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Measures on orthomodular vector space lattices

by

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Abstract. The problem of classifying the measures on an orthomodular lattice $\mathscr{L}$ has its source in the lattice-theoretical approach to quantum mechanics. Here we give a solution in the following case: $\mathscr{L} = \mathscr{L}(V)$ is the lattice of all subspaces of a finite-dimensional vector space V over the field $\boldsymbol{R}((t))$ and the orthocomplementation is implemented by a bilinear form $\Psi \simeq \operatorname{diag}(1, \ldots, 1, t, \ldots, t)$. We prove that every measure on $\mathscr{L}(V)$ can be obtained by lifting measures from the residual space of (V, Ψ). The measures being lifted are known by Gleason's theorem. From the classification we deduce, among other things, that the set of all measures on $\mathscr{L}(V)$ is not separating.

Introduction. Let V be a finite-dimensional vector space (over any field) and $\mathscr{L}(V)$ the lattice of all linear subspaces of V. We suppose that V carries an anisotropic Hermitian form Ψ; then the operation $U \mapsto U^{\perp}$ of taking orthogonals turns $\mathscr{L}(V)$ into an orthomodular lattice. A function μ from $\mathscr{L}(V)$ into the nonnegative reals $\boldsymbol{R}^+$ is called a (*finitely additive*) *measure* if $U_1, U_2 \in \mathscr{L}(V)$, $U_1 \perp U_2$ implies that $\mu(U_1 + U_2) = \mu(U_1) + \mu(U_2)$. The task is to describe the set $\mathscr{M}(V)$ of all measures on $\mathscr{L}(V)$. This mathematical problem has its origin in the lattice-theoretical approach to quantum mechanics (cf. [1], [5], [6]).

The case where (V, Ψ) is a real or complex inner product space is classical and is settled by the famous theorem of Gleason. It says that if $V = \boldsymbol{R}^n$ or $V = \boldsymbol{C}^n$, $n \geqslant 3$, then every measure μ: $\mathscr{L}(V) \to \boldsymbol{R}^+$ arises from a selfadjoint positive linear operator $V \to V$ through the trace formula. This result generalizes immediately to infinite-dimensional separable Hilbert spaces. However, Gleason's line of thought fails when the base field is different from $\boldsymbol{R}$, $\boldsymbol{C}$.

In this paper we deal with spaces V over the field $\boldsymbol{R}((t))$ of formal power series, endowed with forms $\Psi \simeq \operatorname{diag}(1, \ldots, 1, t, \ldots, t)$. In § 2 we construct measures on $\mathscr{L}(V)$ by lifting measures from the (first and second) residual space. The latter are known, for the residual spaces are Euclidean. Our main result states that this construction by lifting produces all of $\mathscr{M}(V)$. The proof, given in § 3, is based on an analysis of measures by means of certain geometrical configurations. In § 4 we consider applications, in particular we obtain a description of the $\mu \in \mathscr{M}(V)$ in terms of real matrices.

We mention that the geometrical methods of § 3 are an efficacious tool for studying the (countably additive) measures on infinite-dimensional, nonclassical orthomodular spaces (see [3], [7]). We will not enter into these generalizations here in order to avoid complications.

1. The quadratic space (V, Ψ).

1.1. Let $K := \mathbf{R}((t))$ be the field of formal power series in the indeterminate t with coefficients in $\mathbf{R}$, and let $w\colon K \to \mathbf{Z} \cup \{\infty\}$ be the usual exponential valuation. Thus for a typical $\xi = \sum_{i\in\mathbf{Z}} a_i t^i$ in K we have $w(\xi) = \min\{i \in \mathbf{Z}\colon a_i \neq 0\}$ if $\xi \neq 0$, $w(\xi) = \infty$ if $\xi = 0$. We order K as follows: if $\xi = \sum_i a_i t^i \neq 0$ and $w(\xi) = l$ then $\xi > 0$ iff $a_l > 0$. The ordering $\leqslant$ on K is compatible with the valuation w in the sense that

(1) $$0 \leqslant \xi \leqslant \xi' \Rightarrow w(\xi) \geqslant w(\xi').$$

Let $K^* := \{\xi \in K\colon \xi \neq 0\}$ be the multiplicative group of K and K^{*2} its subgroup of squares. The valued field (K, w) is henselian (cf. [8], Ch. 2); in particular, if $w(\xi) > 0$ then $1+\xi \in K^{*2}$. It follows that K^* has four classes modulo K^{*2}, represented by ± 1, $\pm t$.

