Sₜ spaces and vanishing of the functor Ext

by

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Abstract. Sₜ(a, r) spaces are introduced by V. V. Kashirin [4]. We derive some properties of this class of Köthe spaces and obtain necessary and sufficient conditions for an Sₜ(a, 1) space E to satisfy Ext¹(E, E) = 0 when E is of type d₁ or d₂. In [7, 8], Lₜ(a) spaces are introduced and shown to be the only dₜ Köthe spaces which satisfy Ext¹(E, E) = 0. We show that the class of Sₜ(a, ω) spaces coincides with the subclass of Lₜ(a) spaces which consists of those Lₜ(a) spaces where $\Phi = (\phi, \varphi, \ldots)$, i.e., generated by a single function $\phi$. V. V. Kashirin, in [4], asked whether every regular nuclear Köthe space E of type dₜ is representable as an Sₜ(b, ω) space, which was answered in the negative by M. Kocatepe (Alpseymen) [5]. We show that even under the additional assumption that Ext¹(E, E) = 0 this is not possible.

Preliminaries. Unless otherwise stated, throughout this work, the letters E, F, . . . etc. will denote nuclear Köthe spaces $K(a_n)$ which have a continuous norm and whose generating matrix $(a_{kl})$ satisfies $0 < a_{kl} < a_{k+1,l}$ $\forall k, n$. Following E. Dubinsky [1] we say that $E = K(a_n)$ is of type $(d_l)$, $l = 0, 1, 2, 5$, if it is generated by a matrix $(a_{kl})$ which satisfies the corresponding condition below:

(dₜ) For each $k$, $a_{k+1,l}/a_{kl}$ is nondecreasing in $n$ (in this case E is also called regular).

(dₜ) $\exists p \forall k \exists m \sup a_{kl}^{1/p} (a_{mn} a_{nl}) < +\infty$.

(dₜ) $\forall k \exists m \forall r \sup a_{kl} a_{nr}^{1/r} < +\infty$.

(dₜ) $\exists M > 0 \forall k \forall n \ a_{k+1,l}/a_{kl} \leq (a_{k+2,l}/a_{k+1,l})^M$.

An $Lₜ(a, r)$ space, also called a Dragilev space, is the Köthe space $E = K(\text{exp}(f(x, a)))$ where $f$ is an increasing, odd, logarithmically convex function (i.e. In $f'(\text{exp}(x))$ is convex for $x \geq 0$, $a = (a_n)$ is an exponent sequence (a nondecreasing sequence of positive real numbers which approaches infinity rapidly enough to make $E$ nuclear), and $(r_k)$ a strictly increasing sequence with limit $r \in R \cup \{+\infty\}$. An $Lₜ(a, r)$ space is isomorphic to an $Lᵦ$ space of type $-1$ (resp. 1) if $r < 0$ (resp. $0 < r < +\infty$). Hence basically there are four types of $Lₜ$ spaces: $r = -1, 0, 1, +\infty$. For any $A > 1$ the function $f'(A x)/f(x)$

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is increasing and either has a finite limit for all $A > 1$ or approaches infinity for all $A > 1$ as $x \to \infty$; accordingly, $f$ is called slowly increasing resp. rapidly increasing. If $f$ is slowly increasing then $L_f(a, r)$ is isomorphic to the power series space of finite type (which is $d_2$) $A_0(a) = L_0(a, 0)$ if $r < +\infty$ or to the power series space of infinite type (which is $d_1$) $A_\infty(a) = L_\infty(a, \infty)$, where id denotes the identity function. If $f$ is rapidly increasing, then for all $A > 1$, $f^{-1}(A x) / x$ decreases to 1, while $(f(x) / x)$ and $f(x) / x$ increase to $+\infty$.

In this case $L_f(a, r)$ is of type $d_1$ (resp. $d_2$) if $r = 1\cdot\infty$ (resp. $r = -1\cdot0$). All Dragilev spaces are regular and independent of the choice of the sequence $(r_0)$.

A regular Köthe space $E = K(a_n)$ is called weakly stable (resp. unstable) if the generating (regular) matrix $(a_{mn})$ satisfies

$$\forall s \exists p \forall q \exists r \sup_{n} a_{k,m} a_{k+1,m} < +\infty$$

(resp. $\exists s \forall p \exists q \forall r \lim_{n} a_{k,m+1} a_{k+1,m+1} = 0$).

For concepts not defined and as general references we refer the reader to [1], [6], or [10].

