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Analogues of Hardy's inequality in R^n

by

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Abstract. For a class of integral operators K, defined for functions on a cone V in R^n, we prove a weighted L^p-norm inequality

$$\int_V (Kf(x))^p \Delta^\gamma(x)\,dx \leqslant C \int_V f^p(x)\Delta^\gamma(x)\,dx$$

where $1 \leqslant p < \infty$, and the weight functions $\Delta^\gamma(x)$ are suitably defined (as n-dimensional analogues of the power functions t^γ, $t \in R$). As special cases of the operator K we consider Hardy's operator and the Laplace transform.

1. Introduction. The purpose of this paper is to find an n-dimensional analogue of the well-known Hardy inequality [8, p. 20]:

If $1 \leqslant p < \infty$ *and* $\gamma < p-1$, *then for positive functions* f

$$(\mathrm{H}_p) \qquad \int_0^\infty \Big(\int_0^x f(y)\,dy\Big)^p x^{\gamma-p}\,dx \leqslant C \int_0^\infty f^p(x)\,x^\gamma\,dx$$

(where C *is a finite constant).*

We shall consider cones as n-dimensional counterparts of the half-line $(0, \infty) = R_+$. Let V be a cone. Throughout the paper V will be assumed to be open, convex, homogeneous and selfadjoint (see Section 2 for the definitions). The cone V defines a partial order in R^n in the following way: $x <_V y$ iff $y-x \in V$. We shall write $\langle a, b\rangle$ for the "interval" $\langle a, b\rangle = \{x \in V\colon a <_V x <_V b\}$ and define the operator ("Hardy's operator") by

$$(1) \qquad Hf(x) = \int_{\langle 0,x\rangle} f(y)\,dy, \qquad x \in V,$$

for positive functions f defined on V. In particular, we shall write

$$(2) \qquad \Delta(x) = \int_{\langle 0,x\rangle} dy.$$

It will be shown that the powers of this function, $\Delta^\gamma(x)$, play the role of the weights x^γ in (H_p).

In the one-dimensional case, it is known that inequalities similar to (H_p) hold for other operators too: for example, for the Laplace transform. So we shall consider a class of operators

(3) $$Kf(x) = \int_V k(x, y) f(y)\,dy, \qquad x \in V,$$

where $k\colon V\times V \to \boldsymbol{R}_+$ is a given function, the *kernel* of K, and $f\colon V \to \boldsymbol{R}_+$. If the integral in (3) is convergent for some function f, the operator is said to be *applicable* to f.

We shall find conditions for the kernel k which ensure that the operator K satisfies an (H_p)-type inequality (i.e. it is a bounded operator on the weighted L^p spaces on the cone, with weights Δ^γ). This will be done in Section 3 (Theorem 1), after introducing some definitions and preliminary results in Section 2. In Section 4 we shall consider some special cases of Theorem 1. It will be shown that Hardy's operator satisfies the conditions of this theorem, and this will give the n-dimensional analogue of (H_p) (Theorem 2). We shall also show that the Laplace operator defined by

(4) $$Lf(x) = \int_V e^{-x^*\cdot y} f(y)\,dy$$

(where $*$ denotes an involution in V, see (6) below) satisfies the conditions of Theorem 1, and in fact satisfies the same type of inequality, with the same weights. (See also [1], where it was shown, in the one-dimensional case, but in spaces more general than L^p, that the operators H and L satisfy inequalities with equal weights.)

Note that in the one-dimensional case the weights x^γ in (H_p) were replaced by a larger class of functions. In fact, Muckenhoupt [3] found a necessary and sufficient condition for the weights to satisfy (H_p). Now, as it was pointed out in [4] the n-dimensional analogue of this condition is necessary but not sufficient. In this paper we show that the weights Δ^γ satisfy Hardy's inequality, but the problem of more general weights remains open.

2. Homogeneous cones. First we introduce some definitions (see [2], [5] or [6]).

Let V be an open convex cone in $\boldsymbol{R}^n$. Then $V^* = \{y \in \boldsymbol{R}^n\colon y\cdot x > 0,\ x \in V\}$ is called the *dual cone* of V. The cone V is *selfadjoint* if $V = V^*$. Let $G(V)$ be the group of automorphisms of V (i.e. the group of all nonsingular linear transformations $A\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that $AV = V$). The cone is said to be *homogeneous* if for any $x, y \in V$ there is an $A \in G(V)$ such that $x = Ay$.

