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An algebra associated with the generalized sublaplacian

by

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Abstract. We construct and investigate a commutative Banach algebra associated with the differential operator

$$L = -\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha-1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}\right),$$

$\alpha \geqslant 1$, acting on $\boldsymbol{R}_+ \times \boldsymbol{R}$. The construction has been inspired by the existence of the well-known algebras of integrable radial functions on the Heisenberg groups. In consequence we also describe an example of Urbanik's generalized convolution which lives on the semigroup $\boldsymbol{R}_+ \times \boldsymbol{R} \cup \{(0, 0)\}$.

1. The Heisenberg group case. For $m = 1, 2, \ldots$ consider the $(2m+1)$-dimensional Heisenberg group $\boldsymbol{H}_m = \boldsymbol{R}^m \times \boldsymbol{R}^m \times \boldsymbol{R}$ with the multiplication law

$$(x, y, t)(x', y', t') = \left(x+x', y+y', t+t'+\sum_{i=1}^{m}(x_i y_i' - x_i' y_i)\right).$$

The Lie algebra of the left-invariant vector fields on $\boldsymbol{H}_m$ is then generated by X_i, Y_i, T, $i = 1, \ldots, m$, where

$$X_i = \frac{\partial}{\partial x_i} - y_i\frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial y_i} + x_i\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

and the only nontrivial commutators are $[X_i, Y_i] = T$, $i = 1, \ldots, m$. We say that a function f on $\boldsymbol{H}_m$ is *radial* if $f(\xi, t) = f(A\xi, t)$ for every $A \in \mathrm{SO}(\boldsymbol{R}^{2m})$, $\xi = (x, y) \in \boldsymbol{R}^{2m}$ and $t \in \boldsymbol{R}$. It is well known that the space of radial integrable functions (with respect to the Haar measure $dx\,dy\,dt$) forms a commutative Banach algebra under convolution. Various properties of this algebra and of the second order differential operator

$$\mathscr{L} = -\sum_{j=1}^{m}(X_j^2 + Y_j^2),$$

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called the *sublaplacian*, have recently been intensively studied by many authors (cf. [2], [4]).

By introducing polar coordinates in $\boldsymbol{R}^{2m}$ it is easy to check that the radial part of the operator $\mathscr{L}$ is given by

$$L = -\left(\frac{\partial^2}{\partial r^2}+\frac{2m-1}{r}\frac{\partial}{\partial r}+r^2\frac{\partial^2}{\partial t^2}\right).$$

This means that $\mathscr{L}f(x, y, t) = Lf_0(\|(x, y)\|, t)$ for every radial function f with radial part f_0, i.e. $f(x, y, t) = f_0(\|(x, y)\|, t)$, where $\|\ \|$ denotes the Euclidean norm in $\boldsymbol{R}^{2m}$.

The aim of this paper is to investigate the operator

$$L = -\left(\frac{\partial^2}{\partial r^2}+\frac{\alpha}{r}\frac{\partial}{\partial r}+r^2\frac{\partial^2}{\partial t^2}\right),$$

$\alpha \geqslant 1$ being an arbitrary real number, which is positive and symmetric in $L^2(X, \mu)$ where $X = \boldsymbol{R}_+\times\boldsymbol{R}$ and $d\mu(r, t) = r^\alpha\,dr\,dt$. In order to do this we construct a commutative convolution structure in $L^1(X, \mu)$ modelled on the Heisenberg group radial case.

2. The operator and the transform. Let $\alpha \geqslant 1$ be an arbitrary fixed parameter. We equip the space $X = \boldsymbol{R}_+\times\boldsymbol{R}$ with the measure $d\mu(x, t) = x^{2\alpha-1}\,dx\,dt$ and denote by $L^p(\mu)$, $1 \leqslant p \leqslant \infty$, the classical Lebesgue spaces with respect to the measure μ endowed with the norm $\|\ \|_p$. The operator

$$L = -\left(\frac{\partial^2}{\partial x^2}+\frac{2\alpha-1}{x}\frac{\partial}{\partial x}+x^2\frac{\partial^2}{\partial t^2}\right) \tag{2.1}$$

is positive, symmetric in $L^2(\mu)$ and homogeneous of degree 2 if X is endowed with the family of dilations $(\delta_r)_{r>0}$, $\delta_r(x, t) = (rx, r^2 t)$.

