The centralizer of Morse shifts induced by arbitrary blocks

by

JAN KWIATKOWSKI and TADEUSZ ROJEK (Toruń)

Abstract. The centralizer $C(T)$ of the Morse shift given by a continuous Morse sequence $x = b^1 \times b^2 \times \ldots$ over $Z_2$ is described. Let $|v| = \lambda$, $v \neq 0$, and let $G$ be the set of $\lambda$-adic integers, \( \lambda \neq (\lambda)^2 \). $C(T)$ can be identified with a set of pairs $(g_0, \varphi)$, where $g_0 \in G$ and $\varphi$ is a measurable function from $G$ to $Z_2$. The set $G_0$ satisfies the following properties:

(a) $m(G_0) = 0$, where $m$ is the normalized Haar measure on $G$.
(b) Either $G_0 = \emptyset$ or $G_0$ is uncountable. $T_\alpha$ is rigid if $G_0$ is uncountable.
(c) There exists a Morse sequence $x$ such that $T_\alpha$ is rigid and $C(T) \neq \emptyset$.

Introduction. There are some reasons to investigate the centralizers of Morse shifts. The Morse shift $(\Omega_{\alpha}, T, \mu_{\alpha})$ is metrically isomorphic to a $Z_2$-extension of the group of $\lambda$-adic integers with the Haar measure $m$ and the translation by 1. Newton's results [5] imply that each automorphism $S$ commuting with $T$ can be identified with $(g_0, \varphi)$, where $g_0 \in G$ and $\varphi$ is a measurable function from $G$ to $Z_2$. The transformation $S$ is an extension of the translation of $G$ by $g_0$ to an automorphism of $C(T)$.

Łemańczyk [4] has proved that the centralizer of the Morse shift $(\Omega_{\alpha}, T, \mu_{\alpha})$ is countable but not trivial assuming that $x$ is a regular Morse sequence and the sequence $|\|v\|\|$ is bounded. In this case $G_0$ coincides with the set of all $\lambda$-adic rational integers. On the other hand, Łemańczyk described some class of Morse shifts having the property to be rigid. The centralizer of such systems is uncountable. The question arose what are $G_0$ and $\varphi$ for an arbitrary Morse shift. The next reason for investigating the commutant of Morse shifts is connected with Walter's question in [7]. He asked whether there is a rigid automorphism $T$ with simple spectrum such that the commutant of $T$ is not the closure of the powers of $T$ in the weak topology. A supposition arose that an example could be found in the class of Morse shifts. In this paper we confirm that supposition.

§ 1. Centralizer of Morse shifts. We first give the necessary preliminaries. For a more complete treatment, the reader is referred to [1], [2], [4], [6].

Let $x = b^1 \times b^2 \times \ldots$ be a 0-1 Morse sequence. There exists an almost periodic two-sided sequence $\omega$ such that $\omega[k] = x[k]$, $k \geq 0$. Putting $\Omega_x$
In order to describe the centralizer $C(T_0)$ of the dynamical system $(G \times Z_2, m \times \frac{1}{2}, T_0)$ we use the isomorphism theorem (see § 4 below). In the sequel we assume that the lengths $\lambda_i$ of $b'$ satisfy the condition
\[
\sum_{i=0}^{n} \frac{1}{\lambda_i} < \infty.
\]

Let $(Q_\infty, T, \mu_\infty)$ be another continuous Morse shift,

\[y = \beta^0 \times \beta^1 \times \ldots \quad |\beta^i| = \beta_i = \lambda_i, \quad t \geq 0.
\]

Newton's results [5] allow to find the form of isomorphisms between the dynamical systems $(Q_\infty, T, \mu_\infty)$ and $(Q_\infty, T, \mu_\infty)$ regarded as $Z_2$-extensions of $G$.

Let $\phi_t : G \to Z_2$ be the function defined by

\[\phi_t(g) = d_i(1 + j_i) \times d_j(j_i), \quad g = (j_i)^{\eta_i}, \quad 0 < j_i < \eta_i - 1,
\]

where $d_t = \beta^0 \times \beta^1 \times \ldots \beta_t, t \geq 0$, and $\phi(\eta_i^{-1})^{\eta_i}$. The system $(Q_\infty, T, \mu_\infty)$ is metrically isomorphic to $(G \times Z_2, m \times \frac{1}{2}, T_0)$. Then each isomorphism $W$ from $(G \times Z_2, m \times \frac{1}{2}, T_0)$ to $(G \times Z_2, m \times \frac{1}{2}, T_2)$ is of the form

\[W(g, \bar{g}) = (g + g_0, i + \phi(g)),
\]

where $g_0$ is a $\bar{g}$-adic integer and $\phi : G \to Z_2$ is a function such that

\[\phi_\delta(g + g_0) + \phi(g) = \phi(g + 1) + \phi(g)
\]

for all $g \in G$.

Applying the isomorphism theorem (see § 4, Th. 5) we will describe $g_0$ and the function $\phi$. Put

\[g_0 = \sum_{i=0}^{n} q_i n_i, \quad \eta_i = \sum_{i=0}^{n} q_i n_i - 1
\]

and for fixed $t$ define

\[\phi_t(g) = (c_i + r_i)(c_i + s_i)[1 + j_i] + d_i(j_i), \quad g = (j_i)^{\eta_i}.
\]

Next let

\[H_i = \{g \in G : \phi_{t_i}(g) = \phi(g)\}.
\]

We estimate $m(H_i)$. Let $G = \sum_{i=0}^{n} q_i n_i - 1$. We have

\[\phi_{t_i}(g) = \phi_{\lambda_1}(g) + (b^{t_i} + r_{t_i})(b^{t_i} + s_{t_i})[1 + q_{t_i} q_{t_i + 1} + \beta^{t_i} [q_{t_i + 1} + \eta_{t_i + 1}]] + r_i
\]

whenever $q_i + \eta_i < \lambda_{t_i} - 1$ and $q_{t_i + 1} + \eta_{t_i} \neq \lambda_{t_i} - 1$. If $q_i + \eta_i \geq \lambda_{t_i}$ and $q_{t_i + 1} + \lambda_{t_i} - 1$ then

\[\phi_{t_i}(g) = \phi_{\lambda_1}(g) + (b^{t_i} + r_{t_i})(b^{t_i} + s_{t_i})[1 + q_{t_i} q_{t_i + 1} + \beta^{t_i} [q_{t_i + 1} + \eta_{t_i}]].
\]

5 – Studia Mathematica 89/2
The above equalities imply
\[ m(H_i) \geq 1 - \frac{2}{\lambda_i} + \frac{2}{\lambda_i} \left[ \left( 1 - \frac{3}{\lambda_i} \right) D_{i+1} + \frac{\beta_i}{\lambda_i} B_{i+1} \right]. \]

where \( D_i, B_i \) are defined in § 4.