Let φ be the "characteristic function" of the cone:

(5) $$\varphi(x) = \int_V e^{-x\cdot y}\,dy, \qquad x \in V.$$

(The integral is convergent, see for example [6]).

Now put

(6) $$x^* = -\operatorname{grad} \log \varphi(x).$$

It was proved in [2, 5] that the function $*$ is an involution of V and satisfies

(7) $$x^{**} = x,$$

(8) $$(Ax)^* = A'^{-1}x^*, \qquad \text{for } A \in G(V),$$

(9) $$\varphi(x)\,\varphi(x^*) = c_1,$$

(10) $$\left|\frac{dx^*}{dx}\right| = c_2\,\varphi^2(x),$$

where the left-hand side denotes the Jacobian of the transformation $x \to x^*$; c_1 and c_2 are constants depending on the cone V.

Throughout the paper the letters C, c, possibly with subscripts, will be used to denote constants, not necessarily the same at each appearance; it is clear from the context on which parameters the constants depend.

Next we introduce a definition.

Definition. A function $f\colon V \to \boldsymbol{R}_+$ is said to be *V-homogeneous of order* α, for some $\alpha \in \boldsymbol{R}$, if

$$f(Ax) = |A|^\alpha f(x)$$

for all $A \in G(V)$.

Since for $\lambda > 0$, $Ax = \lambda x$ is clearly an automorphism of V, we see that V-homogeneous functions are homogeneous in the usual sense, i.e. $f(\lambda x) = \lambda^{\alpha n} f(x)$.

For example, the function φ defined in (5) is V-homogeneous of order -1, and Δ defined in (2) is V-homogeneous of order 1. Both statements are verified by introducing a change of variables in the respective integrals. In the latter case we also use the fact that $x <_V y$ implies $Ax <_V Ay$, for every $A \in G(V)$.

In the next lemma we prove that there are not many V-homogeneous functions. This is the analogue of the fact that Ct^α, $\alpha \in \boldsymbol{R}$, are the only homogeneous functions in $\boldsymbol{R}_+$.

Lemma 1. *Let V be a homogeneous cone and let $f\colon V \to \boldsymbol{R}_+$ be V-homogeneous of order α, $\alpha \in \boldsymbol{R}$. Then there is a constant C such that*

$$f(x) = C\Delta^\alpha(x), \qquad x \in V.$$

Proof. Consider the set $D = \{x \in V\colon \Delta(x) = 1\}$. We claim: if $A \in G(V)$ is such that $Ax_1 = x_2$ for some $x_1, x_2 \in D$, then $|A| = 1$. Indeed, since Δ is

V-homogeneous of order 1, we have

$$1 = \Delta(x_2) = \Delta(Ax_1) = |A|\,\Delta(x_1) = |A|$$

which proves the claim.

Now we prove that every V-homogeneous function is constant on D. Indeed, if $x_1, x_2 \in D$, then there is an $A \in G(V)$ such that $Ax_1 = x_2$ (since the cone V is homogeneous) and such that $|A| = 1$ (by the claim above). Now since f is V-homogeneous, we have

$$f(x_2) = f(Ax_1) = |A|^\alpha f(x_1) = f(x_1)$$

which proves that $f(\bar{x}) = C$, for every $\bar{x} \in D$. Finally, for $x \in V$ we have $\bar{x} = x\Delta^{-1/n}(x) \in D$, so that

$$f(x) = f\big(\Delta^{1/n}(x)\,\bar{x}\big) = \big(\Delta^{1/n}(x)\big)^{\alpha n} f(\bar{x}) = \Delta^\alpha(x)\, f(\bar{x}) = C\Delta^\alpha(x).$$

This proves the lemma.

Corollary 1.

$$1/\varphi(x) = C\Delta(x), \qquad x \in V.$$

This corollary is obvious, since by a previous remark, the function $1/\varphi$ is V-homogeneous of order 1.