Let $L_n^a(x)$, $n = 0, 1, \ldots$, be the Laguerre polynomials of order $a > -1$ defined in terms of the generating function by

$$\sum_{n=0}^{\infty} t^n L_n^a(x) = \frac{1}{(1-t)^{a+1}}\exp\left(\frac{xt}{t-1}\right).$$

For $\lambda \neq 0$ and $n = 0, 1, \ldots$ we put

$$\varphi_{\lambda,n}(x, t) = \frac{n!\,\Gamma(\alpha)}{\Gamma(n+\alpha)} e^{i\lambda t} e^{-|\lambda|x^2/2} L_n^{\alpha-1}(|\lambda|\, x^2). \tag{2.2}$$

By using the differential identity

$$x\frac{d^2}{dx^2}L_n^a+(a+1-x)\frac{d}{dx}L_n^a+nL_n^a = 0$$

it may be easily verified that $\varphi_{\lambda,n}$ is an eigenfunction for the operator L, more precisely,

$$L\varphi_{\lambda,n} = 4|\lambda|(\alpha/2+n)\,\varphi_{\lambda,n}.$$

Moreover (cf. [7]), $|\varphi_{\lambda,n}(x, t)| \leqslant 1$ for $(x, t)\in X$.

On the other hand, the functions

$$\psi_\tau(x, t) = 2^{\alpha-1}\Gamma(\alpha)\frac{J_{\alpha-1}(x\tau)}{(x\tau)^{\alpha-1}}, \tag{2.3}$$

$\tau \geqslant 0$, where J_a denotes the ordinary Bessel function of order a, form the second series of eigenfunctions for L, namely

$$L\psi_\tau = \tau^2\psi_\tau.$$

One can easily verify this using the differential identity

$$x^2\frac{d^2}{dx^2}J_a+x\frac{d}{dx}J_a+(x^2-a^2)J_a = 0.$$

As before, we also have $|\psi_\tau(x, t)| \leqslant 1$ (cf. [1]).

Now, for any function $f\in L^1(\mu)$ we define its transform $\hat{f}$ by putting, for $\lambda \neq 0$ and $n = 0, 1, \ldots$,

$$\hat{f}(\lambda, n) = \int_X f(x, t)\,\varphi_{\lambda,n}(x, t)\,d\mu(x, t),$$

and for $\tau \geqslant 0$,

$$\hat{f}(\tau) = \int_X f(x, t)\,\psi_\tau(x, t)\,d\mu(x, t).$$

3. The generalized translations. We are now in a position to associate with the operator L a family of generalized translations $T^{y,u}$, $y \geqslant 0$, $u\in\boldsymbol{R}$. For an appropriate function f on X and $y \geqslant 0$, $u\in\boldsymbol{R}$ we put: for $\alpha > 1$,

$$\begin{aligned} T^{y,u}f(x, t) &= \frac{\alpha-1}{\pi}\int_0^\pi\int_0^\pi f\big((x^2+y^2-2xy\cos\theta)^{1/2},\ t-u+xy\cos\varphi\sin\theta\big) \\ &\qquad \times(\sin\varphi)^{2\alpha-3}(\sin\theta)^{2\alpha-2}\,d\varphi\,d\theta, \end{aligned} \tag{3.1}$$

and for $\alpha = 1$,

$$T^{y,u}f(x, t) = (2\pi)^{-1}\int_0^{2\pi} f\big((x^2+y^2-2xy\cos\theta)^{1/2},\ t-u-xy\sin\theta\big)\,d\theta. \tag{3.1$'$}$$

Denoting by $\Delta(x, y, z)$, $x, y, z \geqslant 0$, the area of a triangle with sides x, y, z if such a triangle exists and 0 otherwise, we define for $\alpha > 1$ the family of probability measures $W_{x,y}$, $x, y \geqslant 0$, on X by

$$dW_{x,y}(z, s) = 2^{2\alpha-3}\frac{\alpha-1}{\pi}\cdot\frac{\Delta(x, y, z)^{2\alpha-3}}{(xyz)^{2\alpha-2}}(1-s^2)_+^{\alpha-2}\,d\mu(z, s), \tag{3.2}$$

where $\lambda_+ = \max\{0, \lambda\}$. It may be easily checked that by the change of variables $x^2+y^2-2xy\cos\theta = z^2$, $\cos\varphi = s$, we can write

$$T^{y,u}f(x, t) = \int_{|x-y|}^{x+y}\int_{-1}^{1} f\big(z, t-u+2s\Delta(x, y, z)\big)\,dW_{x,y}(z, s) \tag{3.3}$$

when $\alpha > 1$, and in a similar way

$$T^{y,u}f(x, t) = (4\pi)^{-1}\int_{|x-y|}^{x+y}\{f\big(z, t-u-2\Delta(x, y, z)\big) + f\big(z, t-u+2\Delta(x, y, z)\big)\}\frac{z\,dz}{\Delta(x, y, z)} \tag{3.3'}$$

when $\alpha = 1$.