Putting \( E_i = \cap_{H_i \neq E} H_i \) we obtain
\[ m(E_i) \geq 1 - 4 \sum_{i \geq 1} \frac{1}{\lambda_i} \lambda_i - \sum_{i \geq 1} \left( \left[ 1 - \frac{3}{\lambda_i} \right] D_{i+1} + \frac{\beta_i}{\lambda_i} B_{i+1} \right]. \]

Finally we define \( E = \bigcup_{i=0}^{\infty} E_i \) and
\[ \phi(g) = \phi_i(g) \quad \text{for } g \in E_i. \]

We have \( E_i \subseteq E_{i+1}, \ t \geq 0 \), and from Theorem 5 we get \( m(E) = 1 \). It is not hard to see that the function \( \phi_i \) satisfies (4) with \( g_0 \) defined above. It follows from the proofs of [3, Th. 1] and [6, Th. 1] that every isomorphism \( W \) from \( (Q_\alpha, T, \mu_\alpha) \to (Q, T, \mu) \) is of the form described above.

Now we are in a position to describe the centralizer \( C(T) \) of the Morse shift \((G \times Z_2, m \times \lambda, T)\). Taking \( b^i = b^i, \ t \geq 0 \), and applying Theorem 5 we get the following

\[ \textbf{THEOREM 1.} \text{ Each } S \in C(T) \text{ has the form} \]
\[ S(g, h) = (g + g_0, h + \varphi(g)) \]

and \( g_0 = (1/\lambda)^\infty \) satisfies the following condition: there exist \( \bar{r}, \bar{s} \in \mathbb{Z}_2, \ t \geq 0 \), such that
\[ \sum_{i \geq 0} \left( \left[ 1 - \frac{1}{\lambda_i} \right] D_{i+1} + \frac{\beta_i}{\lambda_i} B_{i+1} \right] < \infty \]

where
\[ D_i = d((b^i + \bar{r}_i)(b^i + \bar{s}_i) \bar{r}_i, \bar{s}_i + \lambda_i - 1, b^i + \bar{r}_i - 1), \]
\[ B_i = d((b^i + \bar{r}_i)(b^i + \bar{s}_i) \bar{r}_i + \bar{s}_i, \bar{r}_i + \lambda_i + 1, b^i + \bar{s}_i - 1) \]

(here for blocks \( A, B \) of length \( n \), \( d(A, B) \) denotes the number \( n^{-1} \text{card} \{0 \leq i \leq n-1: A[i] \neq B[i]\} \).

The function \( \varphi \) is the limit a.e. of the sequence of functions \( \varphi_i \).

\[ \varphi_i(g) = (\lambda_i + \bar{r}_i)(\lambda_i + \bar{s}_i) \lambda_i + \lambda_i, \]

Moreover, \( \varphi \) satisfies the condition
\[ \varphi(g + g_0) + \varphi(g) = \varphi(g + \tilde{r}) + \varphi(g). \]

Since \( \sum_{i=0}^{n} \frac{1}{\lambda_i} < \infty \) the condition (6) is equivalent to
\[ \sum_{i=0}^{n} d((b^i + \bar{r}_i)(b^i + \bar{s}_i) \bar{r}_i, \bar{s}_i + \lambda_i - 1, b^i) < \infty, \]

\[ \sum_{i=0}^{n} \min \left( 1 - \frac{\bar{r}_i}{\lambda_i}, \frac{\bar{s}_i}{\lambda_i} \right) \left[ \delta(r_i, \bar{r}_i, b^i) + \delta(r_i, \bar{s}_i, b^i) \right] < \infty. \]

Remark 1. The numbers \( q_i \) in (6) and \( r_i, s_i \in \mathbb{Z}_2, \ t \geq 0 \), may be so chosen that \( q_0 = \tilde{q}_0 \) and
\[ q_i = \left\{ \begin{array}{ll}
\tilde{q}_i, & \text{if } \tilde{q}_i \in \lambda_i - 1, \\
\tilde{q}_i + 1 \pmod{\lambda_i}, & \text{otherwise,}
\end{array} \right. \]

\[ r_i + s_i = \tilde{r}_i + \tilde{s}_i \quad \text{in } \mathbb{Z}_2. \]

The \( \lambda \)-adic integer \( g_0 \) is defined by \( g_0 = \sum_{i=0}^{\infty} \tilde{q}_i n_i^{-1} \).

Thus each \( S \in C(T) \) is an extension of the translation \( T_{g_0} \). \( T_{g_0}(g) = g_0 + g \), of \( G \). Note that \( T_{g_0} \) is an element of the centralizer of \( T_i \). We will prove that extension of \( T_{g_0} \) to an element of \( C(T) \) is possible only if \( g_0 \) runs through a subset of \( G \) of Haar measure zero. In the sequel let \( G_0 \) be the set of all \( g_0 \in G \) satisfying (6).

\[ \textbf{§ 2. Properties of } C(T). \text{ For } S \in C(T) \text{ we will write } S = (g_0, \varphi) \text{ if } g_0 \in G_0 \]

and \( \varphi \) is a function satisfying (8).

I. For every \( g_0 \in G_0 \) there exist exactly two functions \( \varphi, \psi \) satisfying (8).

In fact, if \( \varphi \) and \( \psi \) are such functions then \( \varphi + \psi \) is a \( T_i \)-invariant function so either \( \varphi \equiv \psi \) or \( \varphi \equiv \psi + 1 \). On the other hand, if \( \varphi \) satisfies (8) then so does \( \psi + 1 \). Note that if \( \varphi \) is determined by \( g_0 \) and the sequences \( \{r_i, s_i\} \) which satisfy (6) then \( \varphi + 1 \) is determined by \( g_0 \) and \( \{r_i, s_i\}, r_i = r_i, s_i = s_i + 1, \ t \geq 0 \).

II. The set \( G_0 \) is a \( T \)-invariant subgroup of \( G \).

First we show that \( G_0 \) is measurable. To this end we use the condition (6) in the following form:
\[ \sum_{i=0}^{n} \left( \left[ 1 - \frac{\bar{r}_i}{\lambda_i} \right] D_{i+1} + \frac{\bar{r}_i}{\lambda_i} B_{i+1} \right] < \infty. \]

Put
\[ A_{\alpha \beta} = \left\{ g_0 = \sum_{i=0}^{n} \tilde{q}_i n_i^{-1}: \text{there exist } \tilde{r}_i, \tilde{s}_i \in \mathbb{Z}_2, \ t \leq n + k, \text{ such that } \sum_{i=0}^{n+k} \left[ \left( 1 - \frac{\bar{r}_i}{\lambda_i} \right) D_{i+1} + \frac{\bar{r}_i}{\lambda_i} B_{i+1} \right] < \frac{1}{p_i} \right\}, \]
where \( n, k, p \) are positive integers. Then it is clear that
\[
G_0 = \bigcap_{p=1}^{\infty} \bigcup_{x=0}^{\infty} A_{a_k^x}.
\]
Since \( A_{a_k^x} \) is a finite union of cylinder sets, \( G_0 \) is measurable.

