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Added in proof (September 1987). The author has recently found a new proof of Watson's product formula (4.3) for Laguerre polynomials (Canad. Math. Bull., to appear). The proof is based upon the observation that (4.3) holds for $\alpha = 1, 2, \ldots$ This is established by considering the commutative Banach algebras of radial functions on the Heisenberg groups H_n, $n \geqslant 2$. Then a theorem of Carlson is used to get the validity of (4.3) for all values of α with $\operatorname{Re}\alpha > -1/2$.

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Measures on groups with given projections

by

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Abstract. A theorem of Lindenstrauss asserts that every extreme doubly stochastic measure is singular. This result is extended to the case of locally compact groups.

§ 0. Introduction. Let λ be the Lebesgue measure on $I = [0, 1]$. A Borel measure μ on $I \times I$ is said to be *doubly stochastic* if $\mu(A \times I) = \mu(I \times A) = \lambda(A)$ for each Borel set $A \subseteq I$. The collection of all doubly stochastic measures forms a convex, weakly compact set whose extreme points have been the subject of much study: [1]–[3], [6]–[8]. It was shown by Lindenstrauss [6] that every extreme doubly stochastic measure is singular with respect to the planar measure $\lambda \otimes \lambda$. In [8], this result was generalized.

Let $L_1, \ldots, L_m$ be nontrivial linear subspaces of $\boldsymbol{R}^n$. Suppose that ν is a probability measure on $\boldsymbol{R}^n$ and let E be the convex set of probabilities on $\boldsymbol{R}^n$ whose projections onto $L_1, \ldots, L_m$ agree with those of ν. In [8], it is proved that the extreme points of E are singular with respect to the n-dimensional Lebesgue measure on $\boldsymbol{R}^n$. The fact that homotheties of $\boldsymbol{R}^n$ by a scalar r change the Lebesgue measure by a factor of r^n was an important feature of the proof.

We further generalize this result to the context of a locally compact group. An appropriate convex set is the collection of all measures on the group whose projections onto various quotient groups are prescribed. Under suitable hypotheses, the extreme points of this set will be singular with respect to the Haar measure. For the proof, one must compensate for the fact that homotheties are not available in the context of groups. As in almost all work in such problems, the following result of Douglas and Lindenstrauss is crucial.

Let (X, ν) be a finite measure space and let F be a linear space of ν-integrable functions containing all constant functions. Let $E(\nu)$ be the (convex) set of all finite measures ϱ on X such that

$$\int f\, d\nu = \int f\, d\varrho$$

for each $f \in F$.

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0.1. Lemma. *A measure ν is an extreme point of $E(\nu)$ if and only if F is dense in $L^1(X, \nu)$.*

Indication: See [3] or [6].

Some remarks on notation are as follows. If A is a set, then $|A|$ is its cardinality. Also, 1_A is the indicator (characteristic) function of A. If G is a group, then the symbol e denotes its identity element.

§ **1. Basic results.** Let G be a locally compact group and let H be a closed subgroup of G. Let $\pi\colon G \to G/H$ be the canonical projection onto the (left) coset space G/H. Let $M^+(G)$ be the collection of all finite regular Borel measures on G. If $\nu \in M^+(G)$, then $\pi(\nu)$ denotes the Borel measure on G/H defined by $\pi(\nu)(B) = \nu(\pi^{-1}(B))$. Since π is a quotient map, $\pi(\nu)$ is a regular Borel measure.

1.1. Lemma. *Let H be a closed subgroup of a locally compact group G. Let $\pi\colon G \to G/H$ be the usual projection and suppose that ν and ϱ are elements of $M^+(G)$. The following are equivalent:*

(1) $\pi(\nu) = \pi(\varrho)$.

(2) $\int f\,d\nu = \int f\,d\varrho$ *for all bounded Baire-measurable functions on G that are constant on the cosets of H.*

Proof of the lemma above is omitted; it is standard.

Let $H_1, \ldots, H_n$ be closed subgroups of a locally compact group G and let ν be a finite regular Borel measure on G. Define

$$E(H_1, \ldots, H_n; \nu) = \{\varrho \in M^+(G)\colon \pi_i(\varrho) = \pi_i(\nu),\ 1 \leqslant i \leqslant n\}.$$

Define F_0 to be the linear space of all real functions on G of the form

$$x \to f_1(x) + \ldots + f_n(x),$$

where each f_i is a bounded Baire function, constant on the cosets of H_i.

1.2. Lemma. *Let $H_1, \ldots, H_n$, G and ν be as above. Then ν is an extreme point of the convex set $E(H_1, \ldots, H_n; \nu)$ if and only if F_0 is dense in $L^1(\nu)$.*

The lemma follows from the theorem of Douglas and Lindenstrauss (0.1) and Lemma 1.1.