Without loss of generality in the sequel we will consider the case $\alpha > 1$ only. For $\alpha = 1$ the proofs of all results proceed in the same way with necessary changes.

We now come to the following results.

LEMMA 3.1. *We have $(T^{y,u})^* = T^{y,-u}$, i.e. for all appropriate functions f, g on X*

$$\int_X T^{y,u}f\cdot g\,d\mu = \int_X f\cdot T^{y,-u}g\,d\mu. \tag{3.4}$$

Proof. It is easy to see that (3.4) is a consequence of the following identity:

$$dW_{x,y}(z, s)\,x^{2\alpha-1}\,dx = dW_{z,y}(x, s)\,z^{2\alpha-1}\,dz \tag{3.5}$$

which follows from the definition (3.2).

PROPOSITION 3.2. *For every p, $1 \leqslant p \leqslant \infty$, and $y \geqslant 0$, $u \in \boldsymbol{R}$, the generalized translation $T^{y,u}$ is a submarkovian contraction, i.e. $0 \leqslant T^{y,u}f \leqslant 1$ whenever $0 \leqslant f \leqslant 1$ and $\|T^{y,u}f\|_p \leqslant \|f\|_p$.*

Proof. The submarkovian property follows promptly from the definition of $T^{y,u}$. In order to prove the inequality $\|T^{y,u}f\|_p \leqslant \|f\|_p$ it suffices to use (3.3), (3.5) and the fact that the $W_{x,y}$ are probability measures.

PROPOSITION 3.3. *Let $f \in L^p(\mu)$, $1 \leqslant p < \infty$. Then the mapping*

$$X \ni (y, u) \mapsto T^{y,u}f \in L^p(\mu) \tag{3.6}$$

is continuous.

Proof. It suffices to verify the continuity of the mapping defined by (3.6) for a dense class of functions in $L^p(\mu)$, say, for the continuous functions with compact support. Let f be such a function. Fix $(y_0, u_0) \in X$. By the definition (3.1) there exists a constant $K > 0$ such that if $(y, u) \in X$ satisfies $|y-y_0| < 1$, $|u-u_0| < 1$ then

$$\operatorname{supp} T^{y,u}f \subseteq [0, K]\times[-K, K] = U_K.$$

On the other hand, from (3.1) it follows that the mapping

$$X\times X \ni \big((x, t), (y, u)\big) \mapsto T^{y,u}f(x, t)$$

is continuous. Since

$$\|T^{y,u}f - T^{y_0,u_0}f\|_p \leqslant \mu(U_K)^{1/p}\sup_{(x,t)\in U_K}|T^{y,u}f(x, t) - T^{y_0,u_0}f(x, t)|,$$

we then get the desired result.

4. The convolution. For appropriate functions f, g on X we set

$$f*g(x, t) = \int_X T^{y,u}f(x, t)\cdot g(y, u)\,d\mu(y, u). \tag{4.1}$$

Let us note that for $\alpha = 2m-1$, $m = 1, 2, \ldots$, the formula (4.1) gives the radial part of the convolution on $\boldsymbol{H}_m$ of two functions with radial parts f and g respectively. The result then is

LEMMA 4.1. *Let $f \in L^p(\mu)$, $1 \leqslant p \leqslant \infty$ and $g \in L^1(\mu)$. Then $f*g$ is a well defined element of $L^p(\mu)$ and*

$$\|f*g\|_p \leqslant \|f\|_p\|g\|_1. \tag{4.2}$$

Moreover, $f*g = g*f$, i.e. the convolution is commutative.

Proof. The inequality (4.2) follows easily from the fact that the generalized translations are contractions. In the proof of the commutativity Lemma 3.1 is used.

In 1939 Watson (cf. [7]) established a formula concerning Laguerre polynomials which may be rewritten in the following form: for $\alpha > 1$

$$L_n^{\alpha-1}(x^2)\,L_n^{\alpha-1}(y^2) = \frac{2^{\alpha-3/2}\Gamma(n+\alpha)}{n!\,\pi^{1/2}} \times\int_0^\pi e^{xy\cos\theta}\frac{J_{\alpha-3/2}(xy\sin\theta)}{(xy\sin\theta)^{\alpha-3/2}}L_n^{\alpha-1}(x^2+y^2-2xy\cos\theta)(\sin\theta)^{2\alpha-2}\,d\theta. \tag{4.3}$$

When $\alpha = 1$ this formula takes the form (we write L_n instead of L_n^0)

$$L_n(x^2)\,L_n(y^2) = \pi^{-1}\int_0^\pi e^{xy\cos\theta}\cos(xy\sin\theta)\,L_n(x^2+y^2-2xy\cos\theta)\,d\theta. \tag{4.3'}$$

This gives rise to the following result.

