On duality and interpolation for spaces of polyharmonic functions

by

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Abstract. We extend the results from our previous works on duality and interpolation to the case of $m$-polyharmonic functions, i.e., functions $u$ for which $\Delta^su = 0$. We give an explicit characterization of Sobolev and Besov spaces of such functions for $1 \leq p < \infty$, $-\infty < s < \infty$ and study the duality and interpolation relations between them. We also apply the results obtained to the study of the $\partial$-Neumann problem and the Bergman projection.

I. Introduction. The present paper is the final part of a long series of papers ([18]-[24]) devoted to various aspects of duality and interpolation in spaces of harmonic functions. We generalize the results proved in [18]-[24] to the case of spaces of polyharmonic functions of finite order $m$, i.e., functions $h$ for which $\Delta^m h = 0$. We call these functions $m$-polyharmonic. For information about polyharmonic functions see [23].

We make extended use of the fact, that Sobolev and Besov norms restricted to the space of all $m$-polyharmonic functions on a smooth bounded domain are equivalent. In particular, we use it to get new estimates and duality results in the limit cases $p = 1$ and $p = \infty$ which will be proved in Appendix 2. This also permits us to get an explicit description of the Sobolev spaces $\text{Harm}_s(m)(D)$ of $m$-polyharmonic functions for all $-\infty < s < \infty$, $1 \leq p < \infty$. The equality $\text{Harm}_s(m)(D) = B^s_{p,\infty} \cap \text{Harm}(m)(D)$, $-\infty < s < \infty$, $1 \leq p < \infty$, also explains the slightly astonishing interpolation results, which were proved for the spaces of harmonic functions in [22] and [24] and are here generalized to the spaces of $m$-polyharmonic functions. We hope that the estimates concerning the polyharmonic functions can be useful in the study of the $\partial$-Neumann problem. In fact, this was the main motivation to do the present work. We also discuss the possible generalizations of the above results.

Before we shall state our results more precisely we need to recall some definitions, notation and facts. We always denote by $D$ a bounded domain in $\mathbb{R}^n$ with boundary of class $C^\infty$, and by $g$ its defining function, i.e., $g \in C^\infty(D)$, $D = \{x \in \mathbb{R}^n : g(x) < 0\}$, $\text{grad} g \not\equiv 0$ on $\partial D$. By $W^2_p(D)$, $1 < p < \infty$, $-\infty < s < \infty$, we denote the Sobolev spaces on $D$, defined as follows. If $s \geq 0$ is an
integer then $W^s_p(D)$ is the usual space of functions whose $s$th derivatives belong to $L^p(D)$, with

$$
\|f\|_{W^s_p} = \|f\|_{L^p(D)} + \sum_{|\alpha| = s} \|\partial^\alpha f/\partial x^\alpha\|_{L^p(D)}.
$$

We then denote by $W^1_p(D)$ the closure of $C^\infty_0(D)$ in $W^1_p(D)$. By $W^{-1}_p(D)$, $1 < p < \infty$, we denote the dual space to $W^1_p(D)$ with respect to the usual $L^2$ scalar product $(\cdot, \cdot)_0$ on $D$, $p = 2(p-1)$. It consists of distributions $g$ on $D$ such that

$$g = \sum_{|\alpha| = s} D^\alpha g_\alpha + g_0,$$

where $g_0, g_\alpha \in L^p(D)$ (see [25]). We shall need the following important properties of the above-defined spaces:

(a) The mapping $f \mapsto |f|^{q^*}$ maps continuously $W^1_p(D)$ into $L^p(D)$.

(b) The mapping $f \mapsto Df$ is an isomorphism between $W^1_p(D)$ and $W^{-1}_p(D).

Property (a) follows from Muckenhoupt's inequalities [27] (see [25], Theorem 1.3.1/2). Property (b) follows from [26, Theorem 5.A] (it could also be proved by using the Hölder estimates from [1] and interpolation). For $p = 2$, property (b) is elementary (see [18]).