1.2. We fix two integers n, m with $n \geqslant 3$, $1 \leqslant m \leqslant n-1$. Consider the vector space $V := K^n$ over K and let $\{e_1, \ldots, e_n\}$ be the canonical basis. We define the symmetric bilinear form $\Psi\colon V \times V \to K$ by

$$\Psi(e_i, e_i) := 1 \quad \text{for } 1 \leqslant i \leqslant m, \qquad \Psi(e_i, e_i) := t \quad \text{for } m+1 \leqslant i \leqslant n,$$
$$\Psi(e_i, e_j) := 0 \quad \text{for } i \neq j.$$

The form Ψ is positive-definite. We say that $x, y \in V$ are *orthogonal*, written $x \perp y$, if $\Psi(x, y) = 0$, and for $U \subseteq V$ we let $U^\perp$ denote the orthogonal space of U, $U^\perp = \{x \in V\colon x \perp u \text{ for all } u \in U\}$. The set

$$\mathscr{L}(V) := \{U\colon U \text{ is a linear subspace of } V\}$$

is a modular lattice under $\cap$, $+$. Since Ψ is anisotropic the operation $U \mapsto U^\perp$ is an orthocomplementation in $\mathscr{L}(V)$. The lattice $\mathscr{L}(V)$ is atomistic, the set of atoms is $\mathscr{G}(V) := \{G \in \mathscr{L}(V)\colon \dim G = 1\}$.

For $x \in V$, $x \neq 0$, we let $\langle x \rangle$ be the straight line spanned by x.

1.3. Let x be a nonzero vector in V. The *type* of x, denoted by $T(x)$, is defined to be

$$T(x) := \begin{cases} 0 & \text{if } w(\Psi(x, x)) \text{ is even,} \\ 1 & \text{if } w(\Psi(x, x)) \text{ is odd.} \end{cases}$$

Clearly $T(x) = T(\xi x)$ for all $0 \neq \xi \in K$. Hence there is a type attached to every straight line G; we denote it by $T(G)$.

Remark. The above terminology is suggested by a much more general and elaborate concept of "type of a nonzero vector", which plays a crucial role in the construction of nonclassical infinite-dimensional orthomodular spaces. We refer to [3].

Since Ψ is positive we have $\Psi(x, x) \in K^{*2} \cup t\cdot K^{*2}$ for $0 \neq x \in V$. It is plain that

(2) $$T(x) = 0 \Leftrightarrow \Psi(x, x) \in K^{*2}, \qquad T(x) = 1 \Leftrightarrow \Psi(x, x) \in t\cdot K^{*2}.$$

Hence a straight line G has type 0 [type 1] if and only if G contains a vector x with $\Psi(x, x) = 1$ [resp. $\Psi(x, x) = t$].

1.4. Our constructions of measures will be based on the (first and second) residual space of (V, Ψ) (cf. [9]). We need

LEMMA 1. *For all $x, y \in V$ we have*

(i) $w(\Psi(x+y, x+y)) \geqslant \min\{w(\Psi(x, x)), w(\Psi(y, y))\}$.
(ii) $w(\Psi(x, y)) \geqslant \min\{w(\Psi(x, x)), w(\Psi(y, y))\}$.
(iii) $x \perp y \Rightarrow w(\Psi(x+y, x+y)) = \min\{w(\Psi(x, x)), w(\Psi(y, y))\}$.

Proof. This follows from (1) and the fact that Ψ is positive.

The valuation ring belonging to w is $S = \{\xi \in K\colon w(\xi) \geqslant 0\}$. Let J be the maximal ideal of S, $\hat{K} := S/J$ the residue field and $\theta\colon S \to \hat{K}$ the canonical map. Of course, $\hat{K} \cong \mathbf{R}$ and we shall identify $\hat{K}$ with $\mathbf{R}$. Consider $M := \{x \in V\colon \Psi(x, x) \in S\}$. It follows from Lemma 1 that M is a module over S and that $\Psi(x, y) \in S$ whenever $x, y \in M$. Furthermore, $N := \{x \in V\colon \Psi(x, x) \in J\}$ is a submodule of M. Since $N \supseteq J\cdot M$ the quotient $\hat{V} := M/N$ is a vector space over $\hat{K}$. Let $\pi\colon M \to \hat{V}$ be the canonical map. Using Lemma 1 one verifies that a bilinear form $\hat{\Psi}\colon \hat{V} \times \hat{V} \to \hat{K}$ can be defined by

$$\hat{\Psi}(\pi(x), \pi(y)) := \theta(\Psi(x, y)) \qquad (x, y \in M).$$

$(\hat{V}, \hat{\Psi})$ is called the (*first*) *residual space* of (V, Ψ). There is a canonical map $\pi\colon \mathscr{L}(V) \to \mathscr{L}(\hat{V})$, given by $U \mapsto \hat{U} := \{\pi(x)\colon x \in U \cap M\}$.

LEMMA 2. (i) *Let $U_1, U_2 \in \mathscr{L}(V)$. If $U_1 \perp U_2$ then $\pi(U_1) \perp \pi(U_2)$ and $\pi(U_1+U_2) = \pi(U_1)+\pi(U_2)$.*

(ii) *Let G be a straight line. Then $\pi(G) = \hat{0}$ if and only if $T(G) = 1$.*

Proof. (i) Suppose $U_1 \perp U_2$. Then $\pi(U_1) \perp \pi(U_2)$ by the definition of $\hat{\Psi}$. Clearly $\pi(U_1)+\pi(U_2) \subseteq \pi(U_1+U_2)$. To prove the other inclusion let $x \in (U_1+U_2) \cap M$, $x = x_1+x_2$ with $x_1 \in U_1$, $x_2 \in U_2$. Then

$$0 \leqslant w(\Psi(x, x)) = \min\{w(\Psi(x_1, x_1)), w(\Psi(x_2, x_2))\}$$

by Lemma 1(iii). Thus $x_1, x_2 \in M$, hence $\pi(x) = \pi(x_1)+\pi(x_2) \in \pi(U_1)+\pi(U_2)$ as desired. Claim (ii) follows from (2).