\( G_0 \) is a subgroup of \( G \), because if \( S = \{g_0, \varphi\}, S = (f_0, \varphi) \) then
\[
S \circ S = (g_0 + g_0, \psi_1), \quad \psi_1(g) = \varphi(g) + \varphi(g_0 + g),
\]
\[
S \circ S = (g_0 + g_0, \psi_1), \quad \psi_2(g) = \varphi(g) + \varphi(g_0 + g).
\]
Therefore \( g_0 + g_0 \in G_0 \). Since \( T_n = \{I, \varphi_n\} \) we have \( g_0 + I \in G_0 \) whenever \( g_0 \in G_0 \). In particular,
\[
T_n = (m, \sum_{i=0}^{n-1} \varphi_i(g + i), m = \pm 1, \pm 2, \ldots,
\]
\[
I = T_n = (0, 1), \quad \sigma = (0, 1, 0), \quad S \circ \sigma = (g_0, \varphi + 1).
\]
Properties I and II are valid for every ergodic \( Z_2 \)-extension of \( G \).

III. The set \( \hat{Z} \) of \( \mathbb{Z}_2 \)-adic rational integers is contained in \( G_0 \). Either \( G_0 = \hat{Z} \) or \( G_0 \) is an uncountable subset of \( G \).

The first statement is evident. Suppose that \( g_0 \in G_0 \). Then there exist numbers \( q, r, s \in \mathbb{Z}_2 \) that satisfy (6) and determine \( g_0 \) as in Remark 1. The set \( Z_0 \) of all \( t \) such that \( q, r > 0 \) is infinite. Take any infinite subset \( I \) of \( Z_0 \). We construct a \( \mathbb{Z}_2 \)-adic integer \( g_i \) and \( r_i, s_i, t_i > 0 \), satisfying (6). Namely, we put
\[
g_i = q_i, \quad r_i = r_i, \quad s_i = s_i, \quad \text{for } t \in I,
\]
\[
g_i = 0, \quad r_i = s_i = 0, \quad \text{for } t \notin Z_0 \setminus I.
\]
It is easy to see that \( g_i \) and \( r_i, s_i, t_i > 0 \) satisfy (6) so they determine a \( \mathbb{Z}_2 \)-adic integer \( g_i \notin \hat{Z} \). Taking different subsets \( I \) of \( Z_0 \) we obtain different \( \mathbb{Z}_2 \)-adic integers \( g_i \). Thus \( G_0 \) is uncountable.

Observe that \( G_0 = \hat{Z} \) iff \( C(T_i) = \{T_i^j \alpha_i \} \), \( i = 0, \pm 1, \ldots, \) \( j \in \mathbb{Z}_2 \). The above considerations allow us to generalize Lemma 1.2 from [4, Th. 1].

If \( x = b^0 \times b^1 \times \ldots \) is a regular Morse sequence then the conditions (6) can be written as
\[
\sum_{i=0}^{\infty} (b_i + r_i)(b_i + s_i)(q_i + q_i + 1), b_i < \infty,
\]
(6')
\[
\sum_{i=0}^{\infty} \min(1/l_i, 1 - 1/l_i) < \infty,
\]
where \((l_i)_i \in G \) and the \( l_i \) are defined in Remark 1.

In this case the assumption \( \sum_{i=0}^{\infty} 1/l_i < \infty \) is not necessary [3]. In addition, if the sequence \( l_i \) is bounded then (6) implies \( G_0 = \hat{Z} \).

For convenience we use the following notation:
\[
\text{fr}(00 \lor 11; A) = \text{fr}(00, A) + \text{fr}(11, A),
\]
\[
\text{fr}(01 \lor 10; A) = \text{fr}(01, A) + \text{fr}(10, A).
\]

**Example 1.** Let \( x = b^0 \times b^1 \times \ldots \) be a continuous nonregular Morse sequence. Then \( G_0 \) is uncountable. In fact, if \( x \) is not regular then we can represent it in such a way that
\[
\sum_{i=0}^{\infty} \min(\text{fr}(00 \lor 11; b_i), \text{fr}(01 \lor 10; b_i)) < \infty
\]
(see [3]). Put \( q_i = 1 \) and

\[
r_i = 1, \quad s_i = 0 \quad \text{if } \text{fr}(00 \lor 11; b_i) < \text{fr}(01 \lor 10; b_i),
\]

\[
r_i = s_i = 0 \quad \text{otherwise}.
\]

Then \( g_i = \sum_{i=0}^{\infty} q_i n_{i-1} \in G_0 \) and \( g_i \notin \hat{Z} \). Thus \( G_0 \) is uncountable.

**Theorem 2.** The set \( G_0 \) has Haar measure zero.

**Proof.** Property II implies \( m(G_0) = 0 \) or 1. To prove that \( m(G_0) = 0 \) it suffices to show that \( G \setminus G_0 \neq \emptyset \). If
\[
\sum_{i=0}^{\infty} \min(\text{fr}(01 \lor 10; b_i), \text{fr}(00 \lor 11; b_i)) = +\infty,
\]
then the element \( g = \sum_{i=0}^{\infty} q_i n_{i+1} \), where \( q_i = \lfloor n_i/2 \rfloor \), \( t > 0 \), does not satisfy the second condition of (6'), so \( g \notin G \setminus G_0 \). Therefore we may assume that the condition (9) is not satisfied. In this case we use the inequalities
\[
\text{fr}(00 \lor 11; A \times B) \leq \text{fr}(00 \lor 11; A) + 1/|A|,
\]
\[
\text{fr}(01 \lor 10; A \times B) \leq \text{fr}(01 \lor 10; A) + 1/|A|,
\]
\[
\sum_{i=0}^{\infty} 1/l_i < \infty,
\]
and we group the blocks \{\( b_i \}\} in \( b \) to find a new representation (also denoted by \( x = b^0 \times b^1 \times \ldots \)) such that \( \sum \text{fr}(01 \lor 10; b_i) < \infty \) or \( \sum \text{fr}(00 \lor 11; b_i) < \infty \). Moreover, we may assume that
\[
\frac{1}{2} < \text{fr}(0, b_i) < \frac{3}{4},
\]
because \( \text{fr}(0, c) \rightarrow \frac{1}{2} \) as \( t \rightarrow \infty \).
(A) Suppose first $\sum_{r=0}^{l} \mathfrak{r}(01 \lor 10; b) < \infty$. Let
\[ d(q) = d(b^r b^q [q, q + \lambda_i - 1], b^r), \quad q = 0, 1, \ldots, \lambda_i - 1. \]
It follows from the definition of $d(q)$ that
\[ \lambda_i^{-1} \sum_{q=0}^{\lambda_i-1} d(q) = 2\mathfrak{r}(0, b) \mathfrak{r}(1, b) > 2 \cdot \frac{1}{3} = \frac{2}{3}. \]
Hence
\[ \frac{1}{2} \lambda_i^{-1} \sum_{t=0}^{\lambda_i-1} d(q) > \frac{1}{2} (\frac{2}{3} - \frac{1}{3}) = \frac{1}{3}. \]
Since $d(0) = 0$ and
\[ |d(q + 1) - d(q)| \leq \mathfrak{r}(01 \lor 10; b) + 1/\lambda_i, \quad q = 0, \ldots, \lambda_i - 2, \]
we deduce that there exist $q_0, \frac{1}{2} \lambda_i < q_0 < \frac{3}{2} \lambda_i$, such that $\frac{1}{4} < d(q_0) < \frac{1}{3}$ (for sufficiently large $i$). Note that the sequence $\{q_i\}$ does not satisfy (6). Indeed, the inequality $\frac{1}{4} \lambda_i < q_0 < \frac{3}{4} \lambda_i$ and (A) imply that the second condition of (6) is possible only if $r_i = s_i$ for sufficiently large $i$. Then the first condition of (6) is impossible, because $\frac{1}{4} < d(q_0) < \frac{1}{3}$ and
\[ d(b^r b^q [q, q + \lambda_i - 1], b^r) - 1 = d(q) \]
Thus $g_0 = \sum_{t=0}^{\lambda_i} g_{s,t,0}$ does not belong to $G_0$, where $\tilde{g}_i$ is defined as in Remark 1.