If $s$ is not an integer we define

$$W^s_p(D) = \{W^k_p(D), W^{k+1}_p(D)\}_{k=0}^{\infty},$$

the value of the complex interpolation functor at $\theta = s - [s]$ (for the definitions of the complex interpolation functor [7, 14] and the real interpolation functor [7, 14]) see [7, 14] and [31]. If $s > 0$ then the space $W^s_p(D)$ is the dual of

$$W^s_q(D) \equiv \{W^{k+1}_q(D), W^k_q(D)\}_{k=0}^{\infty}, \quad q = \frac{p}{p-1}.$$ 

It follows from [31], 3.4.3, that $W^s_p(D)$ is equal to the closure of $C^s_0(D)$ in $W^1_p(D)$ for $s \neq k + 1/q$, $k = 0, 1, \ldots$ If $s = k + 1/q$ there is a continuous inclusion $W^s_p(D) = C^s_0(D) \subset W^1_p(D)$. A simple duality and interpolation argument shows that the mapping $f \mapsto |f|^{q^*}$ maps continuously $W^1_p(D)$ into $L^p(D)$. The results from [31] also imply that the spaces $W^s_p(D)$ defined as above are the spaces of the restrictions of functions (distributions) from $W^1_p(R^n)$, the Sobolev spaces on $R^n$. If $s$ is not an integer $W^s_p(R^n)$ is often called "the space of Bessel potentials". Note that $W^s_p(R^n) = F^s_p(R^n)$ in the notation of [31]). We can thus extend the above definition to the case $s \leq 1$, by putting $W^s_p(D) = \text{the restriction of } W^s_p(R^n)$ to $D$, $-\infty < s < \infty$.

If $s > 0$ is not an integer then the Besov space can be defined for $1 < p < \infty$ as follows:

$$B^s_p(D) = \left\{f \in L^p(D) : \|f\|_p = \|f\|_{L^p(D)} \right\} + \sum_{|\alpha| = s} \left\{\frac{\|D^\alpha f(x) - D^\alpha f(y)\|_{L^p(D)}}{|x - y|^{(1-\theta)||\alpha||}}\right\}^{1/s} < \infty.$$ 

If $s > 0$ is an integer then we put

$$B^s_p(D) = \left[ B^{s+1}_p, B^{s-1}_p \right]_{1/2}.$$ 

For $s = 1/p + k$ and $1 < p < \infty$ we define $B^s_p(D)$ as the closure of $C^s_0(D)$ in $B^s_p(D)$, and $B^s_p(D)$, $q = p/(p-1)$, as the dual of $B^s_p(D)$ as in the case of Sobolev spaces. If $s = 1/p + k$ we put

$$B^s_p(D) = \left[ B^{s+1}_p(D), B^{s-1}_p(D) \right]_{1/2},$$ 

Again the results from [31] imply that the spaces defined above are the spaces of the restrictions of functions (distributions) from $B^s_p(R^n)$.

$A_s(D)$ denotes the usual Hölder space and $L^p(D, |q|)$ the $L^p$ space with respect to the measure $|q| dx$ on $D$ if $p < \infty$. $L^p_s(D, |q|)$ is the space of functions $f$ such that $f|q| \in L^p(D)$, with norm $\|f|q|\|_{L^p_s(D, |q|)} = \|f|q|\|_{L^p(D)}$.

Harm(m)(D) will denote the space of all m-polynomial functions on $D$.

Then

$$L^p_s \text{ Harm}(m)(D, |q|) = L^p_s(D, |q|) \cap \text{ Harm}(m)(D),$$ 

$A_s \text{ Harm}(m)(D) = A_s \text{ Harm}(m)(D)$, 

$B^s_p \text{ Harm}(m)(D) = B^s_p(D) \cap \text{ Harm}(m)(D)$, 

$\text{ Harm}^m(D) = W^\infty_p(D) \cap \text{ Harm}(m)(D)$.

We shall also need the space of Bloch m-polynomial functions $\text{ Bl Harm}(m)(D)$, defined as the space of m-polynomial functions $h$ such that $h|\infty| = \sup_{x \in D} (|h(x)| + |\nabla h(x)|) < \infty$.

Now, let $T$ be a differential operator of order $2m$ with $C^\infty$ coefficients on $R^n$ such that $\sigma_T$, the principal symbol of $T$, does not vanish on $D \times \{0\}$. We also assume that the Dirichlet problem $Tu = u$, $u$ vanishes on $D$ up to order $m - 1$, is uniquely solvable. Denote the operator solving this problem by $G_T$. Let $T^* T$ denote the formal adjoint of $T$. Note that $T^* T$ and $T^* T^*$ must have the same properties as $T$.

Let $P_T$ denote the orthogonal projection from $L^2(D)$ onto the space $L^2(D) \cap Ker T$. We have

$$P_T u = u - T^* G_T u = T^* (G_{T^*} u - G_{T^*} Tu).$$
If \( T = \Delta^m \) then \( \text{Ker} \ T \) is the space of \( m \)-polyharmonic functions. In this case we write \( P_r = P_m \) and \( G_r = G_m \).