COROLLARY 1. *$(\hat{V}, \hat{\Psi})$ is isometric to the Euclidean space $\mathbf{R}^m$.*

Proof. Lemma 2 entails that $\{\pi(e_1), \ldots, \pi(e_m)\}$ is an orthonormal basis of $\hat{V}$.

Corollary 2. *Let $U \in \mathscr{L}(V)$. All orthogonal bases of U contain the same number of vectors of type* 0.

Proof. The number in question is equal to $\dim \pi(U)$.

2. Measures on $\mathscr{L}(V)$. We continue with the notation of § 1.

2.1. Recall that a measure on $\mathscr{L}(V)$ is a function μ: $\mathscr{L}(V) \to \boldsymbol{R}^+$ with the property that $U_1 \perp U_2$ implies $\mu(U_1+U_2) = \mu(U_1)+\mu(U_2)$. If, additionally, $\mu(V) = 1$ then μ is called a *probability measure*. The collection of all probability measures on $\mathscr{L}(V)$, denoted by $\mathscr{P}(V)$, is a convex subset of the product space $\boldsymbol{R}^{\mathscr{L}(V)}$. The set of all measures on $\mathscr{L}(V)$, denoted by $\mathscr{M}(V)$, is the cone in $\boldsymbol{R}^{\mathscr{L}(V)}$ generated by $\mathscr{P}(V)$.

Every $\mu \in \mathscr{M}(V)$ is uniquely determined by its values on the straight lines $G \in \mathscr{G}(V)$. We observe that given $a_0, a_1 \in \boldsymbol{R}^+$ there exists a $\mu \in \mathscr{M}(V)$ such that for all $G \in \mathscr{G}(V)$,

$$\mu(G) = a_0 \quad \text{if } T(G) = 0, \qquad \mu(G) = a_1 \quad \text{if } T(G) = 1.$$

Indeed, define μ: $\mathscr{L}(V) \to \boldsymbol{R}^+$ by

$$\mu(U) := (\dim \pi(U)) \cdot a_0 + (\dim U - \dim \pi(U)) \cdot a_1.$$

Then μ is a measure by Lemma 2(i).

2.2. The main tool for constructing measures on $\mathscr{L}(V)$ is given by

Lemma 3. *Every measure ν: $\mathscr{L}(\hat{V}) \to \boldsymbol{R}^+$ can be lifted to a measure μ: $\mathscr{L}(V) \to \boldsymbol{R}^+$ by putting $\mu(U) := \nu(\pi(U))$ $(U \in \mathscr{L}(V))$.*

Proof. This follows immediately from Lemma 2(i).

Since $\hat{V} \cong \boldsymbol{R}^m$ (by Cor. 1) the measures ν on $\mathscr{L}(\hat{V})$ are easily described:

(a) The case $m = 1$ is trivial.

(b) Suppose $m = 2$. The set of all straight lines in $\mathscr{L}(\hat{V})$ is partitioned into orthogonal pairs, and measures on $\mathscr{L}(\hat{V})$ are simply functions ν: $\mathscr{L}(\hat{V}) \to \boldsymbol{R}^+$ such that $\nu(\hat{0}) = 0$, $\nu(\hat{G})+\nu(\hat{G}') = \nu(\hat{V})$ for each of these orthogonal pairs $\hat{G}$, $\hat{G}'$.

(c) In case $m \geq 3$ we have

Theorem (Gleason [2]). *Let $m \geq 3$. For every measure ν: $\mathscr{L}(\boldsymbol{R}^m) \to \boldsymbol{R}^+$ there is a selfadjoint positive linear operator Q: $\boldsymbol{R}^m \to \boldsymbol{R}^m$ such that*

$$\nu(U) = \operatorname{trace}(Q \circ P_U) = \operatorname{trace}(P_U \circ Q)$$

for all $U \in \mathscr{L}(\boldsymbol{R}^m)$, where P_U is the orthogonal projection of $\boldsymbol{R}^m$ onto U.

We put $\mathscr{M}_0(V) := \{\mu \in \mathscr{M}(V)$: μ is obtained by lifting some $\nu \in \mathscr{M}(\hat{V})\}$.

2.3. The above construction can be varied as follows. Replace Ψ by $\Psi_1 := (1/t) \cdot \Psi$. This does not affect the relation of orthogonality, hence the ortholattice $\mathscr{L}(V)$ remains unchanged. But we obtain a new residual space and a new assignment of types. Let $(\hat{V}_1, \hat{\Psi}_1)$ be the residual space of (V, Ψ_1) and write $T_1(x)$ for the type of x with respect to Ψ_1. From (2) we infer that $T_1(x) = 0$ iff $T(x) = 1$; consequently, by the analogue of Lemma 2, $(\hat{V}_1, \hat{\Psi}_1)$ is isometric to $\boldsymbol{R}^{n-m}$. Again every measure ν: $\mathscr{L}(\hat{V}_1) \to \boldsymbol{R}^+$ can be lifted, and we set

$$\mathscr{M}_1(V) := \{\mu \in \mathscr{M}(V)\colon \mu \text{ is obtained by lifting some } \nu \in \mathscr{M}(\hat{V}_1)\}.$$