1.3. Lemma. *Let μ be the (left) Haar measure on a locally compact group G. Suppose that $E \subseteq G$ is a Borel set with $\mu E < \infty$. If U is a neighborhood of $e \in G$ with $\mu U < \infty$, define*

$$f_U(x) = \mu(E \cap Ux)/\mu(Ux).$$

Given $\varepsilon > 0$, there is a neighborhood V of e such that if $U \subseteq V$, then

$$\int_G |f_U(x) - 1_E(x)|\,d\mu(x) < \varepsilon.$$

Indication: [4; pp. 268–9].

§ **2. The main theorem.**

2.1. Theorem. *Let $H_1, \ldots, H_n$ be closed nondiscrete normal subgroups of a locally compact group G. Let $\pi_1, \ldots, \pi_n$ be the canonical projections from G to the quotient groups $G/H_1, \ldots, G/H_n$. Let ν be a finite regular Borel measure on G and let μ be the left Haar measure on G.*

If ν is an extreme point of $E(H_1, \ldots, H_n; \nu)$, then $\nu \perp \mu$.

Proof. Write $\nu = \nu_0 + \nu_1$, where $\nu_0 \perp \mu$ and $\nu_1 \ll \mu$ [5; 14.22, p. 180]. There is a Borel function $F \in L^1(G, \mu)$ such that $d\nu_1 = F\,d\mu$ [5; 14.19, p. 179]. We show that $\nu_1 = 0$. Suppose not. Then there is some $\delta > 0$ such that $\mu\{x\colon F(x) \geqslant \delta\} > 0$. Let $E \subseteq \{x\colon F(x) \geqslant \delta\}$ be a Borel set with $0 < \mu E < \infty$.

We now apply Lemma 1.3. For each $m \geqslant 1$, let V_m be a neighborhood of e with compact closure such that whenever $U \subseteq V_m$, then

$$\int_G |f_U(x) - 1_E(x)|\,d\mu(x) < 1/m.$$

Given a collection $h_1, \ldots, h_n$ of not necessarily distinct elements of G and $1 \leqslant k \leqslant n$, define

$$C_k(h_1, \ldots, h_n) = \{h_{i_1} \ldots h_{i_k}\colon 1 \leqslant i_1 < \ldots < i_k \leqslant n\}$$

and put

$$C_0(h_1, \ldots, h_n) = \{e\},$$

$$C = C(h_1, \ldots, h_n) = C_0(h_1, \ldots, h_n) \cup \ldots \cup C_n(h_1, \ldots, h_n).$$

Now, if $1 \leqslant k \leqslant n$ and $1 \leqslant i_1 < \ldots < i_k \leqslant n$, the subset $S(i_1, \ldots, i_k)$ of $H_1 \times \ldots \times H_n$ defined by

$$S(i_1, \ldots, i_k) = \{(h_1, \ldots, h_n)\colon h_{i_1} \ldots h_{i_k} = e\}$$

is a nowhere dense subset of $H_1 \times \ldots \times H_n$: this follows, since none of the H_i is discrete. For each $m \geqslant 1$, choose a point $(h_1, \ldots, h_n) \in H_1 \times \ldots \times H_n$ so that:

1) $h_i \neq e$ for $i = 1, \ldots, n$.

2) $(h_1, \ldots, h_n) \notin S(i_1, \ldots, i_k)$ for each $(i_1, \ldots, i_k)$.

3) $C(h_1, \ldots, h_n) \subseteq V_m$.

Then choose a neighborhood W of e such that:

a) if $c \in C(h_1, \ldots h_n)$, then $cW \subseteq V_m$;

b) if c and c' are distinct elements of $C(h_1, \ldots, h_n)$, then $cW \cap c'W = \emptyset$.

Put $U_m = \bigcup\{cW\colon c \in C(h_1, \ldots, h_n)\}$, noting that the choices of $h_1, \ldots, h_n$ and W depended on m. Define $f_m = f_{U_m}$. Since $f_m \to 1_E$ in $L^1(G, \mu)$, there is a subsequence of the f_m converging to 1_E a.e. for μ. Select $x_0 \in E$ and m such that

$$f_m(x_0) > \frac{2^n - 1}{2^n}.$$

Define

$$L = \bigcap_c (Wx_0 \cap c^{-1} E),$$

where the intersection is taken over all $c \in C(h_1, \ldots, h_n)$.

CLAIM 1. *We have* $\mu L > 0$.