1. $S_\infty$ spaces. An $S_\infty(a, r)$ is defined similarly to an $L_f(a, r)$, but the logarithmic convexity of $f$ is replaced by the convexity of $g$ for $x \geq 0$. Below some properties of $S_\infty$ spaces are stated. The proofs, being immediate consequences of the convexity of $g$, are omitted.

Proposition 1.1. Let $E = S_\infty(a, r)$, then $E$ is regular and:

(i) If $r < 0$ then $E \cong S_\infty(b, -1)$ and $E = d_2$.

(ii) If $r = 0$ then $E = d_2$.

(iii) If $0 < r < +\infty$ then $E \cong S_\infty(b, 1)$ and $E = d_2$.

(iv) If $r = +\infty$ then $E = d_2$.

Proposition 1.2. Every $L_f(a, r)$ space is an $S_\infty(a, r)$ space for some $g$.

Proof. The proofs for $r = 1\cdot\infty$, $0\cdot\infty$ will be given. The cases $r = -1\cdot0$, $0\cdot-1$ can be similarly proved.

Let $E = L_f(a, \infty)$; $r_\infty + 1\cdot\infty$. Define $g(2^n) = f(2^n)$ and extend $g$ to an odd function on $R$ via linear segments; hence $g$ is strictly increasing. Since

$$g(2^n + 1) - g(2^n) = f(2^n + 1) - f(2^n - 1)$$

and each factor in the latter product is nondecreasing in $n$, it follows that $g$ is also convex. Suppose $2^n \leq r_\infty a_n < 2^{n+1}$. Then

$$f(r_n a_n) = f(2^n + 1) = g(2^n + 1) - g(2^n a_n),$$

and

$$g(r_n a_n) \leq g(2^n + 1) = f(2^n + 1) \
\leq f(2^n a_n),$$

so $L_f(a, \infty) = S_\infty(a, \infty)$ as sets and the topologies coincide.

Let $E = L_f(a, 1)$. Since the case $f = identity$ is trivial, assume $f$ increases rapidly. Let $r_0 \geq 1$. Let $p_n = f^{-1}(2^n)$, define $g(p_n) = f(p_n) = 2^n$ and extend $g$ to an odd function on $R$ via linear segments; hence $g$ is strictly increasing. Then

$$g(r_{n+1} a_n) - g(r_n a_n) = f^{-1}(2^n) / g^{-1}(2^n + 1) - f^{-1}(2^n) / g^{-1}(2^n)$$

and the latter quotient increases with $n$ since $f$ is rapidly increasing. We conclude that $g$ is convex. Now suppose $p_n \leq r_n a_n < p_{n+1}$. Then for $r_2 > r_n$ and for large $n$ we have

$$g(r_n a_n) < g(p_{n+1}) = 2^{n+1} = 2^n / 2^n \leq 2 f(r_n a_n) < g(r_n a_n),$$

$$f(r_n a_n) < f(p_{n+1}) = g(p_{n+1}) \leq g(p_{n+1} / p_n a_n) \leq g(r_n a_n)$$

where the last inequality follows from $p_{n+1} / p_n = 1$. Therefore $L_f(a, 1) = S_\infty(a, 1)$ as sets and the topologies coincide.

Remark 1.3. The above propositions show that $S_\infty$ spaces generalize $L_f$ spaces and moreover exhibit similar properties to $L_f$ spaces with the exception of $S_\infty(a, 1)$ spaces. V. V. Kashirin in [4] gives an example of an $S_\infty(a, \infty)$ space which is not isomorphic to any $L_f$ space and concludes that $S_\infty$ spaces form a strictly larger class. A close inspection reveals that $S_\infty(a, 1)$ spaces may be of type $d_1$ or $d_2$ (in which case they are isomorphic to some power series space of finite type; see [1]), or even a cross product of a $d_1$ space by a $d_2$ space, which cannot occur for any $L_f$ space.

An $L_f(a, 1)$ space is $d_1$ (if $f$ is rapidly increasing). The next lemma shows that if an $S_\infty(a, 1)$ space is $d_1$ then $g$ resembles a rapidly increasing function, but only at the points $a_n$ somewhat "locally".

Lemma 1.4. Let $E = S_\infty(a, 1)$, then $g$ satisfies:

(i) $\forall M > 0 \forall a \in S_\infty(a, 1) \exists n_0$ such that $g(a_n a) \leq M$ for all $n \geq n_0$.