2.4. We can now state our main result.

Theorem 1. *Let (V, Ψ) be as in § 1. Every measure μ: $\mathscr{L}(V) \to \boldsymbol{R}^+$ can be written uniquely as $\mu = \mu_0+\mu_1$ where $\mu_0 \in \mathscr{M}_0(V)$, $\mu_1 \in \mathscr{M}_1(V)$.*

We first observe that the decomposition $\mu = \mu_0+\mu_1$ is unique because every $\mu_0 \in \mathscr{M}_0(V)$ is identically zero on $\{G \in \mathscr{G}(V)$: $T(G) = 1\}$, and similarly for $\mu_1 \in \mathscr{M}_1(V)$.

3. Proof of Theorem 1.

3.1. We may assume that $m \geq 2$, for otherwise we replace (V, Ψ) by (V, Ψ_1). We consider an arbitrary, but fixed 3-dimensional subspace W of V with $\dim \pi(W) = 2$. Let $\mathscr{G}(W) := \{G \in \mathscr{L}(W)$: $\dim G = 1\}$. Our aim is to first establish

Theorem 2. *Every measure μ: $\mathscr{L}(W) \to \boldsymbol{R}^+$ is constant on $\{G \in \mathscr{G}(W)$: $T(G) = 1\}$.*

The proof of Th. 2 is divided into several steps; it will cover the next five sections.

A triple (G_1, G_2, G_3) of pairwise orthogonal straight lines in $\mathscr{G}(W)$ will be called a *frame*. Every frame contains exactly two members of type 0 (cf. Cor. 2).

3.2. Suppose we are given two straight lines G, H in $\mathscr{G}(W)$ such that

$$T(G) = 1, \qquad T(H) = 0, \qquad G \text{ is not orthogonal to } H.$$

Let $F, E \in \mathscr{G}(W)$ be determined by $F \perp G$, $F \perp H$ and $E \perp G$, $E \perp F$. Notice that $T(E) = T(F) = 0$. Let $\mathscr{F}$ be the set of all frames (A, B, C) which satisfy the conditions

(i) $A \perp G$, $B \perp H$,

(ii) $A \neq E$, $A \neq F$.

We ask: for which $(A, B, C) \in \mathscr{F}$ does there exist a $(A^*, B^*, C^*) \in \mathscr{F}$ such that $C \perp C^*$? In other words, we are looking for configurations made up by two frames (A, B, C), (A^*, B^*, C^*) in $\mathscr{F}$ linked by the condition that $C \perp C^*$.

Pick $e \in E$, $f \in F$, $g \in G$ with $\Psi(e, e) = \Psi(f, f) = 1$, $\Psi(g, g) = t$ (cf. 1.3). It is helpful to introduce the "translated plane" $\Sigma := \{e + \xi f + \eta g \colon \xi, \eta \in K\}$ and to project from the origin onto Σ. We indicate orthogonality between lines by connecting the corresponding points in Σ and get a picture like this:

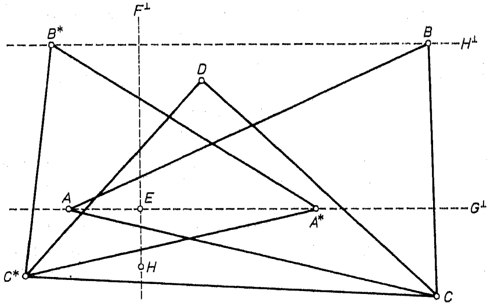

Fig. 1. A typical configuration

Here the straight line D with $D \perp C$, $D \perp C^*$ has been added for reference later on.

Lemma 4. *Let $(A, B, C) \in \mathscr{F}$ and let $A = \langle e + \alpha f \rangle$ where $0 \neq \alpha \in K$. For the existence of a frame $(A^*, B^*, C^*) \in \mathscr{F}$ with $C \perp C^*$ it is sufficient that*

$$w(\alpha) = 0. \tag{3}$$

Proof. There is a unique $\eta \in K$ such that $K = \langle e + \eta g \rangle$. We have $\eta \neq 0$ as $H \not\perp G$, and the hypothesis $T(H) = 0$ implies

$$w(\eta^2 t) > 0. \tag{4}$$

Consider any frame (A^*, B^*, C^*) in $\mathscr{F}$. Then $A^* = \langle e + \alpha^* f \rangle$ for some $0 \neq \alpha^* \in K$. A routine computation shows that C and C^* are spanned by

$$c = e - \alpha^{-1} f - (1 + \alpha^{-2}) \eta g, \qquad c^* = e - \alpha^{*^{-1}} f - (1 + \alpha^{*^{-2}}) \eta g$$

respectively. The requirement that $c \perp c^*$ is equivalent to $1 + \alpha^{-1} \alpha^{*^{-1}} + (1 + \alpha^{-2})(1 + \alpha^{*^{-2}}) \eta^2 t = 0$, or

$$(\alpha^2 + (1 + \alpha^2) \eta^2 t) \cdot \alpha^{*^2} + \alpha \cdot \alpha^* + (1 + \alpha^2) \eta^2 t = 0. \tag{5}$$

This is an equation of second degree in α^* with discriminant

$$\Delta = \alpha^2 (1 - 4\varrho) \quad \text{where } \varrho = (1 + \alpha^2)(1 + (1 + \alpha^{-2}) \eta^2 t) \eta^2 t.$$

Using (3), (4) we see that $w(\varrho) > 0$. Hence Δ is a square in K, so equation (5) has a solution. The lemma is proved.