The estimates from [1] and [26] imply that for every strongly elliptic \( T, P_r \) maps continuously \( W^{s,1}(D) \) into \( W_r^{s,1}(D) \), \( s \geq 0 \), and \( A_r \) into \( A_r \). By using the real interpolation functor one can easily check that \( P_r \) also maps \( P_r \) into \( P_r \), \( s \geq 0 \). Let us now define the family of Bell's operators \( L_r \) in the following manner:

\[
L_r u = u - T^* \left( \sum_{k=0}^{r-1} \frac{\partial^k}{|\partial|} \right) u,
\]

where \( \phi \) is an arbitrarily chosen \( C^\infty \) function equal to 1 in a neighborhood of \( \partial D \) and to 0 in a neighborhood of the set \( \{ \partial D \} \).

It follows directly from the definition of \( L_r \) that if \( u \in C^\infty(\bar{D}) \) then \( P_r u = P_r T_r \) and \( L_r u \) vanishes on \( \partial D \) up to order \( r-1 \). The operators \( L_r \) were defined for \( T = \Delta \) by S. Bell in [5]. Earlier S. Bell defined similar operators for \( \Delta \) and the Bergman projection \( B \) onto the \( L^2 \) space of holomorphic functions in [4]. These operators were used to study the duality relations between spaces of holomorphic functions in [4], [6], [13] and [3], and between spaces of harmonic and pluriharmonic functions in [5], [18], [19], [20], [21], [22] and [24]. E. Straube in [28] and [29] constructed a single operator \( L_r \) which maps continuously \( W_r^{s,1}(D) \) into \( L^2(D) \), \( 0 \leq s < \infty \), such that \( P_r u = P_r L_r u \). However, the construction of \( L \) is more complicated than the explicit construction of Bell's operators.

We denote by \( \langle \cdot, \cdot \rangle_{T, \alpha} \) the sesquilinear pairing defined as follows:

\[
\langle u, v \rangle_{T, \alpha} = \langle u, L_r^{\alpha} \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the usual \( L^2 \) scalar product.

If \( T = \Delta^m \) then we denote \( L_r^{\alpha} \) by \( L_r^{\alpha} \) and \( \langle \cdot, \cdot \rangle_{T, \alpha} \) by \( \langle \cdot, \cdot \rangle_{T} \). We shall show that \( L_r^{\alpha} \) maps \( L_r \text{Har}_{m}(\partial D) \) into \( L^2(D) \), and thus if \( u, v \in L_r \text{Har}_{m}(\partial D) \) then \( \langle u, v \rangle_{T, \alpha} = \langle u, v \rangle_{T} \).

II. Statement of the results.

Proposition 1. For every integer \( m > 0 \) the projection \( P_m \) maps continuously:

\[
\begin{align*}
(a) & \ L^\alpha(D, \partial D^m) \to A_r \text{Har}_{m}(D), \quad \alpha > 0; \\
(b) & \ L^\alpha(D, \partial D^m) \to \text{Har}_{m}(D, \partial D^m), \quad 0 < \beta < 1; \\
(c) & \ L^\alpha(D) \to \text{Bl Har}_{m}(\partial D) \\
(d) & \ L^\alpha(D, \partial D^m) \to \text{Har}_{m}(\partial D), \quad -1+1/p < s < \infty, \quad 1 < p < \infty.
\end{align*}
\]

Proposition 2. (a) For any integers \( m \geq 1 \) and \( k \geq 0 \) the mapping \( \mathcal{T}_r u = \chi u \) maps continuously:

\[
\begin{align*}
1) & \ \text{Har}_{m}(\partial D) \to W_r^\alpha(D), \quad -\infty < s < \infty, \quad 1 < p < \infty; \\
2) & \ A_r \text{Har}_{m}(D) \to A_r \text{Har}_{m}(D); \\
3) & \ L^\alpha(D, \partial D^m) \to \text{Har}_{m}(D, \partial D^m), \quad 0 < \beta < 1; \\
4) & \ W_r^\alpha(d \partial D^m) \to \text{Bl Har}_{m}(\partial D).
\end{align*}
\]

(b) For any integers \( m \geq 1 \) and \( r \geq 1 \) the operator \( L_r^{\alpha} \) maps continuously:

\[
\begin{align*}
1) & \ \text{Har}_{m}(D) \to W_r^\alpha(D) \text{Har}_{m}(D); \\
2) & \ A_r \text{Har}_{m}(D) \to \text{Har}_{m}(D), \quad r > 1; \\
3) & \ W_r^\alpha(D) \to L^\alpha(D, \partial D^m).
\end{align*}
\]

Propositions 1 and 2 imply the following duality theorem.