LEMMA 4.2. *The functions $\varphi_{\lambda,n}$ and ψ_τ defined by* (2.2) *and* (2.3) *satisfy the following formulas*:

$$\varphi_{\lambda,n}(x, t)\varphi_{\lambda,n}(y, u) = T^{y,-u}\varphi_{\lambda,n}(x, t), \tag{4.4}$$

$$\psi_\tau(x, t)\psi_\tau(y, u) = T^{y,-u}\psi_\tau(x, t). \tag{4.5}$$

Proof. To verify (4.4) it suffices, by homogeneity, to consider the case $\lambda = 1$ only. By the identity

$$\int_0^\pi \cos(a\cos\varphi)(\sin\varphi)^{2\alpha-3}\,d\varphi = \pi^{1/2}2^{\alpha-3/2}\Gamma(\alpha-1)\frac{J_{\alpha-3/2}(a)}{a^{\alpha-3/2}},$$

which is valid for $\alpha > 1$, the formula (4.4) follows directly from (4.3). Similarly, (4.5) is a consequence of Watson's other formula concerning Bessel functions (cf. [1], p. 351).

As a corollary of (4.4) and (4.5) we obtain the following

LEMMA 4.3. *Let either $\xi = (\lambda, n)$ for some $\lambda \neq 0$ and $n = 0, 1, \ldots$, or $\xi = \tau$ for a $\tau \geqslant 0$. Then for any $f, g \in L^1(\mu)$ we have*

$$(f*g)\hat{\ }(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

The convolution of two functions given by (4.1) may be readily extended to the convolution of two measures. In fact, setting for $\alpha > 1$

$$\Phi((x, t), (y, u), (z, s)) = 2^{2\alpha-4}\frac{\alpha-1}{\pi}\cdot\frac{\Delta(x, y, z)^{2\alpha-4}}{(xyz)^{2\alpha-2}}\left(1-\left(\frac{u+s-t}{2\Delta(x, y, z)}\right)^2\right)_+^{\alpha-2},$$

for two measures λ_1, λ_2 on X we define their convolution $\lambda_1*\lambda_2$ by

$$\lambda_1*\lambda_2(E) = \int_X\int_X\left\{\int_E\Phi((x, t), (y, u), (z, s))\,d\mu(x, t)\right\}d\lambda_1(y, u)\,d\lambda_2(z, s), \tag{4.6}$$

for every Borel subset $E \subseteq X$. Clearly, (4.6) agrees with (4.1) in the case when λ_1 and λ_2 have densities f and g with respect to μ.

The following lemma generalizes the result in [5].

LEMMA 4.4. *Suppose $\alpha > 1$ and λ_1, λ_2 are measures on X. Then $\lambda_1*\lambda_2 \in L^1(\mu)$.*

Proof. Let $E \subseteq X$ be a set of measure zero, $\mu(E) = 0$. Then the innermost integral in (4.6) is zero for every (y, u) and (z, s), and so $\lambda_1*\lambda_2(E) = 0$ which completes the proof.

5. The heat kernel associated with L. The operator L defined by (2.1) is clearly hypoelliptic on X, and so is the heat operator associated to L,

$$L+\partial_s = L_{(x,t)}+\frac{\partial}{\partial s},$$

defined on $X\times\mathbf{R}$.

Repeating, in a suitable way, the arguments from the proof of Hunt's famous theorem (cf. e.g. the proof of Theorem 3.4, [3]) it is not difficult to establish the existence of a unique semigroup, in the sense of our convolution, of probability measures $\{\nu_t\}_{t>0}$ on X with infinitesimal generator L; this means that we have

$$\nu_s*\nu_t = \nu_{s+t}, \quad s, t > 0, \tag{5.1}$$

$$\partial_s(u*\nu_s) = -(Lu)*\nu_s, \tag{5.2}$$

for every $u \in C_0(X)$.

The hypoellipticity of the heat operator implies in the standard way that, in fact, the measures ν_t, $t > 0$, have smooth densities. Moreover, we have

PROPOSITION 5.1. *There is a unique C^∞-function $p((x, t), s) = p_s(x, t)$ on $X\times(0, \infty)$ with the following properties*:

(i) $(\partial_s+L)p = 0$ on $X\times(0, \infty)$.