We put $\mathscr{A} := \{A = \langle e + \alpha f \rangle \colon \alpha \text{ satisfies (3)}\}$.

3.3. Remarks. (i) Let $A \in \mathscr{A}$. Let $A' \in \mathscr{G}(W)$ be such that $A' \perp G$, $A' \perp A$. Then $A' \in \mathscr{A}$. For if $A = \langle e + \alpha f \rangle$ then $A' = \langle e - \alpha^{-1} f \rangle$ and $w(-\alpha^{-1}) = -w(\alpha) = 0$.

(ii) Let $(A, B, C) \in \mathscr{F}$ with $A \in \mathscr{A}$. Note that $T(A) = 0$. Using (3), (4) we see that $c = e - \alpha^{-1} f - (1 + \alpha^{-2}) \eta g$ has type 0. Thus $T(C) = 0$ and consequently $T(B) = 1$.

(iii) It is readily verified that for one of the solutions of eq. (5), say α_1^*, we have $w(\alpha_1^*) = w(-\alpha^{-1})$. Thus α_1^* satisfies (3). We have therefore shown that every $A \in \mathscr{A}$ gives rise to a configuration consisting of (A, B, C), $(A^*, B^*, C^*) \in \mathscr{F}$ with $C \perp C^*$ such that, additionally, A^* is also in $\mathscr{A}$.

3.4. For a measure μ on $\mathscr{L}(W)$ we put

$$s_j(\mu) := \sup\{\mu(L) \colon L \in \mathscr{G}(W),\ T(L) = j\} \qquad (j = 0, 1)$$
$$r_j(\mu) := \inf\{\mu(L) \colon L \in \mathscr{G}(W),\ T(L) = j\} \qquad (j = 0, 1).$$

The claim of Th. 2 is that $s_1(\mu) = r_1(\mu)$ for all $\mu \in \mathscr{M}(W)$. We first prove

Lemma 5. *Let μ be a measure on $\mathscr{L}(W)$ and suppose that $r_0(\mu) = r_1(\mu) = 0$. Then $s_1(\mu) = 2(\mu(W) - s_0(\mu))$.*

Proof. We write $s_j = s_j(\mu)$, $j = 0, 1$.

(a) To show that $s_1 \geq 2(\mu(W) - s_0)$ let $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$. By hypothesis there are $G, H \in \mathscr{G}(W)$ with $T(G) = 1$, $T(H) = 0$, such that

$$\mu(G) < \varepsilon, \qquad \mu(H) < \varepsilon.$$

Suppose $G \perp H$. We complete G, H to a frame (G, H, L) and note that $T(L) = 0$, thus $s_0 \geq \mu(L) = \mu(W) - \mu(G) - \mu(H) > \mu(W) - 2\varepsilon$ and therefore $s_1 \geq 0 > 2(\mu(W) - s_0) - 4\varepsilon$.

Now suppose that G and H are not orthogonal. Let $\mathscr{F}$, $\mathscr{A}$ be defined as in 3.2. Put

$$q := \sup\{\mu(A) \colon A \in \mathscr{A}\}$$

and pick $A \in \mathscr{A}$ such that

$$\mu(A) > q - \varepsilon. \tag{6}$$

The line A gives rise to a configuration (A, B, C), $(A^*, B^*, C^*) \in \mathscr{F}$ with $C \perp C^*$; by 3.3(iii) we may assume that $A^* \in \mathscr{A}$. Let $D \in \mathscr{G}(W)$ be such that

$D \perp C$, $D \perp C^*$ (see Fig. 1). Now

$$\mu(W) = \mu(A)+\mu(B)+\mu(C) = \mu(A^*)+\mu(B^*)+\mu(C^*) = \mu(C)+\mu(C^*)+\mu(D),$$

from which

(7) $$\mu(D) = \mu(A)+\mu(A^*)+\mu(B)+\mu(B^*)-\mu(W).$$

We have $T(D) = 1$ since $T(C) = T(C^*) = 0$ (cf. 3.3(ii)), hence

(8) $$s_1 \geqslant \mu(D).$$

By 3.3(i) the line A' with $A' \perp G$, $A' \perp A^*$ is in $\mathscr{A}$, so $\mu(A') \leqslant q$. Since (G, A^*, A') is a frame we obtain

(9) $$\mu(A^*) > \mu(W)-q-\varepsilon.$$

Since $B \perp H$, $B^* \perp H$ there are frames (B, H, L) and (B^*, H, L^*). Here $T(L) = 0 = T(L^*)$ as $T(B) = T(B^*) = 1$ (cf. 3.3(ii)). Thus $\mu(L), \mu(L^*) < s_0$ and therefore

(10) $$\mu(B) > \mu(W)-s_0-\varepsilon, \quad \mu(B^*) > \mu(W)-s_0-\varepsilon.$$

Substituting (6), (8), (9), (10) into (7) we get $s_1 \geqslant 2(\mu(W)-s_0)-5\varepsilon$. We conclude that $s_1 \geqslant 2(\mu(W)-s_0)$.