(ii) $\forall A > 1 \exists a_0$ such that $g(a_0) / g(a_0) = \lim_{n} g(a_n a) / g(a_n a) = 0$.

(iii) $g(a_n a) / a_n + \infty$.

Proof. The proofs for (ii) and (iii) will be omitted since they follow from (i). To prove (i) let $E = d_1$. Then we may assume $2 g(r_n a_n) \leq g(r_{n+1} a_n)$ for all $n \geq n_0$. By successive applications we obtain

$$g(r_{n+1} a_n) \leq g(r_n a_n) + 2 g(r_n a_n) / 2^n \leq g(r_n a_n) + 2 g(r_{n+1} a_n) \leq \cdots$$

$$\leq g(r_n a_n) + 2^{-j} + 2^{-j} g(r_{n+j} a_n)$$

for all $j \geq 1$, $n \geq n_0$. By the convexity of $g$ and using the above estimate we have for all $n \geq n_0$

$$r_{n+1} g(r_n a_n) / r_n \leq g(r_{n+1} a_n) \leq r_n g(r_n a_n) + 2^{-n} g(r_{n+1} a_n).$$
Therefore with \( p = k + m + 1 \) where \( m \) is chosen large enough to satisfy \( M < 2^k (r_{k+1} - r_k)^{m} \) and for \( n_0 = n_0 \) we obtain the result. 

We next show that the classes of \( S_n \) spaces for \( r = 1, r = \infty \) are not disjoint. In the next section this iteration will be completely characterized.

**Proposition 1.5.** Any unstable \( S_j(a, \infty) \) space \( E \) is diagonally isomorphic to some \( S_j(b, 1) \) space.

**Proof.** The unstability of \( E \) is equivalent to \( \lim_{n \to \infty} a_n/a_{n-1} = +\infty \). Let \( d_n = \min \{ a_n, a_{n+1}/2 \} \). Then \( d_n \to +\infty \). By modifying \( a_n \) if necessary assume \( d_n > 1/2 \). Define \( r_n \) by \( r_1 = 1/4, r_2 = 1/2, r_3 = 1 - d_3^{-1/2} \) for \( k > 2 \). Then \( (1 - r_k)^{k-2} < d_2 \leq a_{k+1}/a_k \) for \( k > 2 \). Let \( s_n \) by \( s_1 = 0, s_2 = (1 - r_2)^{-1} \) for \( k > 1 \). Then \( s_k \) is a strictly increasing sequence of \( k \) increases with \( k \).

(2) \( s_k \leq a_k / (s_{k+1}) \) for \( k > 1 \).

Let \( b_1 = 1, b_{n+1} = 4r_n b_n \) for \( n > 1 \). Clearly \( b_n \to +\infty \). Define \( g \) successively by the points

\[
 r_2 b_2 - r_3 b_2 - \ldots - r_2 b_2 - r_2 b_2 - \ldots
\]

By

\[
 g(r_2 b_2) = \sum_{j=1}^{n-1} f(s_j a_j) + f(s_j a_j)
\]

for \( k = 1, \ldots, n \)

and extend \( g \) to an odd function on \( R \) via linear segments. Clearly \( g \) is strictly increasing and \( g(r_2 b_2) = g(r_2 b_2) \) since \( s_1 = 0 \). Let \( X = (x, y) \) denote the slope of the expression \( f(x, y) \) for \( y < x \). To show that \( g \) is convex we distinguish two cases:

**Case 1:** \( k + 1 < n \). Using (1) and the convexity of \( f \) we have

\[
 (r_{k+1} - r_k)^m (r_{k+1} - r_k) \leq (s_{k+1} - s_k)/(s_{k+1} - s_k)
\]

\[
 s_k \leq f(s_2 a_2) - f(s_2 a_2) - f(s_1 a_1) - f(s_1 a_1)
\]

from which it follows that

\[
 m(r_{k+1} - b_2 - b_2) \leq m(r_{k+1} + b_2 + b_2).
\]

**Case 2:** \( k + 1 = n \). We have

\[
 b_{k+1} - r_k = r_k - r_{k-1} \leq d_k \leq a_{k+1}/(s_{k+1} - s_k)^{-1}
\]

By the convexity of \( f \), this is

\[
 f(t a_{k+1}) = \frac{f(r_{k+1} - b_2) - f(r_2 b_2)}{f(s_2 a_2) - f(s_1 a_1)}
\]

from which \( m(r_{k+1} - b_2 - r_2 b_2) \leq m(r_{k+1} + b_2 + b_2) \) follows.