(ii) $p((x, t), s) \geqslant 0$, $p_s(x, -t) = p_s(x, t)$, $\int_X p_s\,d\mu = 1$.

(iii) $p_{s_1}*p_{s_2} = p_{s_1+s_2}$, $s_1, s_2 > 0$.

(iv) $p_{r^2s}(\delta_r(x, t)) = r^{-(2\alpha+2)}p_s(x, t)$.

Proof. The proposition is proved by a well-known technique (cf. e.g. [2], p. 56), but for the sake of completeness we include here the suitable argument just taken from there. Define the distribution p on $X\times(0, \infty)$ by

$$\langle p, u\otimes v\rangle = \int_0^\infty\int_X u(x, t)v(s)\,d\nu_s(x, t)\,ds,$$

where $u \in C_0(X)$, $v \in C_0(0, \infty)$ and $\{\nu_s\}_{s>0}$ is the unique semigroup of measures which satisfies (5.1) and (5.2). So we get

$$\begin{aligned}\langle p, Lu\otimes v\rangle &= \int_0^\infty\int_X Lu(x, t)v(s)\,d\nu_s(x, t)\,ds = \int_0^\infty Lu*\nu_s(0, 0)v(s)\,ds\\ &= -\int_0^\infty\partial_s(u*\nu_s)(0, 0)v(s)\,ds = \int_0^\infty u*\nu_s(0, 0)\,\partial_s v(s)\,ds\\ &= \int_0^\infty\int_X u(x, t)\,\partial_s v(s)\,d\nu_s(x, t)\,ds = \langle p, u\otimes\partial_s v\rangle.\end{aligned}$$

But this clearly means that p is a distribution solution of $(L+\partial_s)p = 0$ and, by the hypoellipticity of $L+\partial_s$, p is a C^∞-function on $X\times(0, \infty)$. Obviously $dv_s(x, t) = p_s(x, t)\,d\mu(x, t)$ and therefore the properties (ii), (iii) follow from the corresponding properties of v_s. Finally, (iv) follows from the fact that L is homogeneous of degree 2 with respect to the dilations δ_r, $r > 0$. This completes the proof of the proposition.

Now, for a function $f\in L^p(\mu)$ we can define its *Gauss–Weierstrass integral* F by

$$F((x, t), s) = f * p_s(x, t), \tag{5.3}$$

and we then get

PROPOSITION 5.2. *Let $f\in L^p(\mu)$, $1 \leqslant p \leqslant \infty$. Then the function F defined by* (5.3) *satisfies the following*:

(i) *F is a C^∞-function on $X\times(0, \infty)$ and $(\partial_s+L)F = 0$.*

(ii) *$\|F(\cdot, s)-f\|_p \to 0$ as $s\to 0$ if $p < \infty$.*

(iii) *If $p = \infty$ and $f\in C_0(X)$ then $\|F(\cdot, s)-f\|_\infty \to 0$ as $s\to 0$.*

Proof. Using Proposition 5.1 it is not difficult to show that $(\partial_s+L)F = 0$ in distribution sense and so, once again by hypoellipticity, F is a C^∞-function. This gives (i). The verification of (ii) and (iii) is routine.

The transform of the heat kernel p_s, $s > 0$, is now described by the following

LEMMA 5.3. *For any $s > 0$ and $\tau \geqslant 0$, $\lambda \neq 0$, $n = 0, 1, \ldots,$*

$$\hat{p}_s(\tau) = \exp(-s\tau^2), \qquad \hat{p}_s(\lambda, n) = \exp(-4|\lambda|(\alpha/2+n)s). \tag{5.4}$$

Proof. The equalities $\partial_s(p_s * u) = -Lu * p_s$ and $(Lu)\hat{}\,(\tau) = \tau^2\hat{u}(\tau)$, $(Lu)\hat{}\,(\lambda, n) = 4|\lambda|(\alpha/2+n)\hat{u}(\lambda, n)$ show that the functions $\hat{p}_s(\tau)$ and $\hat{p}_s(\lambda, n)$ satisfy the differential equations $(d/ds)w = -\tau^2 w$ and $(d/ds)w = -4|\lambda|(\alpha/2+n)w$ respectively. Clearly, this completes the proof.