Thus $1/\varphi$ and Δ are equal up to a constant, which was not at all obvious from the definitions (2) and (5) of Δ and φ. Now we can use both formulas (2) and (5) to derive the properties of these two functions. First we see that Δ satisfies formulas analogous to (9) and (10):

(11) $$\Delta(x)\,\Delta(x^*) = C_1,$$

(12) $$\left|\frac{dx^*}{dx}\right| = \frac{C_2}{\Delta^2(x)}.$$

On the other hand, the following properties are easily deduced from (2):

(13) *Δ is continuous and $\Delta(t) > 0$, $t \in V$.*

(14) *$\Delta(t) \to 0$ as t approaches the boundary of the cone.*

(15) *If B is a compact set in V, then there are constants C_1 and C_2 such that $0 < C_1 \leqslant \Delta(x) \leqslant C_2 < \infty$, for $x \in B$.*

3. Operators on cones. In this section we consider operators of the type (3). The kernel $k\colon V \times V \to \mathbf{R}_+$ of the operator K is said to be *V-homogeneous of order α* if

$$k(Ax, Ay) = |A|^\alpha k(x, y)$$

for every $A \in G(V)$.

Examples of operators with V-homogeneous kernels are (1) and (4). Their kernels are

(16) $$k(x, y) = \begin{cases} 1, & y <_V x, \\ 0, & \text{for other } y \in V, \end{cases}$$

and

(17) $$k(x, y) = e^{-x^* \cdot y}$$

respectively. It is easy to check that both kernels are V-homogeneous of order 0 (for (17) we have to use (8)).

As a matter of fact, it is enough to consider V-homogeneous kernels of order 0 only, since if k is V-homogeneous of order α we can take $k_1(x, y) = \Delta^{-\alpha}(x)\,k(x, y)$, which is V-homogeneous of order 0.

Lemma 2. *Let V be a homogeneous selfadjoint cone, and let the operator K have a V-homogeneous kernel of order 0 and be applicable to Δ^α, for some $\alpha \in \mathbf{R}$. Then*

$$K\Delta^\alpha(x) = C\Delta^{\alpha+1}(x).$$

Proof. We shall show that $K\Delta^\alpha(x)$ is V-homogeneous of order $\alpha+1$; then the lemma will follow from Lemma 1. Now, if $A \in G(V)$, we have by introducing a change of variables and making use of the V-homogeneity of k and Δ:

$$K\Delta^\alpha(Ax) = \int_V k(Ax, y)\,\Delta^\alpha(y)\,dy = \int_V k(Ax, Au)\,\Delta^\alpha(Au)\,|A|\,du$$
$$= |A|^{\alpha+1} \int_V k(x, u)\,\Delta^\alpha(u)\,du = |A|^{\alpha+1} K\Delta^\alpha(x).$$

This proves the lemma.

Next we consider the adjoint operator

$$K'f(y) = \int_V k(x, y)\,f(x)\,dx.$$

In the proof of Theorem 1 we shall have to deal with both operators K and K' simultaneously. It will make matters easier if we impose a further condition upon the kernel $k(x, y)$, which relates it to the adjoint kernel $k(y, x)$:

(18) $$k(x^*, y^*) = k(y, x).$$

Let a kernel which satisfies (18) be called *$*$-symmetric*. The lemma below shows for $*$-symmetric kernels the relation between the domains of K and K'.

Our main two examples, Hardy's and Laplace's operators, have $*$-symmetric kernels. Indeed, for the kernel (16) we have

$$k(x^*, y^*) = 1 \Leftrightarrow y^* <_V x^* \Leftrightarrow x <_V y \quad \text{(cf. [2])}$$
$$\Leftrightarrow k(y, x) = 1,$$

which means that k satisfies (18). Also for the kernel (17) we have (by making use of (7))

$$k(x^*, y^*) = e^{-x^{**}\cdot y^*} = e^{-x\cdot y^*} = e^{-y^*\cdot x} = k(y, x).$$

Lemma 3. *Let V be a homogeneous selfadjoint cone. If the kernel k of the operator K is $*$-symmetric, then*

$$K'\Delta^\alpha(y^*) = CK\Delta^{-\alpha-2}(y).$$

Proof. We have

$$K'\Delta^\alpha(y^*) = \int_V k(x, y^*)\Delta^\alpha(x)\,dx.$$

If we introduce the change of variables $x = u^*$, we have, according to (12), $dx = Cdu/\Delta^2(u)$ and therefore

$$K'\Delta^\alpha(y^*) = C\int_V k(u^*, y^*)\Delta^\alpha(u^*)\frac{du}{\Delta^2(u)}.$$

Now an application of (18) and (11) yields

$$K'\Delta^\alpha(y^*) = C_1\int_V k(y, u)\Delta^{-\alpha}(u)\Delta^{-2}(u)\,du = C_1 K\Delta^{-\alpha-2}(y).$$

This proves the lemma.