(B) Suppose now $\sum_{r=0}^{l} \mathfrak{r}(00 \lor 11; b) < \infty$. Let
\[ d(q) = d(b^r b^q [q, q + \lambda_i - 1], b^r), \quad q = 0, 1, \ldots, \lambda_i - 1. \]
If infinitely many of the integers $\lambda_i$ are even, then by grouping the blocks $\{b^5, b^7, \ldots\}$ we may assume that the $\lambda_i$ are even for $t > t_0$. It is easy to verify that
\[ \tilde{d}(q) = 1 - \tilde{d}(q - q), \quad q = 1, \ldots, \lambda_i - 1. \]
In particular, $\tilde{d}(\lambda/2) = 1 - \tilde{d}(\lambda/2)$ with implies $\tilde{d}(\lambda/2) = 1/2$. Put $q_i = 0$ for $t < t_0$ and $q_i = \lambda_i/2$ for $t > t_0$. Just as above, we check that the sequence $\{q_i\}$ does not satisfy (6).

Finally, suppose that the $\lambda_i$ are odd for $t > t_0$. We reduce this case to (A). Since $x = b^p \times b^q \times \cdots$ is a continuous Morse sequence, we have
\[ \sum_{t \geq 0} \min_{b \geq 0} (d(b^t, 010 \ldots 010), d(b^t, 101 \ldots 101)) = + \infty. \]
Now we use the following equality:
\[ 1 - 2d(A_i \times \cdots \times A_{s}, B_i \times \cdots \times B_{s} + b) = \prod_{i=1}^{n} (1 - 2d(A_i, B_i + b)). \]
where $l, \lambda_1, \ldots, \lambda_{l-1} \in \mathbb{Z}_2$ and $l = 1 + l_1 + \cdots + l_{l-1}, |A| = |B|$, which can be easily proved by induction. Take $A_1 = b^{m_1}, \ldots, A_l = b^{m_l}, B_1 = 010 \ldots 010$ (with 010 repeated $l_{l-1}$ times), $B_2 = 010 \ldots 010 (l_{l-1}$-fold repetition), and $l = 0$ if $d(b^{m_1} \times \cdots \times b^{m_l}, 010 \ldots 010) \leq 1$ and $l = 1$ otherwise. In the same way we choose $l_i, i = 1, \ldots, u - 1$. For $t \geq t_0$ we obtain
\[ 1 - 2\min(d(b^{m_1} \times \cdots \times b^{m_l}, 010 \ldots 010), d(b^{m_1} \times \cdots \times b^{m_l}, 101 \ldots 101)) = \prod_{i=1}^{n} (1 - 2a_{l_i}), \]
where $a_i = \min(d(b^l, 010 \ldots 010), d(b^l, 101 \ldots 101))$.

Since $\sum_{i=1}^{n} a_i = \infty$, we have $\prod_{i=1}^{n} (1 - 2a_{l_i}) = 0$ for every $t \geq t_0$. Thus
\[ \min(d(b^{m_1} \times \cdots \times b^{m_l}, 010 \ldots 010), d(b^{m_1} \times \cdots \times b^{m_l}, 101 \ldots 101)) \geq \frac{1}{2}. \]
By grouping the blocks $\{b^q\}$ we can find a representation of $x, x = b^p \times b^q \times \cdots$, such that
\[ \frac{1}{4} < d(b^p, 010 \ldots 010) < \frac{1}{3} \quad \text{for } t \geq 1. \]
Putting $E_i = b^p \times \cdots \times b^q (l_i$-fold repetition) we have $\frac{1}{4} < \mathfrak{r}(0, E_i) < \frac{1}{3}, i \geq 1$. Since $\mathfrak{r}(01 \lor 10; b) = \mathfrak{r}(00 \lor 11; b)$ it follows that $y = b^p \times E_i \times E_j \times \cdots$ is a continuous Morse sequence satisfying (A). Repeating the considerations in (A) we obtain a sequence $\{q_i\}, \frac{1}{2} \lambda_i < q_i < \frac{3}{2} \lambda_i$ and $\frac{1}{4} < d(q_i) < \frac{1}{3}$, where $d(q_i)$ is defined by $y$. By easy computations we establish that
\[ \min(d(q_i), 1 - d(q_i)) = \min(d(q_i), 1 - d(q_i)), \quad 0 \leq q_i < \lambda_i - 1. \]
The last conditions imply $\frac{1}{4} \leq d(q_i) \leq \frac{1}{3}$. As above, we show that the sequence $\{q_i\}$ does not satisfy (6). This finishes the proof of the theorem.

§ 3. Topology of $C(T_0)$. We start with the following remark: if $S, S_0 \in C(T_0), S_0 = (\phi_0, \phi_0), S = (\phi_0, \phi_0), g_0, g_0 \in G_0$ then $S_0 \rightarrow S$ in the weak topology iff $g_0 \rightarrow g$ in $G$ and $\phi_0 \rightarrow \phi$ in Haar measure $m$.

**Theorem 3.** $C(T_0)$ is the closure of $\{T_0^s\}, s = 0, 1, \ldots, \in \mathbb{Z}_2$, in the weak topology. $T_0$ is a rigid transformation iff $G_0$ is uncountable.