(b) The reverse inequality is proved in a similar way. This time one starts with lines G, H satisfying $\mu(G) > s_1-\varepsilon$, $\mu(H) > s_0-\varepsilon$.

3.5. Let G be any line in $\mathscr{G}(W)$ with $T(G) = 1$. Pick $0 \neq g \in G$ and extend $\{g\}$ to an orthogonal basis $\{g, x, y\}$ of W with $\Psi(x, x) = \Psi(y, y) = 1$. The linear transformation $W \to W$ defined by $g \mapsto g$, $x \mapsto y$, $y \mapsto -x$ is an isometry ("rotation with axis G through a right angle") and therefore it induces an orthoautomorphism $\Omega\colon \mathscr{L}(W) \to \mathscr{L}(W)$ with the properties

(11) $$\Omega(G) = G,$$

(12) $$L \perp G \Rightarrow \Omega(L) \perp L \quad \text{for all } L \in \mathscr{G}(W).$$

Note also that Ω does not change types.

3.6. We are now ready to prove Th. 2.

(a) Suppose, indirectly, that there exists a measure μ^* on $\mathscr{L}(W)$ with $s_1(\mu^*) > r_1(\mu^*)$. Put

$$\varepsilon := (s_1(\mu^*)-r_1(\mu^*))/26$$

and pick $G \in \mathscr{G}(W)$ with $T(G) = 1$ such that

$$\mu^*(G) > s_1(\mu^*)-\varepsilon.$$

Let Ω be the automorphism of $\mathscr{L}(W)$ induced by the rotation with axis G through an angle 90° (cf. 3.5). Define $\mu' \in \mathscr{M}(W)$ by

$$\mu'(U) := \tfrac{1}{2}\big(\mu^*(U)+\mu^*(\Omega(U))\big) \quad (U \in \mathscr{L}(W)).$$

According to the observation in 2.1 there exists a $\mu'' \in \mathscr{M}(W)$ such that, for all $L \in \mathscr{G}(W)$, $\mu''(L) = r_j(\mu')$ if $T(L) = j$ $(j = 0, 1)$. Clearly $\mu := \mu'-\mu''$ is a measure. Using (11), (12) one verifies that μ has these properties:

(13) $$r_0(\mu) = r_1(\mu) = 0,$$

(14) $$\mu \text{ is constant on } \{L \in \mathscr{G}(W)\colon L \perp G\},$$

(15) $$\mu(G) > s_1(\mu)-\varepsilon,$$

(16) $$s_1(\mu) > 12\varepsilon.$$

From now on we write s_j instead of $s_j(\mu)$, $j = 0, 1$. In view of (13) we have $s_1 = 2(\mu(W)-s_0)$ by Lemma 5.

(b) Pick $H \in \mathscr{G}(W)$ with $T(H) = 0$ such that

$$\mu(H) > s_0-\varepsilon.$$

If we had $H \perp G$ then (14) would entail $\mu(W) = \mu(G)+\mu(H)+\mu(H)$, thus $\mu(W) > s_1+2s_0-3\varepsilon$. But $s_1 = 2(\mu(W)-s_0)$, hence $\mu(W) < 3\varepsilon$ which is impossible because of (16).

Therefore G and H are not orthogonal and we are in the situation of 3.2. Consider any configuration $(A, B, C), (A^*, B^*, C^*) \in \mathscr{F}$ with $C \perp C^*$ and $A, A^* \in \mathscr{A}$. Let $D \perp C$, $D \perp C^*$ (see Fig. 1). Then, as in 3.4, we have

(7) $$\mu(D) = \mu(A)+\mu(A^*)+\mu(B)+\mu(B^*)-\mu(W).$$

Since $B \perp H$, $B^* \perp H$ and $\mu(H) > s_0-\varepsilon$ we have

(17) $$\mu(B) < \mu(W)-s_0+\varepsilon, \quad \mu(B^*) < \mu(W)-s_0+\varepsilon.$$

Next, from (14) we infer $\mu(A) = \mu(A^*) = \tfrac{1}{2}(\mu(W)-\mu(G))$, thus

(18) $$\mu(A)+\mu(A^*) < \mu(W)-s_1+\varepsilon.$$

Combining (17), (18) and (7) and recalling that $s_1 = 2(\mu(W)-s_0)$ we get the inequalities

(19) $$\mu(D) < 3\varepsilon,$$

(20) $$\mu(B) > \mu(W)-s_0-2\varepsilon.$$

(c) We determine successively L, M, B' in $\mathscr{G}(W)$ by $L \perp H$, $L \perp D$ and $M \perp L$, $M \perp D$ and $B' \perp L$, $B' \perp H$. The line B' belongs to a unique frame (A', B', C') in $\mathscr{F}$ (see Fig. 2; to obtain a full picture of the situation one should superpose Fig. 1 and Fig. 2).