Corollary 2. *Let the kernel of the operator K be $*$-symmetric. If K is applicable to Δ^α, for some $\alpha\in \mathbf{R}$, then K' is applicable to $\Delta^{-\alpha-2}$.*

Now we come to the main theorem of the paper.

Theorem 1. *Let $1 \leqslant p < \infty$. Let V be a homogeneous selfadjoint cone in $\mathbf{R}^n$. Let k be a V-homogeneous kernel of order 0, which is $*$-symmetric and such that for a given $\alpha\in\mathbf{R}$ the operator K is applicable to Δ^α. Then*

$$\int_V (Kf(x))^p \Delta^{\gamma-p}(x)\,dx \leqslant C\int_V f^p(x)\Delta^\gamma(x)\,dx \tag{19}$$

where $\gamma = -\alpha p - 1$.

Proof. First, since K is applicable to Δ^α, we see from Lemma 2 that

$$K\Delta^\alpha(x) = C\Delta^{\alpha+1}(x) \tag{20}$$

and from Corollary 2 it follows that the operator K' is applicable to $\Delta^{-\alpha-2}$, so that again by Lemma 2 we have

$$K'\Delta^{-\alpha-2}(x) = C\Delta^{-\alpha-1}(x). \tag{21}$$

Now we apply Hölder's inequality to the integral defining Kf (the integrand of which was multiplied by $\Delta^\beta\Delta^{-\beta}$; $\beta\in\mathbf{R}$ will be chosen later):

$$\begin{aligned} Kf(x) &= \int_V k(x, y) f(y)\,dy \\ &= \int_V k^{1/p}(x, y) f(y)\Delta^\beta(y) k^{1/p'}(x, y)\Delta^{-\beta}(y)\,dy \\ &= \Big(\int_V k(x, y) f^p(y)\Delta^{\beta p}(y)\,dy\Big)^{1/p}\Big(\int_V k(x, y)\Delta^{-\beta p'}(y)\,dy\Big)^{1/p'}. \end{aligned} \tag{22}$$

Now if we choose $\beta = -\alpha/p'$, i.e. $-\beta p' = \alpha$, then we can apply (20) to the last integral in (22) and obtain

$$\begin{aligned} Kf(x) &= \Big(\int_V k(x, y) f^p(y)\Delta^{-\alpha p/p'}(y)\,dy\Big)^{1/p}\big(K\Delta^\alpha(x)\big)^{1/p'} \\ &= \big(C\Delta^{\alpha+1}(x)\big)^{1/p'}\Big(\int_V k(x, y) f^p(y)\Delta^{-\alpha p/p'}(y)\,dy\Big)^{1/p}. \end{aligned} \tag{23}$$

Now we can substitute (23) into the left-hand side of (19) (and use $p/p' = p-1$):

$$\begin{aligned} &\int_V (Kf(x))^p\Delta^{-\alpha p-1-p}(x)\,dy \\ &\leqslant C\int_V \Delta^{-\alpha p-1-p}(x)\Delta^{(\alpha+1)(p-1)}(x)\int_V k(x, y) f^p(y)\Delta^{-\alpha(p-1)}(y)\,dy\,dx. \end{aligned} \tag{24}$$

By an application of Fubini's Theorem the last integral equals

$$\int_V f^p(y)\Delta^{-\alpha(p-1)}(y)\int_V k(x, y)\Delta^{-\alpha-2}(x)\,dx\,dy \tag{25}$$

(since $-\alpha p - 1 - p + (\alpha+1)(p-1) = -\alpha-2$), and the inner integral is $K'\Delta^{-\alpha-2}(y)$ so that by (21) we deduce that (25) equals

$$C\int_V f^p(y)\Delta^{-\alpha(p-1)}(y)\Delta^{-\alpha-1}(y)\,dy = C\int_V f^p(y)\Delta^{-\alpha p-1}(y)\,dy. \tag{26}$$

Now the theorem follows from (24)–(26).