6. The Gelfand space of the Banach algebra $L^1(\mu)$. As we have shown, $L^1(\mu)$ equipped with the convolution (4.1) forms a commutative Banach algebra. Moreover, if $\xi = (\lambda, n)$ for some $\lambda \neq 0$ and $n = 0, 1, \ldots,$ or $\xi = \tau$ for some $\tau \geqslant 0$ then $f\mapsto \hat{f}(\xi)$ is a multiplicative functional on $L^1(\mu)$. Now we show the converse.

PROPOSITION 6.1. *Every multiplicative functional on $L^1(\mu)$ is of the form $f\mapsto\hat{f}(\xi)$, where either $\xi = (\lambda, n)$ for some $\lambda \neq 0$ and $n = 0, 1, \ldots,$ or $\xi = \tau$ for some $\tau \geqslant 0$.*

Proof. Suppose Φ is a nontrivial multiplicative functional on $L^1(\mu)$. Clearly, Φ is of the form $\Phi(f) = \int_X f\varphi\,d\mu$ for a function φ, $|\varphi(x, t)| \leqslant 1$, which satisfies

$$\varphi(x, t)\cdot\varphi(y, u) = T^{y,-u}\varphi(x, t) \tag{6.1}$$

$\mu\times\mu$-almost everywhere. We will show that φ is an eigenfunction for L, i.e.

$$L\varphi = k\varphi \tag{6.2}$$

for a $k\in\boldsymbol{R}$. Indeed, take the Gauss–Weierstrass integral of φ: $u((x, t), s) = \varphi * p_s(x, t)$. Since (6.1) easily implies

$$\varphi * p_s = \varphi * p_s(0, 0)\cdot\varphi \tag{6.3}$$

and $Lu = -\partial_s u$, we obtain

$$\varphi * p_s(0, 0)\,L\varphi = -\varphi\cdot\partial_s(\varphi * p_s(0, 0)).$$

But by (6.3), φ is continuous on X and therefore $\lim_{s\to 0}\varphi * p_s(0, 0) = \varphi(0, 0)$ as well as $\lim_{s\to 0}\partial_s(\varphi * p_s(0, 0))$ exist. Thus $L\varphi = k^2\varphi$ for a $k \geqslant 0$. By (6.1) it also follows that φ has separated variables:

$$\varphi(x, t) = T^{0,0}\varphi(x, t) = T^{0,-t}\varphi(x, 0) = \varphi(x, 0)\,\varphi(0, t) = g(x)\,h(t).$$

Consequently, the above, the fact that $\|h\|_\infty = 1$ and (6.2) show that, first, $h(t) = e^{i\lambda t}$ for a $\lambda\in\boldsymbol{R}$. Next, when $\lambda = 0$, we have

$$g'' + \frac{2\alpha-1}{x}g' = -k^2 g$$

and thus, since $\|g\|_\infty = 1$, $g(x) = 2^{\alpha-1}\Gamma(\alpha)J_{\alpha-1}(xk)(xk)^{1-\alpha}$. In the case $\lambda \neq 0$, g satisfies the equation

$$g'' + \frac{2\alpha-1}{x}g' + (k^2 - x^2\lambda^2)g = 0,$$

which implies that $k\in N$ and, since $\|g\|_\infty = 1$,

$$g(x) = \frac{k!\,\Gamma(\alpha)}{\Gamma(k+\alpha)}e^{-|\lambda|x^2/2}L_k^{\alpha-1}(|\lambda|x^2).$$

This completes the proof of the proposition.

7. An example of a generalized convolution. K. Urbanik in his papers (cf. [6] for references) developed the theory of generalized convolutions. This theory treats measures which live on the positive half-line $\boldsymbol{R}_+$ equipped with the usual family of dilations. Here we present an example of a generalized convolution living on the semigroup $\boldsymbol{R}_+\times\boldsymbol{R}\cup\{(0, 0)\}$ endowed with a family of dilations, which satisfies all the axioms of Urbanik's theory.

Consider the semigroup $\bar{X} = \boldsymbol{R}_+\times\boldsymbol{R}\cup\{(0, 0)\}$ endowed with the family of dilations $(\delta_r)_{r>0}$, $\delta_r(x, t) = (rx, r^2 t)$. We denote by $\mathscr{P}$ the set of all probability measures defined on the Borel subsets of $\bar{X}$ and we equip $\mathscr{P}$ with the topology of weak convergence. For $P \in \mathscr{P}$ and $r > 0$ we define the measure $T_r P$ by

$$T_r P(E) = P(\delta_{r^{-1}}(E)),$$

for every Borel subset $E \subseteq \bar{X}$.