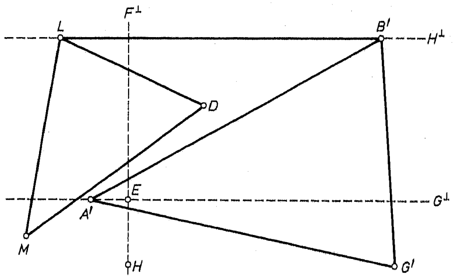

Fig. 2

It is not difficult to verify that the configuration (A, B, C), (A^*, B^*, C^*) from which we started can be so chosen that the resulting line A' falls into $\mathscr{A}$. Indeed, if $A = \langle e+\alpha f\rangle$ then a straightforward (but somewhat lengthy) computation shows that $A'\in\mathscr{A}$ provided that $w(\alpha-1) = 0 = w(\alpha+1)$. We therefore assume that $A'\in\mathscr{A}$. Note that in order to derive (20) we only used the fact that A is in $\mathscr{A}$. Hence, correspondingly, $\mu(B') > \mu(W)-s_0-2\varepsilon$ holds true. Looking at the frame (H, L, B') we get

(21) $$\mu(L) < 3\varepsilon.$$

Substituting (19), (21) into $\mu(W) = \mu(D)+\mu(L)+\mu(M)$ yields

(22) $$\mu(M) > \mu(W)-6\varepsilon.$$

Now $T(D) = 1$, so $T(M) = 0$. Thus (22) implies that $s_0 > \mu(W)-6\varepsilon$. Hence $s_1 = 2(\mu(W)-s_0) < 12\varepsilon$. But this contradicts (16). The proof of Th. 2 is complete.

3.7. We are going to derive Th. 1 from Th. 2.

COROLLARY 3. *Let $P\in\mathscr{L}(V)$ be a plane that contains vectors of both types. Every measure μ: $\mathscr{L}(V)\to \mathbf{R}^+$ is constant on $\{G\in\mathscr{G}(V)$: $G\subset P$, $T(G) = 1\}$, hence also on $\{G\in\mathscr{G}(V)$: $G\subset P$, $T(G) = 0\}$.*

Proof. P is contained in a subspace W of V with the specifications of 3.1. Apply Th. 2 to the restriction $\mu|\mathscr{L}(W)$.

COROLLARY 4. *If $n-m = 1$ then every $\mu\in\mathscr{M}(V)$ is constant on $\{G\in\mathscr{G}(V)$: $T(G) = 1\}$.*

We are now able to establish a connection between measures $\mu\in\mathscr{M}(V)$ and the reduction map π: $\mathscr{L}(V)\to\mathscr{L}(\hat{V})$.

LEMMA 6. *Let $\mu\in\mathscr{M}(V)$. Let $H, H'\in\mathscr{G}(V)$, $T(H) = T(H') = 0$. If $\pi(H) = \pi(H')$ then $\mu(H) = \mu(H')$.*

Proof. We may assume that $H\neq H'$ and that the plane $P := H+H'$ contains only vectors of type 0, for otherwise the assertion follows from Cor. 3. Let F be the line in P orthogonal to H. We choose a $G\in\mathscr{G}(V)$ such that $G\perp P$ and $T(G) = 1$. Pick $h\in H$, $f\in F$, $g\in G$ with $\Psi(h, h) = \Psi(f, f) = 1$, $\Psi(g, g) = t$. We have $H' = \langle h+\eta f\rangle$ for some $\eta\in K$. The hypothesis $\pi(H) = \pi(H')$ implies that $w(\eta) > 0$, so $w(\eta/t)\geqslant 0$. Consequently, the lines $E := \langle h+\eta f+g\rangle$ and $L := \langle f-(\eta/t)g\rangle$ have type 0. Clearly $E\perp L$. From Theorem 2, applied to $W := P+G$ and $\mu|\mathscr{L}(W)$, we infer that

$$\mu(E)+\mu(L) = \mu(H)+\mu(F).$$

Both the planes $P_1 := H'+E$ and $P_2 := F+L$ satisfy the hypothesis of Cor. 3, hence $\mu(E) = \mu(H')$, $\mu(L) = \mu(F)$, and the assertion follows.

COROLLARY 5. *Let $\mu\in\mathscr{M}(V)$. Let U, $U'\in\mathscr{L}(V)$ and suppose that U, U' contain only vectors of type 0. If $\pi(U) = \pi(U')$ then $\mu(U) = \mu(U')$.*

Proof. By induction on $k := \dim U$ $(= \dim U')$ it is easily shown that we can write $U = H_1+\ldots+H_k$, $U' = H'_1+\ldots+H'_k$ where $H_i\perp H_j$, $H'_i\perp H'_j$ for $1\leqslant i<j\leqslant k$ and $\pi(H_i) = \pi(H'_i)$ for $1\leqslant i\leqslant k$. The claim then follows from Lemma 6.