Proof of claim. Observe that

$$\sum_c \mu(Wx_0 \cap c^{-1} E) = \sum_c \mu(cWx_0 \cap E) = \mu(U_m x_0 \cap E)$$
$$> \frac{2^n - 1}{2^n} \mu(U_m x_0) = \frac{2^n - 1}{2^n} \sum_c \mu(cWx_0)$$
$$= \frac{2^n - 1}{2^n} |C| \mu(Wx_0) \quad (C = C(h_1, \ldots, h_n))$$
$$\geqslant (|C| - 1) \mu(Wx_0),$$

since $|C| \leqslant 2^n$.

For each $c \in C$, define a linear mapping $l_c \colon L^1(G, \nu) \to \boldsymbol{R}$ by

$$l_c(f) = \int_{cL} f \, d\mu.$$

CLAIM 2. *Each* l_c *is a bounded linear functional on* $L^1(G, \nu)$.

Proof of claim. Because $cL \subseteq E$, we have

$$\Big|\int_{cL} f \, d\mu\Big| \leqslant \delta^{-1} \int_{cL} |f| F \, d\mu \leqslant \delta^{-1} \int_G |f| \, d\nu.$$

Define $l \colon L^1(G, \nu) \to \boldsymbol{R}^{|C|}$ by $l = (l_c)_{c \in C}$.

CLAIM 3. *The range of* l *is all of* $\boldsymbol{R}^{|C|}$.

Proof of claim. Suppose that a is an element of $\boldsymbol{R}^{|C|}$ whose components are given by $a(c)$ for $c \in C$. Define $f \colon G \to \boldsymbol{R}$ by

$$f(x) = \begin{cases} a(c)/\mu L & \text{if } x \in cL, \\ 0 & \text{otherwise.} \end{cases}$$

Then $l(f) = a$. (Note that $c \neq c'$ implies $cL \cap c'L = \emptyset$.)

Since ν is extreme in $E(H_1, \ldots, H_n; \nu)$, the class F_0 (Lemma 1.2) is dense in $L^1(\nu)$. So $l(F_0) = \boldsymbol{R}^{|C|}$.

CLAIM 4. *The dimension of* $l(F_0)$ *does not exceed* $|C| - 1$.

Proof of claim. Given a nonempty set $A \subseteq \{1, \ldots, n\}$, write $A = \{i_1, \ldots, i_k\}$ where $1 \leqslant i_1 < \ldots < i_k \leqslant n$. Put $c(A) = h_{i_1} \ldots h_{i_k}$ and $c(\emptyset) = e$. Then suppose $1 \leqslant i \leqslant n$ and that g_i is a bounded Baire function on G, constant on the cosets of H_i. We show that

$$(*) \qquad \sum n(A) \, l_{c(A)}(g_i) = 0,$$

where the summation is taken over all subsets A of $\{1, \ldots, n\}$ and where

$$n(A) = \begin{cases} +1 & \text{if } |A| \text{ is even,} \\ -1 & \text{if } |A| \text{ is odd.} \end{cases}$$

To see this, write the left member of $(*)$ as

$$\sum\nolimits_1 n(A) \, l_{c(A)}(g_i) + \sum\nolimits_2 n(A) \, l_{c(A)}(g_i),$$

where $\sum_1$ (resp. $\sum_2$) indicates summation over A such that $i \in A$ (resp. $i \notin A$). If $i \in A$, write $\bar{A} = A \setminus \{i\}$. The pairing $A \to \bar{A}$ gives a one-one correspondence between terms in $\sum_1$ and those in $\sum_2$.

FACT. *Suppose* $i \in A$ *and put* $c = c(A)$ *and* $c' = c(\bar{A})$. *Then* $l_c(g_i) = l_{c'}(g_i)$.

To see this, suppose $c = k_1 h_i k_2$ and $c' = k_1 k_2$. Because H_i is normal, we have $h_0 = k_1 h_i k_1^{-1} \in H_i$. Then

$$l_c(g_i) = \int_{k_1 h_i k_2 L} g_i(x) \, d\mu(x) = \int_{h_0 k_1 k_2 L} g_i(x) \, d\mu(x)$$
$$= \int_{k_1 k_2 L} g_i(h_0 x) \, d\mu(x) = \int_{k_1 k_2 L} g_i(x) \, d\mu(x) = l_{c'}(g_i),$$

as desired.

Consider now the correspondence between terms $n(A) \, l_{c(A)}(g_i)$ in $\sum_1$ and terms $n(\bar{A}) \, l_{c(\bar{A})}(g_i)$ in $\sum_2$. We have seen that $l_{c(A)}(g_i) = l_{c(\bar{A})}(g_i)$. Upon noting that $n(\bar{A}) = -n(A)$, we see that $(*)$ obtains.