4. Applications to some special operators. In this section we shall apply Theorem 1 to some special operators including Hardy's (1) and Laplace's (4). In fact we only have to prove that these operators are applicable to some Δ^α (all the other conditions of Theorem 1 were verified earlier).

Now, let $\Sigma = \{x \in V\colon |x| = 1\}$ be the part of the unit sphere contained in V. Consider the integral

(27) $$\int_{\Sigma} \Delta^{\alpha}(t')\,dt'.$$

Since Δ is a bounded continuous function on Σ (see (13)) it is obvious that (27) is convergent for all $\alpha \geqslant 0$. And if (27) converges for some $\alpha_0 < 0$, then it converges for all $\alpha > \alpha_0$, by Hölder's inequality. Let $\sigma_0 = \sigma_0(V)$ be the infimum of all α such that (27) is convergent. For example, the cone $\boldsymbol{R}^n_+ = \{(x_1, \ldots, x_n)\colon x_1 > 0, \ldots, x_n > 0\}$ has $\sigma_0 = -1$, and the cone $V^n_+ = \{(x_0, x_1, \ldots, x_n)\colon x_0^2 > x_1^2 + \ldots + x_n^2\}$ has $\sigma_0 = -2/(n+1)$.

Now put

(28) $$\sigma = \max(-1, \sigma_0).$$

Then we can prove the following two lemmas in which it is shown that the operators H and L are applicable to Δ^{α}.

Lemma 4. *Let $\alpha > \sigma$. Then*

(29) $$\int_{\langle 0,x\rangle} \Delta^{\alpha}(t)\,dt = C\Delta^{\alpha+1}(x).$$

Lemma 5. *Let $\alpha > \sigma$. Then*

(30) $$\int_{V} e^{-x^*\cdot t} \Delta^{\alpha}(t)\,dt = C\Delta^{\alpha+1}(x).$$

Proof of Lemma 4. We have the obvious majorization

(31) $$H\Delta^{\alpha}(x) = \int_{\langle 0,x\rangle} \Delta^{\alpha}(t)\,dt \leqslant \int_{|t|<|x|} \Delta^{\alpha}(t)\,dt.$$

Now we can introduce the polar coordinates $r = |t|$, $t' = t/r$. Then

(32) $$\int_{|t|<|x|} \Delta^{\alpha}(t)\,dt = \int_{\Sigma}\int_{0}^{|x|} r^{n-1}\Delta^{\alpha}(rt')\,dr\,dt',$$

and if we make use of the homogeneity of Δ the last integral equals

(33) $$\int_{\Sigma} \Delta^{\alpha}(t')\,dt' \int_{0}^{|x|} r^{n-1} r^{\alpha n}\,dr.$$

The first integral in (33) is convergent by (27) and (28) since $\alpha > \sigma > \sigma_0$ and the second is convergent since $\alpha > \sigma > -1$ (i.e. $\alpha n + n > 0$).

A combination of (31), (32) and (33) proves that $H\Delta^{\alpha}(x)$ is finite for all $x \in V$. An application of Lemma 2 completes the proof of the lemma.

Proof of Lemma 5. We introduce the polar coordinates $r = |t|$, $t' = t/r$ in the integral

$$L\Delta^{\alpha}(x) = \int_{V} e^{-x^*\cdot t}\Delta^{\alpha}(t)\,dt = \int_{\Sigma} dt' \int_{0}^{\infty} e^{-x^*\cdot rt'}\Delta^{\alpha}(rt')\,r^{n-1}\,dr$$

$$= \int_{\Sigma} \Delta^{\alpha}(t') \int_{0}^{\infty} e^{-r(x^*\cdot t')} r^{\alpha n+n-1}\,dr\,dt'.$$

Now if we introduce the change of variables $r(x^*\cdot t') = \varrho$, we obtain