**Proof.** First we show $C(T_0) = \{T_0^s\}$. Assume that $G_0$ is uncountable and let $S = (\phi_0, \phi_0), g_0 = (l_0)^{\phi_0}, g_0 \in \mathbb{Z}_{2}$. Take $\bar{t}_0, \bar{t}_0 \in \mathbb{Z}_2$ satisfying (6). Define
\[ i_t = \frac{t}{t_0} \quad \text{if } l_t \leq n_t - l_t - 1, \quad T_t = \left\{ \begin{array}{ll} \bar{t}_0 \quad & \text{if } l_t \leq n_t - l_t - 1, \\
\bar{t}_0 \quad & \text{otherwise,}
\end{array} \right. \]
Now we show that $T_0^s \circ \sigma^h \rightarrow S$ in the weak topology. We have $T_0^s = (\phi_0, \phi_0)$ and
\[ \phi_{\omega}(\phi) = c_{i} \cdot \varphi_{i}(\phi) + c_{i} \cdot \varphi_{i}(\phi). \]
for sufficiently large \( t \) (see § 2, II). Put \( \varphi_m(g) = c_i [j_i + m] + c_j [j_j] \). Notice that the equality

\[
\varphi_m(g) = \varphi_m(g)
\]

holds on a set of Haar measure greater than \( 1 - m/n_i \). At the same time, \( \varphi(t) \) is the limit of the sequence of the functions \( \varphi_i \), defined by (7). From the preceding considerations, if \( g \in E_t \), then

\[
\varphi_i(g) = \varphi_i(g) = \begin{cases} 
  c_i [j_i + l_i] + c_j [j_j] & \text{if } j_i + l_i \leq n_i - 1,
  c_i [j_i + l_i - n_i] + c_j [j_j] + \bar{r}_i & \text{if } j_i + l_i \geq n_i.
\end{cases}
\]

Suppose that \( j_i + l_i = j_i + b_i \), where \( 0 \leq l \leq j_i + 1. \) It follows from (11) that

\[
\varphi_i(g) = \varphi_i(g) = \begin{cases} 
  c_i [j_i + l_i] + c_j [j_j] & \text{if } j_i + l_i \leq n_i - 1,
  c_i [j_i + l_i - n_i] + c_j [j_j] + b_i [j_i] + b_i [1] + [1] & \text{otherwise}
\end{cases}
\]

except on a set of Haar measure \( l/n_i < 1/2 \). Assume that \( l \leq n_i - 1 \). Then

\[
\varphi_i(g) = \varphi_i(1/g) + \bar{r}_i
\]

whenever \( j_i + l_i \geq n_i \) and \( b_i [j_i] + b_i [1] + [1] = \bar{r}_i + \bar{s}_i \), except on a set of Haar measure \( l/n_i < 1/2 \). By using (9) with \( l = l_i \) instead of \( q_i \), the equality \( r_i + s_i = \bar{r}_i + \bar{s}_i \) (see Remark 1) gives

\[
\varphi_i(g) = \varphi_i(1/g) + \bar{r}_i
\]

except on a set of measure \( \geq 1 - e_i \), where \( e_i \to 0 \).

If \( l_i > n_i - 1 \) then by similar arguments we establish that

\[
\varphi_i(g) = \varphi_i(1/g) + \bar{r}_i
\]

on a set of measure \( \geq 1 - e_i \), where \( e_i \to 0 \).

Since \( T_{\sigma_i} \circ \sigma_i = (\bar{r}_i, r_i + l_i), l_i \to 0 \) and \( \varphi_i + l_i \to \varphi \) in measure, we conclude that

\[
T_{\varphi_i} \circ \sigma_i = \bar{S}_i.
\]

In this way the equality \( C(T_{\varphi_i}) = \bar{S}_i = \bar{S}_i \), \( i \in Z, j \in Z_2 \), is proved.

To show \( T_{\varphi_i} \) rigid it suffices to remark that there exists a subsequence of \( \{T_i\} \) (denote it by \( \{T_i\} \) again) satisfying \( |T_i| \to \infty \). Then \( T_{\varphi_i} \circ \sigma_i = S_i \) or \( T_{\varphi_i} \circ \sigma_i \to S_i \), hence \( T_{\varphi_i} \circ \sigma_i \to \bar{I}_{\varphi_i} \) or \( T_{\varphi_i} \circ \sigma_i \to \bar{I}_{\varphi_i} \), and so \( T_{\varphi_i} \) is rigid.

**Theorem 4.** Let \( x = b^{0} \times b^{1} \times \ldots \) be a continuous Morse sequence such that

\[
\min_{0 \leq k \leq i} d(b^k [0, j_k - q_k - 1], b^k [q_k, j_k - 1]) \geq 0
\]

for each \( t \geq 0 \). Then \( \sigma_i \in \{T_i\} \), \( i \in Z \).

Proof. Suppose that \( T_{\varphi_i} \to \sigma_i \). Then \( m_i = 0 \) in \( G \) and \( \varphi_{m_i} \to 1 \) in measure \( m_i \), where \( \varphi_{m_i} \) is defined by (10). We may assume \( m_i = 0 \mod n_i \), \( t \to 0 \), i.e., \( m_i = m_i, n_i, t_i \to 1 \). Write

\[
e_i = m_i \{ g \in G : \varphi_{m_i}(g) = 0 \}, \quad t_i \to 0.
\]

Then \( \varepsilon_i \to 0 \). Next let

\[
A_{m_i} = \{ g = (j_k) \in G : 0 \leq j_k \leq n_i - m_i - 1 \}, \quad u > t.
\]

We have \( m(A_{m_i}) = 1 - m_i/n_i \) and \( \varphi_{m_i}(g) = c_i [j_i + m_i] + c_j [j_i] \) for \( g \in A_{m_i} \). Further,

\[
m_i \{ g \in A_{m_i} : \varphi_{m_i}(g) = 0 \} \leq \varepsilon_i + m(A_{m_i}) \leq \varepsilon_i + m_i/n_i.
\]

We choose \( u = u(t) \) such that \( u > t_0 \) and \( m_i/n_i > 0 \). Then (16) implies

\[
m_i \{ g \in A_{m_i} : \varphi_{m_i}(g) = 0 \} < 2 \varepsilon_i \quad \text{for } u = u(t).
\]

Write

\[
\varepsilon_i = e_i + e_i < e_i + e_i.
\]

We have

\[
m_i \{ g \in A_{m_i} : \varphi_{m_i}(g) = 0 \} = \frac{1}{n_i} \text{card} \{ 0 \leq j \leq 1 - m_i - 1 : c_i [j + m_i] = c_i [j] \}.
\]

It follows that

\[
\frac{1}{n_i} \text{card} \{ 0 \leq j \leq n_i - 1 : c_i [j] = c_i [j] \} \leq \varepsilon_i
\]

because \( m_i/n_i = m_i/n_i < e_i \). Here \( e_i \) is an arbitrary element of \( Z_2 \).