Let $g = g_1 + \ldots + g_n$, where each g_i is a bounded Baire function on G, constant on the cosets of H_i. Write

$$(**) \qquad 0 = \sum_A n(A) \, l_{c(A)}(g) = \sum_{c \in C} \Big(\sum n(A)\Big) l_c(g),$$

where the inner sum is taken over all $A \subseteq \{1, \ldots, n\}$ such that $c(A) = c$. We assert that not all of the coefficients $\sum n(A)$ are zero. In fact, we show that

If $c(A) = c = e$, *then* $A = \emptyset$,

so that the coefficient of $l_e(g)$ is unity. To see this, suppose $A = \{i_1, \ldots, i_k\}$ with $1 \leqslant i_1 < \ldots < i_k \leqslant n$. Since $(h_1, \ldots, h_n) \notin S(i_1, \ldots, i_k)$, it follows that $c(A) \neq e$.

So $(**)$ establishes a nontrivial linear relation between the components of any vector in $l(F_0)$. This establishes Claim 4. Claims 3 and 4 provide the desired contradiction. So $v_1' = 0$. ∎

The hypothesis that the subgroups in Theorem 2.1 be nondiscrete cannot be eliminated, as the following example demonstrates.

2.2. EXAMPLE. Define G_1 to be the two-element group and G_2 to be the circle group. Let μ_1 and μ_2 be the Haar measures on G_1 and G_2, respectively Set $G = G_1 \oplus G_2$, $H_1 = G_1 \times \{1\}$, $H_2 = \{0\} \times G_2$. Then $\mu = \mu_1 \otimes \mu_2$ is the Haar measure on G.

Let v be an extreme point of $E(H_1, H_2; \mu)$. The following statements are not hard to prove. There is a Borel set $A \subseteq G_2$ with $\mu_2 A = 1/2$ such that if $F\colon G \to \boldsymbol{R}$ is defined by

$$F(u) = \begin{cases} 2 & \text{if } u \in (\{0\} \times A) \cup (\{1\} \times (G_2 \setminus A)), \\ 0 & \text{if } u \in (\{0\} \times (G_2 \setminus A)) \cup (\{1\} \times A), \end{cases}$$

then $dv = F\,d\mu$. Thus every element of $E(H_1, H_2; \mu)$ is absolutely continuous with respect to μ.

A more intriguing question is whether the subgroups in Theorem 2.1 must be normal. The proof requires this assumption, but in the examples worked out by the author, normality does not seem to be required.

2.3. CONJECTURE. The hypothesis that the H_i be normal subgroups may be dropped from the statement of Theorem 2.1.

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Quadratic operators and invariant subspaces

by

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Abstract. It is shown that every operator has an invariant subspace if and only if every pair of operators satisfying a given pair of quadratic equations has a common invariant subspace. On the other hand, it is also shown that there exists a pair of operators satisfying a cubic and a quadratic equation which generate the algebra of all operators as a strongly closed algebra.

In [2] it is shown that every operator on Hilbert space has a nontrivial invariant subspace if and only if every pair of idempotents has a common nontrivial invariant subspace.

Does the corresponding result hold for pairs of nilpotents of index two? For a nilpotent of index two and an idempotent? More generally, is the existence of common invariant subspaces for pairs of operators satisfying given polynomial equations of degree 2 equivalent to the invariant subspace problem? The purpose of this note is to answer these and certain related questions about generators of the algebra of all bounded linear operators.

We consider bounded linear operators on separable infinite-dimensional complex Hilbert spaces; some remarks on other spaces are given at the end.

The following theorem generalizes [2].

THEOREM 1. *Let p and q be polynomials of degree 2. Then every operator has a nontrivial invariant subspace if and only if every pair $\{A, B\}$ of operators satisfying $p(A) = q(B) = 0$ has a common nontrivial invariant subspace.*

Proof. By dividing by the leading coefficients and completing the square, we can assume that the polynomials have the forms $p(x) = (x-a)^2 - \lambda^2$ and $q(x) = (x-b)^2 - \mu^2$ for complex numbers a, λ, b and μ. Since $A - aI$ and $B - bI$ have a common invariant subspace if and only if A and B do, we can assume that $p(x) = x^2 - \lambda^2$ and $q(x) = x^2 - \mu^2$.

Suppose first that every operator has a nontrivial invariant subspace and $A^2 = \lambda^2 I$, $B^2 = \mu^2 I$. We must show that A and B have a common invariant subspace.

Let $\mathscr{R}$ denote the closure of the range of $A + \lambda I$. Since

$$(AB+BA)(A+\lambda I) = ABA + B(\lambda^2) + \lambda(AB+BA)$$