Letting

\[
 t_n = \exp \left( \frac{1}{2} \sum_{i=1}^{n-1} f(s_i a_i) \right)
\]

we obtain \( t_n \exp \left( \frac{1}{2} \sum_{i=1}^{n-1} f(s_i a_i) \right) = \exp \left( \frac{1}{2} \sum_{i=1}^{n-1} f(s_i a_i) \right) \) and hence conclude that \( S_j(a, \infty) \)

\[
 \cong S_j(b, 1) \text{ diagonally.}
\]

Kashin in [4] has shown the following result:

**Proposition 1.6** Any unstable \( d \) K"{o}the space is diagonally isomorphic to some \( S_j(a, \infty) \) space.

Let \( K(a_n) \) be a K"{o}the space, and \( (\eta) \) a subsequence of \( N \). The space \( K(b_k) \) where \( b_k = a_{\eta k} \) for \( n \leq n < n_{k+1} \) will be called a repeated form of \( K(a_n) \). In the case of a \( S_j(a, \infty) \) space this is equivalent to repeating terms of \( a = (a_k) \). That is, \( b_k = a_k \) if \( n \leq n < n_{k+1} \), and \( b_k = a_k \) will be called a repeated form of \( (a_k) \). Combining Proposition 1.6 with Proposition 1.5 we obtain

**Proposition 1.7.** Any repeated form of an unstable \( S_j \) K"{o}the space is diagonally isomorphic to some \( S_j(a, 1) \) and \( S_j(b, \infty) \) space.

**Proof.** Since in Propositions 1.5 and 1.6 the isomorphisms are diagonal, repeating a coordinate a finite number of times does not disturb the isomorphism if we repeat the corresponding coordinate of \( (a_k) \) resp. \( (b_k) \) the same number of times. This, in turn, still gives us an \( S_j \) space of the same type.

2. **Vanishing of the functor Ext for \( S_j \) spaces.** For the definition of the functor \( \text{Ext}(E, F) = \text{Ext}^1(E, F) \) on the category of Fréchet spaces we refer the reader to Vogt [12]. Consider the following conditions for two K"{o}the spaces \( E = K(a_n), F = K(b_n) \):

\[
 (S_1) \quad \exists p \; \forall u \; \exists k \; \forall m, K, R > 0 \; \exists n, S > 0 \; \forall i, j
\]

\[
 a_{n_0} / b_{n_0} \leq \max \left( a_{n_0} / b_{n_0}, a_{n_0} / b_{n_0} \right)
\]

\[
 (S_2) \quad \forall u \; \exists k \; \forall m, K, R > 0 \; \exists S > 0 \; \forall i, j
\]

\[
 a_{n_0} / b_{n_0} \leq \max \left( a_{n_0} / b_{n_0}, a_{n_0} / b_{n_0} \right)
\]

Write \( (E, F) \in S_j \) resp. \( (E, F) \in S \) if the corresponding condition is satisfied. To simplify notation we shall write \( \text{Ext}(E) \) for \( \text{Ext}(E, E) \). We state the following result of Krone-Vogt [8].

**Theorem 2.1.** Let \( E = K(a_n) \) be a Schwartz space satisfying \( a_{n_0} > 0 \)

\[
 \forall k, n. \text{Then the following conditions are equivalent:}
\]

(i) \( (E, E) \in S \).

(ii) \( \text{Ext}(E) = 0 \).

(iii) Every exact sequence \( 0 \to E \xrightarrow{F} E \to 0 \), where \( F \) is a Fréchet space, splits.
If $E$ is further $d_1$ then these conditions are also equivalent to [3]:
(iv) $(E, E) \equiv S_5$.

Combining Theorem 2.1 and modified forms of some results of M. Kočatep [5] resp. J. Hebecker [3] with Proposition 1.7 we obtain the main result of this section:

Theorem 2.2. Let $E = S_5(a, 1)$ be $d_1$. Then the following conditions are equivalent:
(i) $\text{Ext}(E) = 0$.
(ii) $E \equiv S_5(b, \infty)$.
(iii) $E \equiv S_5(c, 1)$ where $c$ is a repeated form of some $d$ with
\[ \lim_{d_1+/d_1-} d_1 = 1. \]