Let us write, for $x \in \boldsymbol{R}$ and $\beta > 0$,

$$x^{(\beta)} = \operatorname{sgn} x \cdot |x|^{\beta}.$$

For arbitrary but fixed $\alpha \geqslant 1$ and $\beta > 0$, define the generalized convolution $P \circ_{\alpha,\beta} Q$ of two measures $P, Q \in \mathscr{P}$ by means of the functional $\int_{\bar{X}} F\, d(P \circ_{\alpha,\beta} Q)$ (F a bounded continuous function on $\bar{X}$):

$$\int_{\bar{X}} F(x, t)\, d(P \circ_{\alpha,\beta} Q)(x, t)$$

$$= \frac{\alpha-1}{\pi} \int_{\bar{X}} \int_{\bar{X}} \int_0^{\pi} \int_0^{\pi} F\big((x^{2\beta}+y^{2\beta}-2x^{\beta} y^{\beta} \cos\theta)^{1/(2\beta)},$$

$$(u^{(\beta)}+t^{(\beta)}+x^{\beta} y^{\beta} \cos\varphi \sin\theta)^{(1/\beta)}\big)$$

$$\times (\sin\varphi)^{2\alpha-3} (\sin\theta)^{2\alpha-2}\, d\varphi\, d\theta\, dP(x, t)\, dQ(y, u) \quad \text{for } \alpha > 1,$$

$$= (2\pi)^{-1} \int_{\bar{X}} \int_{\bar{X}} \int_0^{2\pi} F\big((x^{2\beta}+y^{2\beta}-2x^{\beta} y^{\beta} \cos\theta)^{1/(2\beta)},$$

$$(u^{(\beta)}+t^{(\beta)}-x^{\beta} y^{\beta} \sin\theta)^{(1/\beta)}\big)\, d\theta\, dP(x, t)\, dQ(y, u) \quad \text{for } \alpha = 1.$$

The result then is (we write $\circ$ instead of $\circ_{\alpha,\beta}$):

PROPOSITION 7.1. *The operation $\circ$ is a commutative and associative $\mathscr{P}$-valued binary operation satisfying the following conditions:*

(i) $\delta_{0,0}$ *is the unit element.*

(ii) $(aP+bQ) \circ R = a(P \circ R)+b(Q \circ R)$, $a+b = 1$, $a, b > 0$, $P, Q, R \in \mathscr{P}$.

(iii) $T_r(P \circ Q) = (T_r P) \circ (T_r Q)$, $r > 0$, $P, Q \in \mathscr{P}$.

(iv) *If* $P_n \to P$ *then* $P_n \circ Q \to P \circ Q$, $P_n, P, Q \in \mathscr{P}$.

In order to introduce the notion of characteristic function let us first define two kinds of kernels by

$$\Omega_1(y) = 2^{\alpha-1} \Gamma(\alpha) J_{\alpha-1}(y^{\beta})\, y^{\beta(1-\alpha)}, \tag{7.1}$$

and for $n = 0, 1, \ldots$,

$$\Omega_{2,n}(y, u) = \frac{n!\,\Gamma(\alpha)}{\Gamma(n+\alpha)} \exp(iu^{(\beta)}) \exp(-y^{2\beta}/2)\, L_n^{\alpha-1}(y^{2\beta}). \tag{7.2}$$

Then for $\tau \geqslant 0$, $\lambda \neq 0$, $n = 0, 1, \ldots$, and $P \in \mathscr{P}$ define the characteristic

functions $\hat{P}(\tau)$, $\hat{P}(\lambda, n)$ by

$$\hat{P}(\tau) = \int_{\bar{X}} \Omega_1(\tau x)\, dP(x, t), \tag{7.3}$$

$$\hat{P}(\lambda, n) = \int_{\bar{X}} \Omega_{2,n}(|\lambda|^{1/2} x, \lambda t)\, dP(x, t). \tag{7.4}$$

Now, it may be checked, by using Lemma 4.2, that the characteristic functions satisfy the following properties:

$$\big(cP+(1-c)Q\big)\hat{}\,(\xi) = c\hat{P}(\xi)+(1-c)\hat{Q}(\xi), \tag{7.5}$$

$$(P \circ Q)\hat{}\,(\xi) = \hat{P}(\xi) \cdot \hat{Q}(\xi), \tag{7.6}$$

where $0 \leqslant c \leqslant 1$, $P, Q \in \mathscr{P}$ and $\xi = \tau$ or $\xi = (\lambda, n)$. Furthermore,

$$(T_r P)\hat{}\,(\tau) = \hat{P}(r\tau), \qquad (T_r P)\hat{}\,(\lambda, n) = \hat{P}(r^2 \lambda, n). \tag{7.7}$$

Now we are in a position to prove the central limit theorem for the convolution $\circ_{\alpha,\beta}$. We denote by $E_{x,t}$ the probability measure concentrated at the point $(x, t) \in \bar{X}$. The power $P^{\circ n}$, $P \in \mathscr{P}$, will be taken here in the sense of the operation $\circ_{\alpha,\beta}$, and p_s, $s > 0$, will denote the heat kernel associated with the operator L defined by (2.1), as in Section 5.