(34) $$L\Delta^{\alpha}(x) = \int_{\Sigma} \Delta^{\alpha}(t')\frac{dt'}{(x^*\cdot t')^{\alpha n+n}} \int_{0}^{\infty} e^{-\varrho}\varrho^{\alpha n+n-1}\,d\varrho.$$

Now the last integral is convergent since $\alpha > -1$ and we only have to prove that the first integral in (34) is also convergent. Let $d(a)$, for $a \in V$, denote the distance of a from the boundary of V. Then

(35) $$a\cdot y' > d(a)$$

for every $y' \in \Sigma$ (see [2]). An application of (35) to (34) yields (since $\alpha n + n > 0$)

$$L\Delta^{\alpha}(x) \leqslant C\frac{1}{d(x^*)^{\alpha n+n}} \int_{\Sigma} \Delta^{\alpha}(t')\,dt'.$$

The last integral is convergent by (27), and $d(x) > 0$ for every $x \in V$ (recall that V is an open cone), so that $L\Delta^{\alpha}(x)$ is finite, and this, by Lemma 2, completes the proof of the lemma.

Now we can easily prove the n-dimensional Hardy inequality, and also a similar inequality for the Laplace transform.

Theorem 2. *Let $1 \leqslant p < \infty$. Let V be a homogeneous selfadjoint cone in $\boldsymbol{R}^n$. Then for $\gamma < -\sigma p - 1$ we have*

$$\int_{V}\Big(\int_{\langle 0,x\rangle} f(y)\,dy\Big)^p \Delta^{\gamma-p}(x)\,dx \leqslant C\int_{V} f^p(x)\Delta^{\gamma}(x)\,dx.$$

Theorem 3. *Let $1 \leqslant p < \infty$. Let V be a homogeneous selfadjoint cone in $\boldsymbol{R}^n$. Then for $\gamma < -\sigma p - 1$ we have*

$$\int_{V} (Lf(x))^p \Delta^{\gamma-p}(x)\,dx \leqslant C\int_{V} f^p(x)\Delta^{\gamma}(x)\,dx.$$

Note that for the cone $\boldsymbol{R}^n_+$ (where $\sigma = -1$) we have $\gamma < p-1$ in Theorems 2 and 3, just as in (H_p).

Proof of Theorem 2. By Lemma 4 we see that the operator H is applicable to Δ^{α} for $\alpha > \sigma$. Also, by some previous remarks, its kernel is V-homogeneous of order 0 and $*$-symmetric. Thus H satisfies all the conditions of Theorem 1 and we have

(36) $$\int_V \Delta^{-\alpha p-1-p}(x)\big(Hf(x)\big)^p\,dx \leqslant C\int_V \Delta^{-\alpha p-1}(x)\,f^p(x)\,dx$$

for $\alpha > \sigma$. Now we only have to put $\gamma = -\alpha p - 1$; then $\gamma < -\sigma p - 1$ and (36) gives the statement of the theorem.

Proof of Theorem 3. The proof follows in a similar way from Lemma 5 and Theorem 1.

Note that we could easily obtain generalizations of Theorems 2 and 3 which would deal with some other operators. For example, the following Lemma 6 is proved along the same lines as Lemma 4, and similarly the proof of Lemma 7 can be obtained by imitating the proof of Lemma 5.

LEMMA 6. *Let $\alpha > \sigma$. Let the kernel k of the operator K be V-homogeneous of order 0 and let there exist a function h: $\mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ such that*

$$k(x, y) \leqslant h(|x|, |y|)$$

and

$$\int_0^\infty r^{\alpha n+n-1}\,h(|x|, r)\,dr < \infty$$

for all $x \in V$. Then

$$K\Delta^\alpha(x) = C\Delta^{\alpha+1}(x).$$

LEMMA 7. *Let $\alpha > \sigma$. Let the kernel of the operator K be of the form $k(x, y) = \varphi(x^* \cdot y)$, where φ is such that*

$$\int_0^\infty \varphi(r)\,r^{\alpha n+n-1}\,dr < \infty.$$

Then

$$K\Delta^\alpha(x) = C\Delta^{\alpha+1}(x).$$

Now it is easy to see that the operators from Lemmas 6 and 7 also satisfy weighted norm inequalities as in Theorems 2 and 3.

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