3.8. Consider now any measure μ: $\mathscr{L}(V)\to\mathbf{R}^+$. We define a function ν: $\mathscr{L}(\hat{V})\to\mathbf{R}^+$ as follows. Every element of $\mathscr{L}(\hat{V})$ can be written in the form $\pi(U)$ for some $U\in\mathscr{L}(V)$ which contains only vectors of type 0; we put $\nu(\pi(U)) := \mu(U)$. Corollary 5 ensures that ν is well defined. It is easily checked that ν is a measure. Let $\mu_0\in\mathscr{M}_0(V)$ be obtained by lifting ν. By construction, μ_0 coincides with μ on $\{G\in\mathscr{G}(V)$: $T(G) = 0\}$ and μ_0 is identically zero on $\{G\in\mathscr{G}(V)$: $T(G) = 1\}$. It is obvious that we can also find a $\mu_1\in\mathscr{M}_1(V)$ such that μ_1 coincides with μ on $\{G$: $T_1(G) = 0\} = \{G$: $T(G) = 1\}$. In fact, in case $n-m\geqslant 2$ we repeat the above arguments with (V, Ψ_1) in place of (V, Ψ); if $n-m = 1$ then μ is constant on $\{G$: $T(G) = 1\}$ (by Cor. 4) and the claim is trivial. Now μ and $\mu_0+\mu_1$ take the same values on all $G\in\mathscr{G}(V)$. We conclude that $\mu = \mu_0+\mu_1$. The proof of Th. 1 is complete.

4. Some applications. We retain the notation of §§ 1, 2.

4.1. We begin with a remark on the set $\mathscr{P}(V)$ of probability measures on $\mathscr{L}(V)$. Put $\mathscr{P}_j(V) := \mathscr{P}(V)\cap\mathscr{M}_j(V)$, $j = 0, 1$. Theorem 1 implies that every $\mu\in\mathscr{P}(V)$ can be expressed uniquely as $\mu = a_0\mu_0+a_1\mu_1$ where $\mu_0\in\mathscr{M}_0(V)$, $\mu_1\in\mathscr{M}_1(V)$ and a_0, $a_1\in\mathbf{R}^+$, $a_0+a_1 = 1$. We conclude that $\mathscr{P}_0(V)$ and $\mathscr{P}_1(V)$ are faces of the convex set $\mathscr{P}(V)$.

4.2. As a consequence of Th. 1 we have the following rather unexpected result.

LEMMA 7. *The set of all measures on $\mathscr{L}(V)$ is not separating* (cf. [1], p. 116).

Proof. Take any two different straight lines H, H' with $T(H) = T(H') = 0$, $\pi(H) = \pi(H')$. By construction, $\mu_0(H) = \mu_0(H')$ and $\mu_1(H) = 0 = \mu_1(H')$ for all $\mu_0 \in \mathcal{M}_0(V)$, $\mu_1 \in \mathcal{M}_1(V)$. Now Th. 1 implies that $\mu(H) = \mu(H')$ for all $\mu \in \mathcal{M}(V)$.

COROLLARY 6. *Let E be any real or complex inner product space. The lattice $\mathcal{L}(V)$ cannot be orthoisomorphically embedded into $\mathcal{L}(E)$.*

Proof. The set of all measures on $\mathcal{L}(E)$ is separating, but $\mathcal{M}(V)$ is not. Hence the assertion.

This corollary illustrates that an insight into measures can provide an answer to purely lattice-theoretical problems (compare Problem C in [4]).

4.3. We now assume that $m \geqslant 3$ and $n-m \geqslant 3$. We write $\hat{V}_0$ instead of $\hat{V}$. Consider an arbitrary $\mu \in \mathcal{M}(V)$ and decompose $\mu = \mu_0 + \mu_1$ where μ_j results from lifting a measure $\nu_j \in \mathcal{M}(\hat{V}_j)$, $j = 0, 1$. Recall that $\hat{V}_0 \cong \boldsymbol{R}^m$ and $\hat{V}_1 \cong \boldsymbol{R}^{n-m}$. By Gleason's theorem, ν_j arises from a selfadjoint positive operator $Q_j\colon \hat{V}_j \to \hat{V}_j$. Let A_0 be the matrix of Q_0 with respect to the orthonormal basis $\{\pi(e_1), \ldots, \pi(e_m)\}$ of $\hat{V}_0$ and define the matrix A_1 similarly. We then associate μ with the matrix

$$(*) \qquad A = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}$$

where A_0, A_1 are real symmetric positive matrices of size $m \times m$ and $(n-m) \times (n-m)$ respectively. The correspondence $\mu \mapsto A$ between $\mathcal{M}(V)$ and the set of all matrices of shape $(*)$ is bijective and compatible with convex linear combinations.

Denote the entries of A by a_{ij} $(i, j = 1, \ldots, n)$ and define the linear operator $Q\colon V \to V$ by $Q(e_j) := \sum_{i=1}^{n} a_{ij} e_i$. It is then easy to check that μ and Q are interrelated by the formula

$$\mu(U) = \theta\big(\mathrm{trace}(Q \circ P_U)\big) \qquad (U \in \mathcal{L}(V)),$$

where P_U is the orthogonal projection onto U and $\theta\colon S \to \hat{K} \cong \boldsymbol{R}$ the canonical map. We have thereby reached a description of the measures on $\mathcal{L}(V)$ which is remarkably resembling the classical case.

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