Proof. (iii) is equivalent to the condition that $1$ is an isolated point among the limit points of $\{d_1/d_1\} : n, m \in N$; hence in view of isomorphism also of $\{a_1/n : n, m \in N\}$. For a discussion of the limit points of $(a_1/n)$ see [2]. In [3], Satz 2.6, this condition is shown to be equivalent to (i) for $L_3(a, 1)$ spaces, but in view of Lemma 1.4(i) this result generalizes to $d_1$ $S_5(a, 1)$ spaces. (i) $\iff$ (iii) (ii) may also be obtained from Proposition 2 in [5] with obvious modifications for $d_1$ $S_5(a, 1)$ spaces. (ii) $\iff$ (i) is Proposition 1.7, and finally (ii) $\iff$ (i) follows from the corresponding results for $L_3$ spaces in [5] or [7] by using Lemma 1.4.

Corollary 2.3. Let $E = S_5(a, \infty)$. Then $E \equiv S_5(b, 1)$ for some $b$ and $f$ $S_5(a, \infty) = S_5(c, \infty)$ where $c$ is a repeated form of some $d$ with
\[ \lim_{d_1+/d_1-} d_1 = +\infty. \]

Proof. If $E \equiv S_5(b, 1)$ then $\text{Ext}(S_5(b, 1)) = 0$ and by Theorem 2.2(iii) it follows that $S_5(b, 1)$ is a repeated form of some unstable $S_5(c, 1)$. Then the same is true for $E$. The converse implication is Proposition 1.7.

Corollary 2.4. The intersection class (up to isomorphism) of $S_5$ spaces of type 1 resp. $\infty$ consists precisely of repeated forms of unstable $d_1$ Köthe spaces.

Proof. This follows by Proposition 1.7 and Theorem 2.2.

The corresponding results in the case of $d_2 S_5(a, 1)$ space are already known (see Nyberg [9] and in particular Hebecker [3] for a complete treatment); in fact, a $d_2 S_5(a, 1)$ space, being also of type $d_2$ by Proposition 1.1(iii), is isomorphic to some $A_4(b)$ (see [3]). We cite Satz 1.11 in [3] in the language of Theorem 2.2:

Theorem 2.5. $\text{Ext}(A_4(b)) = 0$ iff $a = (a_1)$ is equivalent to a repeated form of some unstable exponent sequence.

3. $\lambda_0$ spaces and $S_5$ spaces. The $\lambda_0(a)$ spaces are introduced in [7].

Definition. Let $\Phi = \{\varphi_n\}$ be a sequence of positive increasing functions satisfying for all $x > 0$ and $k$
\[ \varphi_{k+1}(x) \geq \varphi_k(x) \geq \varphi_1(x) \geq x^2. \]

Then the Köthe space $K(a_n)$ with $a_{k+1} = \varphi_k(a_n), a_{k+1} = \varphi_k(a_n), k \geq 1,$
where $a = (a_n)$ is some exponent sequence, is denoted by $\lambda_0(a)$.

Clearly a $\lambda_0(a)$ space is of type $d_1$ and moreover, by replacing $\varphi_k(x)$ by $\varphi_0(x)$ if necessary, one can assume that $\lambda_0(a)$ is regular. Krone in [7] proves the following theorem:

Theorem 3.1. Let $E$ be a $d_1$ Schwartz Köthe space. Then $\text{Ext}(E) = 0$ iff $E \equiv \lambda_0(a)$ for some $\Phi$ and $a$.

We next show that $\lambda_0$ and $S_5$ spaces of infinite type are closely related.

Theorem 3.2. The class of $S_5(a, \infty)$ spaces coincides with the subclass of $\lambda_0(a)$ spaces where $\Phi$ consists of a single function, i.e. $\Phi = \{\varphi, \varphi, \ldots\}$. Without loss of generality assume that $\varphi(x) \geq x^2$ for all $x$ and let $\varphi_0(x)$ denote $\varphi \circ \varphi \circ \ldots \circ \varphi(x)$ ($k$-fold composition). Let $a_n = \varphi_0(a_n)$. Define $p_n \equiv p_0 = 2, p_{n+1} = p_n \varphi(a_n)$ for $n \geq 0.$ Then $(p_n)$ increases strictly. Let $b_n = 2^n p_n$. Finally define $g(2^n) = \log p_n$, extend via linear segments, and complete to an odd function. Then $g$ increases strictly. To show that $g$ is convex we consider
\[ g(2^{n+1}) - g(2^n)(2^{n+1} - 2^n) = 2^{-n} \log(2^{n+1}/p_n). \]

Since
\[ (2^{n+1}/p_n)^3 = (\varphi(a_n)/p_n)^3 \leq (\varphi_0(p_n)^3 \leq \varphi(\varphi(a_n)/p_n)) = p_{n+1}/p_n = p_{n+1}, \]
we see that the right-hand side of the considered equation increases with $n$, and hence $g$ is convex.