PROPOSITION 7.2. *Let $c_m = m^{-1/2}$. Then for every $a > 0$*

$$T_{c_m}(E_{a,0}^{\circ m}) \to \gamma \tag{7.8}$$

weakly as $m \to \infty$, where γ is the probability measure on $\bar{X}$ defined by

$$\gamma(A) = \beta^2 \int_A p_{1/(4\alpha)}(x^{\beta}, t^{(\beta)})\, x^{2\alpha\beta-1}\, t^{(\beta-1)}\, dx\, dt.$$

Proof. To simplify the calculations we consider the case $\beta = 1$ and $a = 1$ only. Clearly

$$\lim_{m \to \infty} (T_{c_m} E_{1,0}^{\circ m})\hat{}\,(\tau) = \exp(-\tau^2/4). \tag{7.9}$$

Similarly, since

$$\frac{n!\,\Gamma(\alpha)}{\Gamma(n+\alpha)} \exp(-y^2/2)\, L_n^{\alpha-1}(y^2) = 1-(\tfrac{1}{2}+n/\alpha)\, y^2+o(y^2),$$

we get

$$\lim_{m \to \infty} (T_{c_m} E_{1,0}^{\circ m})\hat{}\,(\lambda, n) = \exp\big(-|\lambda|\,(\tfrac{1}{2}+n/\alpha)\big). \tag{7.10}$$

Thus, in virtue of Lemma 5.3, the identities (7.9) and (7.10) imply (7.8) in the standard way, which completes the proof of the proposition.

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Added in proof (September 1987). The author has recently found a new proof of Watson's product formula (4.3) for Laguerre polynomials (Canad. Math. Bull., to appear). The proof is based upon the observation that (4.3) holds for $\alpha = 1, 2, \ldots$ This is established by considering the commutative Banach algebras of radial functions on the Heisenberg groups H_n, $n \geqslant 2$. Then a theorem of Carlson is used to get the validity of (4.3) for all values of α with $\mathrm{Re}\,\alpha > -1/2$.

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Measures on groups with given projections

by

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Abstract. A theorem of Lindenstrauss asserts that every extreme doubly stochastic measure is singular. This result is extended to the case of locally compact groups.

§ 0. **Introduction.** Let λ be the Lebesgue measure on $I = [0, 1]$. A Borel measure μ on $I \times I$ is said to be *doubly stochastic* if $\mu(A \times I) = \mu(I \times A) = \lambda(A)$ for each Borel set $A \subseteq I$. The collection of all doubly stochastic measures forms a convex, weakly compact set whose extreme points have been the subject of much study: [1]–[3], [6]–[8]. It was shown by Lindenstrauss [6] that every extreme doubly stochastic measure is singular with respect to the planar measure $\lambda \otimes \lambda$. In [8], this result was generalized.

Let $L_1, \ldots, L_m$ be nontrivial linear subspaces of $\boldsymbol{R}^n$. Suppose that ν is a probability measure on $\boldsymbol{R}^n$ and let E be the convex set of probabilities on $\boldsymbol{R}^n$ whose projections onto $L_1, \ldots, L_m$ agree with those of ν. In [8], it is proved that the extreme points of E are singular with respect to the n-dimensional Lebesgue measure on $\boldsymbol{R}^n$. The fact that homotheties of $\boldsymbol{R}^n$ by a scalar r change the Lebesgue measure by a factor of r^n was an important feature of the proof.

We further generalize this result to the context of a locally compact group. An appropriate convex set is the collection of all measures on the group whose projections onto various quotient groups are prescribed. Under suitable hypotheses, the extreme points of this set will be singular with respect to the Haar measure. For the proof, one must compensate for the fact that homotheties are not available in the context of groups. As in almost all work in such problems, the following result of Douglas and Lindenstrauss is crucial.

Let (X, ν) be a finite measure space and let F be a linear space of ν-integrable functions containing all constant functions. Let $E(\nu)$ be the (convex) set of all finite measures ϱ on X such that

$$\int f\,d\nu = \int f\,d\varrho$$

for each $f \in F$.

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