definition 2). He showed that a symmetric operator on a Hilbert space is selfadjoint if and only if it has a total set of vectors of uniqueness. In the same paper, and in subsequent papers, the selfadjointness of certain operators is shown by proving that certain classes of vectors are always vectors of uniqueness (see [3]–[5]).

Nussbaum defined vectors of uniqueness in terms of the classical moment problem. He defines x to be a vector of uniqueness for A if the moment sequence $\{\langle A^n x, x\rangle\}_{n=0}^{\infty}$ is determined. We use the equivalent definition given in [6], vol. 2, p. 201 (Definition 2).

It is often advantageous, when considering questions of essential selfadjointness, to focus on vectors of uniqueness. It is precisely in this setting, when the domain of A equals span $\{A^n x \mid n = 0, 1, \ldots\}$, for a fixed x (see Definition 2), that the spectral theorem says that selfadjointness is equivalent to A being unitarily equivalent to multiplication by $f(t) \equiv t$ on $L^2(\boldsymbol{R}, \mu)$, for some measure μ.