Now suppose $p_n < a_n \leq p_{n+1}$ (so in particular $b_n = 2^n$). Then
\[ a_{n+1} = \varphi_0(a_n) \leq \varphi_0(a_{n+1}) = p_{n+1} = \exp[g(2^{n+1})] = \exp[g(2^{n+1})] \]
and
\[ \exp[g(2^{n+1})] = p_{n+1} = \varphi_0(p_n) = \varphi_0(a_n) = a_{n+1} \leq \varphi_0(a_{n+1}). \]

We conclude that $E \equiv S_5(b, \infty)$. Conversely, given $F \equiv S_5(b, \infty)$ define for $x > 0, \varphi(x) = \exp[f(2^{n+1} \log x)]$. Then it easily follows that $F \equiv \lambda_0(a)$ where $a_n = \exp[f(2^{n+1})]$.

Remark 3.3. It now follows easily that the popular Köthe space $K(a_n)$ where $a_n = \varphi \circ \varphi \circ \ldots \circ \varphi(a)$ is not only an $S_5(b, \infty)$ space but also an $L_3(c, \infty)$ space. Here it is natural to ask whether every $\lambda_0(a)$ space is isomorphic to some $S_5(b, \infty)$ space. Our next example shows that this is not so and hence that even if a $d_1$ Köthe space $E$ satisfies $\text{Ext}(E) = 0$, it still does not follow that $E \equiv S_5(b, \infty)$. This gives a sharper negative answer to a question of Kashirin [4] whether every regular $d_1$
Köthe space is isomorphic to some $S_{\lambda}(b, \infty)$ space, than that given by M. Kocatepe (Alpseymen) [5].

Example 3.4. In this example we construct a $\lambda_{a}(a)$ space $E$ which is not isomorphic to any $S_{\lambda}(b, \infty)$ space. Here we would like to thank the referee for suggesting this proof which is considerably shorter than the original one. Let $\Phi = (\varphi_{n})$ where $x^{2} \varphi_{1}(x) \leq \varphi_{2}(x) \leq \ldots$ and

$$\lim_{x \to +\infty} \varphi_{m}^{(x)}(x) = 0 \quad \text{for all } k, m$$

where $\varphi_{m}^{(x)}$ denotes the $m$-fold composition. Let $(a_{n})$ satisfy $a_{n} < a_{n+1}^{2}$ (e.g. $a_{n} = 2^{n}$), and put $E = \lambda_{a}(a)$. Suppose $E \cong S_{\lambda}(b, \infty)$ for some $b$ and $\phi$. In view of Theorem 3.2 this means $E \cong F = \lambda_{\psi}(\psi)$ for some $\lambda_{\psi}(\psi, \ldots)$. Now $E \cong F$ implies that the diametral dimensions $\Delta(E)$ and $\Delta(F)$ are equal, and hence $E$ and $F$ both being $G_{m}$ spaces (see [11]), coincide as sets. Therefore we may assume that $a_{n} \leq \psi(b_{n})$ and $\varphi^{(2)}(b_{n}) \leq \varphi(a_{n})$ for large $n$ (in particular $b_{n} \leq \varphi(a_{n})$ where $\varphi = \varphi_{k_{0}} \circ \ldots \circ \varphi_{1}$ for some suitable $k_{0}$. We obtain:

1) Given $x$, choose $n$ such that $a_{n-1} \leq x \leq a_{n}$; hence $x \leq a_{n} \leq a_{n+1}^{2}$ and

$$\psi(x) \leq \psi(a_{n}) \leq \psi^{(2)}(b_{n}) \leq \varphi(a_{n}) \leq \varphi(a_{n+1}^{2}) \leq \varphi(x^{2}) \leq \varphi^{(2)}(x).$$

2) $\varphi_{k_{0}+1}(b_{n}) \leq \varphi_{k_{0}+1}(\varphi_{0}(a_{n})) \leq \varphi^{(2)}(b_{n}) \leq \varphi^{(2m)}(b_{n}) \leq \varphi_{k_{0}+m}(b_{n})$

for some suitable $m$ which contradicts the growth assumption on the $\varphi_{k}$'s.

References