In this way we obtain the following property:

\( A \) For every \( t \) there exists \( u = u(t) > t \) such that for every \( u \geq u(t) \)

\[
\frac{1}{n_i} \text{card} \{ 0 \leq j \leq n_i - 1 : c_i [j + m_i] = c_i [j] \} < 3 \varepsilon_i
\]

and

\[
\bar{m} < n_i/2.
\]
Now we show that (A) implies

(B) For every \( t \) there exists \( v = \varepsilon(t) \geq t \) such that

\[
\min_{0 \leq q \leq 2} d(b^{[0, \lambda_q - q - 1], (b^{[0, \lambda_q - 1]})} < 2 \sqrt{3} q,
\]

To do this we represent the number \( \lambda_q \) as

\[
\lambda_q = \frac{q}{2} \cdot n_q^{(q-1)} + r_q,
\]

and \( d_0(n_q^{(q-1)} + r_q) \) the lengths \( n_q^{(q-1)} \). Using (A) we get the following possibilities:

(a) \( d(b^n[0, \lambda_q - q - 1], (b^{[0, \lambda_q - 1]}) < 2 \sqrt{3} q \),

(b) \( d(c^{(q-1)}_0(n_q^{(q-1)} + r_q) < 2 \sqrt{3} q \),

for some \( i_q \in \mathbb{Z}_2 \). Case (a) implies (B). If (b) holds then we can repeat the above reasoning taking \( u = u + 1 \) and \( \lambda_q = \lambda_{q+1} \). If \( q > 0 \) then we obtain (a) or (b) again with \( q = q+1 \) and \( r = \lambda^{(q-1)} - r \). If \( q = 0 \) and \( r < \lambda^{(q-1)} - 2 \) then (b) holds. If \( q = 0 \) and \( r > \lambda^{(q-1)} - 2 \) then we obtain (a) with \( q = 1 \). Proceeding in this way either we choose \( u = u + 1 \) satisfying (a) or (b) is satisfied for \( u = u + 1 \). The last means (a) for \( u = u + 1 \). We have shown (B) and consequently Theorem 4.

**Example 2.** Take \( b^t = 01101 \ldots 011 \), where 011 is repeated \( 2^{t+1} \) times, \( t \geq 0 \). The sequence \( x = b^0 \times b^1 \times \ldots \) is regular. It satisfies the assumption of Theorem 3 with \( \beta = 1 \). At the same time, \( C_T \) is uncountable. Indeed, we have

\[
d(b^{(3, \lambda_2 + 2), b^t} = 0, t \geq 0.
\]

Taking \( \lambda = 3, t \geq 0 \), we obtain

\[
l = 3 \sum_{0}^{t} n_{t-1} \geq 6n_{t-1},
\]

which implies \( l/n < 1/2 \). Thus \( \min(l/n, 1 - l/n) < 1/2 \) and (6') is satisfied with \( r = 5 = 0 \).

Hence \( C(T^x) \) is rigid. Theorem 3 implies \( C(T_x) \preceq \langle T_x \rangle \). In this way we obtain a negative answer to Walter’s question [7], because \( T_x \) has simple spectrum.

**§ 4. Isomorphism theorem.** To describe the centralizer of Morse shifts we have used a certain form of isomorphisms between such systems. Here we give a modification of isomorphism theorems from [3] and [6], omitting the details, which can be found in the above-mentioned papers.

**Theorem 5.** If \( x = b^0 \times b^1 \times \ldots, y = \beta^0 \times \beta^1 \times \ldots \) are continuous Morse sequences such that \( \lambda_i = |b^i| = |\beta^i|, t \geq 0, \lambda_i \leq \lambda, \) then the Morse dynamical systems \( \theta(x) \) and \( \theta(y) \) are metrically isomorphic if and only if there are sequences of integers \( \{ \lambda_i \}, \{ \lambda_i \}, r_i, a_i \in \mathbb{Z}_2, t \geq 0, \) and an element \( g_0 = (b^0) = \sum_{t=0}^{\infty} g_t \in \mathbb{Z}_2, g_0 = 0 \)

\[
\sum_{t=0}^{\infty} \left[ \begin{array}{c}
\frac{1}{t} \sum_{n=1}^{t} D_{n+1} + \frac{1}{n} \sum_{n=1}^{t} d_{n+1} + 1
\end{array} \right] < \infty,
\]

where

\[
D_t = d((b^0 \times \beta^0), (b^0 \times \beta^0))(a_t, b_t, a_{t+1}, b_{t+1}),
\]

\[
D_t = d((b^0 \times \beta^0), (b^0 \times \beta^0))(a_t, b_t, a_{t+1}, b_{t+1}).
\]

**Proof.** It suffices to show the necessity of the theorem. Indeed, the condition (17) enables us to define a function \( \varphi : G \to \mathbb{Z}_2 \) by (5) (see § 1). Then \( \varphi \) satisfies the condition (4).

**Necessity.** Let us group the blocks \( b^0, b^0, \ldots, b^0, \beta^1, \beta^1, \ldots \) to obtain new representations \( x = b^0 \times b^1 \times \ldots, y = \beta^0 \times \beta^1 \times \ldots \) such that

\[
\varphi(x) = \varphi(y),
\]

\[
\varphi(x) = \varphi(y).
\]

It follows from [6, Lemma 5] that there exist sequences of blocks \( \{a_i\}, \{\lambda_i\}, r_i \geq 0, |a_i| = \lambda_{a_i}, |a_i| = \lambda_{a_i + 1}, t \geq 0, \) and sequences of integers \( \{\lambda_i\}, \{\lambda_i\}, 0 \leq w_i \leq \lambda_{a_i+1}, t \geq 1, 0 \leq \lambda_i \leq \lambda_{a_i+1}, t \geq 0, \) such that

\[
\sum_i d(b^i \times \beta^i)(w_i, a_i) < \infty,
\]

\[
\sum_i d(b^i \times \beta^i)(w_i, a_i) < \infty.
\]

The proof of Lemma 5 shows that \( p_i, w_i \) may be chosen in such a way that if \( w_i \neq w_i \) then \( w_i \neq \lambda_{w_i+1} \lambda_{w_i+1} - \lambda_i + 1 \), and if \( p_i \neq p_i \) then \( p_i \neq \lambda_{p_i+1} \lambda_{p_i+1} - \lambda_{a_i} \).

Now we are in a position to define numbers \( \pi_i, \iota_i \in \mathbb{Z}_2 \). Write

\[
\eta_i = \max(d(b^i \times \beta^i)(w_i, \lambda_{w_i+1} \lambda_{w_i+1} - a_i), \lambda_{a_i} < \infty).
\]

\[
d(b^i \times \beta^i)(p_i, \lambda_{p_i+1} \lambda_{p_i+1} - \lambda_1) < \infty,
\]

Put \( k_i = \lambda_{w_i+1} \lambda_{w_i+1} - w_i, t \geq 1, m_i = \lambda_{w_i+1} \lambda_{w_i+1} - p_i, t \geq 0. \) Then \( 1 \leq k_i \leq \lambda_{w_i+1} - 1 \) (if \( w_i \neq w_i \)) and \( 1 \leq m_i \leq \lambda_i \) (if \( p_i \neq p_i \)). Suppose that \( w_i \neq w_i \). It is easy to verify that

\[
d(b^i \times \beta^i)(w_i, \lambda_{w_i+1} \lambda_{w_i+1} - a_i) < \infty,
\]

\[
d(b^i \times \beta^i)(p_i, \lambda_{p_i+1} \lambda_{p_i+1} - \lambda_i) < \infty.
\]

\[
\sum_{i=0}^{\infty} \left[ \begin{array}{c}
\frac{1}{t} \sum_{n=1}^{t} D_{n+1} + \frac{1}{n} \sum_{n=1}^{t} d_{n+1} + 1
\end{array} \right] < \infty,
\]

where

\[
D_t = d((b^0 \times \beta^0), (b^0 \times \beta^0))(a_t, b_t, a_{t+1}, b_{t+1}),
\]

\[
D_t = d((b^0 \times \beta^0), (b^0 \times \beta^0))(a_t, b_t, a_{t+1}, b_{t+1}).
\]
where $a_i' = a_i a_i[0]$. Notice that

\[ d(\bar{B}, \bar{a}_{i-1} \times a_i' [k_i, k_i + \lambda_{2i-1} - \lambda_{2i-1}]) \]

\[ = \left(1 - \frac{k_i}{\lambda_{2i-1}}\right) d(\beta^{2i-1}[0, k_i - 1] \times \beta^{2i}, \bar{a}_{i-1} [k_i, k_i + \lambda_{2i-1} - 1] \times a_i) \]

\[ + \frac{k_i}{\lambda_{2i-1}} d(\beta^{2i-1} [k_i - 1, k_i + \lambda_{2i-1} - 1] \times \beta^{2i}, \bar{a}_{i-1} [0, k_i - 1] \times a_i a_i [1, \lambda_{2i}]). \]

Let $\bar{r}_{2i-1}, \bar{s}_{2i-1} \in Z_2$ be integers such that

\[ d((\beta^{2i-1} + F_{2i-1}) (\beta^{2i} + F_{2i-1}) [\lambda_{2i-1} - k_i, 2 \lambda_{2i-1} - k_i - 1], \bar{a}_{i-1}) \]

is smallest possible. If $k_i = 0$ then we define $\bar{r}_{2i-1}, \bar{s}_{2i-1}$ in such a way that $\bar{r}_{2i-1} = \bar{s}_{2i-1} = 1$.

To define $\bar{r}_i, \bar{s}_i$ for $i$ even we use the following inequalities:

\[ d(A_i, B_i + \tilde{a}) \leq d(A_i \times A_2, B_i \times B_2), \quad d(A_2, B_2 + \tilde{a}) \leq 3 d(A_1 \times A_2, B_1 \times B_2). \]

Here $A_i, A_2, B_i, B_2$ are blocks, $|A_i| = |B_i|, |A_2| = |B_2|$ and $\tilde{a} \in Z_2$ is such that $d(A_i, B_i + \tilde{a}) \leq \frac{1}{2}$. The above inequalities are simple properties of the distance. Applying them to (19) and using (18) we obtain

\[ d((\beta^{2i-1} + F_{2i-1}) (\beta^{2i} + F_{2i-1}) [\lambda_{2i-1} - k_i, 2 \lambda_{2i-1} - k_i - 1], \bar{a}_{i-1}) \leq \eta_i + \frac{1}{\lambda_{2i}}, \]

\[ (20) \quad \frac{k_i}{\lambda_{2i-1}} d(\beta^{2i} + F_{2i-1}, a_i a_i [1, \lambda_{2i}]) \]

\[ + \left(1 - \frac{k_i}{\lambda_{2i-1}}\right) d(\beta^{2i} + \bar{s}_{2i-1}, a_i) \leq 3 \left(\eta_i + \frac{1}{\lambda_{2i}}\right). \]

Notice that the last inequalities are also valid for $w_i = 0$ if we put $k_i = 0$.

Suppose that $p_i \neq 0$. Let $\bar{r}_{2i}, \bar{s}_{2i} \in Z_2$ be integers such that

\[ d((\beta^{2i} + \bar{s}_{2i}) (\beta^{2i} + \bar{r}_{2i}) [\lambda_{2i} - m_i, 2 \lambda_{2i} - m_i - 1], \tilde{a} i) \]

is smallest possible. If $m_i = 0$ then we can define $\bar{r}_{2i}, \bar{s}_{2i}$ in such a way that $\bar{r}_{2i} = \bar{s}_{2i}$. As above, we establish the inequalities

\[ d((\beta^{2i} + \bar{s}_{2i}) (\beta^{2i} + \bar{r}_{2i}) [\lambda_{2i} - m_i, 2 \lambda_{2i} - m_i - 1], \tilde{a} i) \leq \eta_i + \frac{1}{\lambda_{2i}}, \]

\[ (21) \quad \left(1 - \frac{m_i}{\lambda_{2i}}\right) d(\beta^{2i+1} + \bar{r}_{2i}, \tilde{a} i) \]

\[ + \frac{m_i}{\lambda_{2i}} d(\beta^{2i+1} + \bar{s}_{2i}, \tilde{a} i [1, \lambda_{2i+1}]) \leq 3 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right). \]

These inequalities are also true for $p_i = 0$ if we put $m_i = 0$.

Now we can define numbers $\tilde{q}_i, t \geq 0$. Namely we set $\tilde{q}_0 = 0$, and for $t \geq 1$

\[ \tilde{q}_{2i-1} = \begin{cases} \lambda_{2i-1} - k_i \text{ (mod } \lambda_{2i-1}) & \text{ if } m_{2i-1} \neq 0, \text{ or } m_{2i-1} = 0 \text{ and } \tilde{q}_{2i-2} = 0, \\ \lambda_{2i-1} - k_i - 1 & \text{ if } m_{2i-1} = 0 \text{ and } \tilde{q}_{2i-2} = \lambda_{2i-2} - 1, \\ \lambda_{2i-1} - k_i - 1 & \text{ if } k_i \neq 0, \text{ or } k_i = 0 \text{ and } \tilde{q}_{2i-2} = \lambda_{2i-2} - 1, \\ \lambda_{2i} & \text{ if } k_i = 0 \text{ and } \tilde{q}_{2i-2} = 0. \end{cases} \]

To check that the numbers $\tilde{q}_i$ and $\bar{r}_i, \bar{s}_i \in Z_2$ satisfy (17) we use the following, easily verified, equalities:

\[ d((\beta^{2i} + \bar{s}_{2i}) (\beta^{2i} + \bar{r}_{2i}) [\lambda_{2i} - m_i, 2 \lambda_{2i} - m_i - 1], \tilde{a} i) \]

\[ = d((\beta^{2i} + \bar{s}_{2i}) (\beta^{2i} + \bar{r}_{2i}) [m_i + \lambda_{2i} - 1], \tilde{a} i) + \]

\[ = d((\beta^{2i} + \bar{s}_{2i}) (\beta^{2i} + \bar{r}_{2i}) [m_i + \lambda_{2i} - 2], \tilde{a} i) + \]

\[ = d((\beta^{2i} + \bar{s}_{2i}) (\beta^{2i} + \bar{r}_{2i}) [m_i + \lambda_{2i} - 1], \tilde{a} i) + \]

\[ + d((\beta^{2i} + \bar{s}_{2i}) (\beta^{2i} + \bar{r}_{2i}) [m_i + \lambda_{2i}], \tilde{a} i). \]

The above, (20) and (21) imply

\[ \left(1 - \frac{m_i}{\lambda_{2i}}\right) D_{2i} + \frac{m_i}{\lambda_{2i}} D_{2i+1} \leq 2 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right) + 3 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right) \]

\[ \text{if } m_i \neq 0 \text{ or } m_i = 0 \text{ and } \tilde{q}_2 = 0, \]

\[ \tilde{B}_{2i+1} \leq 2 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right) + \frac{1}{\lambda_{2i+1}} + 3 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right) \]

\[ \text{if } m_i = 0 \text{ and } \tilde{q}_2 = \lambda_{2i} - 1, \]

and

\[ \frac{k_i}{\lambda_{2i-1}} D_{2i} + \left(1 - \frac{k_i}{\lambda_{2i-1}}\right) \tilde{D}_{2i} \leq 2 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right) + 3 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right) \]

\[ \text{if } k_i \neq 0 \text{ or } k_i = 0 \text{ and } \tilde{q}_{2i-1} = \lambda_{2i-1} - 1, \]

\[ D_{2i} < 2 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right) + 3 \left(\eta_i + \frac{1}{\lambda_{2i+1}}\right) \text{ if } k_i = 0 \text{ and } \tilde{q}_{2i-1} = 0. \]

To obtain (17) we observe that if $m_i \neq 0$ and $k_i \neq 0$ then

\[ \frac{m_i}{\lambda_{2i}} \tilde{q}_{2i} \leq \frac{1}{\lambda_{2i}} \left(\frac{1}{\tilde{q}_{2i+1}}\right) \left(\frac{1}{\tilde{q}_{2i-1}}\right) \leq \frac{1}{\lambda_{2i}}, \]

and the series $\sum_{0}^{n} \frac{1}{\lambda_{2i}}, \sum_{0}^{n} \eta_i$ are convergent.
This finishes the proof of the theorem.

Remark 2. One can show that the assumption \( \sum_{k=0}^\infty 1/\lambda_k < \infty \) may be omitted. This can be proved by using Theorem 5 and not difficult, but laborious computations.

We finish this paper with another version of Theorem 5 we have used in \( \S \) 1.

**Theorem 5'.** Let \( x = b^0 \times b^1 \times \cdots, y = b^0 \times b^1 \times \cdots \) be continuous Morse sequences, \( |b^0| = |b^1| = \lambda_1, t \geq 0 \) and \( \sum_{k=0}^\infty 1/\lambda_k < \infty \). Then \( \theta (x) \) and \( \theta (y) \) are metrically isomorphic if there exist numbers \( q_k, 0 \leq q_k \leq \lambda_k - 1 \), and \( r_k, s_k \in \mathbb{Z} \) such that

\[
\sum_{k=0}^\infty d((b^k + r_k)(b^k + s_k)[q_k, q_k + \lambda_k - 1], \beta^k) < \infty,
\]

\[
\sum_{k=0}^\infty \min(1 - \frac{q_k}{\lambda_k}, \frac{q_k}{\lambda_k}) d(r_k, s_k + \lambda_k - 1, b^k + 1) < \infty.
\]

The proof of this theorem follows immediately from Theorem 5. Namely, let

\[
r_0 = r_0, \quad r_{k+1} = \begin{cases} 
\lambda_{k+1} + r_k & \text{if } q_k < \lambda_k - q_k - 1, \\
r_{k+1} + s_k & \text{otherwise},
\end{cases}
\]

\[
s_0 = s_0, \quad s_{k+1} = \begin{cases} 
\lambda_{k+1} + s_k & \text{if } q_k < \lambda_k - q_k - 1, \\
q_{k+1} + s_k & \text{otherwise},
\end{cases}
\]

\[
q_0 = \bar{q}_0, \quad q_k = \begin{cases} 
\bar{q}_k & \text{if } q_k < \lambda_k - q_k - 1, \\
q_k + 1 \mod \lambda_k & \text{otherwise}.
\end{cases}
\]

Then it is easy to verify that

\[
\left(1 - \frac{q_k}{\lambda_k}\right) D_{k+1} + \frac{q_k}{\lambda_k} \beta_{k+1} \leq d((b^l + r_k)(b^l + s_k)[q_k, q_k + \lambda_k - 1], \beta^l) + \min \left(\frac{q_k}{\lambda_k}, 1 - \frac{q_k}{\lambda_k}\right) \{D_{k+1} - \beta_{k+1}\},
\]

\[
\left(1 - \frac{q_k}{\lambda_k}\right) D_{k+1} + \frac{q_k}{\lambda_k} \beta_{k+1} \geq \frac{1}{2} \left(1 - \frac{q_k}{\lambda_k}\right) d((b^l + r_k)(b^l + s_k)[q_k, q_k + \lambda_k - 1], \beta^l) + \min \left(\frac{q_k}{\lambda_k}, 1 - \frac{q_k}{\lambda_k}\right) \{D_{k+1} + \beta_{k+1}\}.
\]

It remains to observe that

\[
D_{k+1} + \beta_{k+1} \geq d((r_k, s_k)[q_k, q_k + \lambda_k - 1], \beta^l) + \min \left(\frac{q_k}{\lambda_k}, 1 - \frac{q_k}{\lambda_k}\right) \{D_{k+1} + \beta_{k+1}\},
\]

\[
|D_{k+1} - \beta_{k+1}| \leq d((r_k, s_k)[q_k, q_k + \lambda_k - 1], \beta^l) + \min \left(\frac{q_k}{\lambda_k}, 1 - \frac{q_k}{\lambda_k}\right) \{D_{k+1} + \beta_{k+1}\}, \quad t \geq 0.
\]

References


Received October 8, 1986

(2224)