Theorem 1. (a) \( \text{Har}_{m}(D) \) and \( \text{Har}_{m}(D) \) are mutually dual via the pairing \( \langle \cdot, \cdot \rangle_{T}, \quad 0 < s \leq r \) and \( 1/p + 1/q = 1, \quad 1 < p < \infty \). \( A_r \text{Har}_{m}(D) \) represents the dual of the space \( L^r(D, \partial D^m) \), which is the closure of \( L_r \text{Har}_{m}(D, \partial D^m) \) in \( L^r(D, \partial D^m) \), via the pairing \( \langle \cdot, \cdot \rangle_{T} \) if \( r > x \).

\( B_r \text{Har}_{m}(D) \) represents the dual of \( L^r(D, \partial D^m) \) via the pairing \( \langle \cdot, \cdot \rangle_{T} \) if \( r > 1 \).

(b) If \( s < 1/p \), then \( B_r \text{Har}_{m}(D) = L^r(D, \partial D^m) \), with equivalent norms.

Propositions 1 and 2 and duality arguments permit us to get the following.

Theorem 2. For any \( -\infty < s < \infty, \quad 1 < p < \infty \) and any integer \( m \geq 1 \)

\[
B_r^p \text{Har}_{m}(D) = \text{Har}_{m}(D)
\]

with equivalent norms.

Remark 1. Theorem 2 is of some interest only for small \( s \), namely for \( s \leq 1 \). Note that if \( s > m + 1/2 \), then the spaces \( B_r^p \text{D} \) and \( W_r^\alpha(D) \) have the same traces on \( \partial D \) equal to \( \prod_{j=1}^{m} B_r^p \partial D^{m-j-1/2} \). Thus, roughly speaking, the \( m \)-polyharmonic extension of these traces must be the same. Interpolation with \( L^r \text{Har}_{m}(D) \) will prove our theorem for \( s > m + 1/2 \). (See also the proof of Proposition 3 below.)

Remark 2. Theorem 2 together with part (b) of Theorem 1 gives us an explicit description of all Sobolev spaces of \( m \)-polyharmonic functions. If \( s > 0 \) is an integer, \( \text{Har}_{m}(D) \) is the classical Sobolev space. If \( s > 0 \) is not
an integer then the equivalent Besov norm is given by an explicit formula (see Introduction). If \( s < 1/p \) then the norm in \( \text{Harm}_m^s(m)(D) \) is the \( L^p(D, |\cdot|^{-m}) \) norm. Note that if \( 0 < s < 1/p \) then the Besov and \( L^p(D, |\cdot|^{-m}) \) norms are equivalent on \( m \)-polyharmonic functions for every \( m \). Thus we get the following

**Corollary 1.** Let \( f \) be a polyharmonic function of finite order on \( D \). For any \( 0 < \beta < 1 \) and \( 1 < p < \infty \) the following conditions are equivalent:

\[
\begin{align*}
(a) & \quad \int_B |f|^q dV < \infty, \\
(b) & \quad \int_D \frac{|f(x)-f(y)|^p}{|x-y|^{1+eta}} dV < \infty.
\end{align*}
\]

We now state the following

**Theorem 3.** Let \( A_p^1, A_p^2 \) be equal to \( \text{Harm}_m^s(m)(D) \) for \( 1 < p < \infty \), \( -\infty < s < \infty \), to \( A_1^s, \text{Harm}_m(m)(D) \) for \( s > 0 \), \( p = \infty \), to \( \text{BlHarm}_m(m)(D) \) for \( s = 0 \), \( p = \infty \), and to \( L^\infty(D, |\cdot|^q) \) for \( 0 > s > -\infty \), \( p = \infty \). Then

\[
\begin{align*}
[A_p^{1}, A_{p_2}] & = A_p^1 \quad \text{if} \quad \min(p_1, p_2) < \infty, \\
[A_{p_1}, A_{p_2}] & = A_{p_1, p_2}^s \quad \text{if} \quad s \in (1-\theta)s_1 + \theta s_2, \quad 0 < \theta < 1,
\end{align*}
\]

where

\[
1 - \frac{1 - \theta}{p_1} - \frac{\theta}{p_2} = s \in (1-\theta)s_1 + \theta s_2, \quad 0 < \theta < 1,
\]

and \([A, B]^*\) denotes the completion of \([A, B]\) with respect to \( A + B \).

**Theorem 3** extends the results of [22] and [24].

**Remark 3.** Theorem 2 and some interpolation results similar to Theorem 3 are valid in a more general case. Let \( T \) be a differential operator of order \( 2m \) with \( C^\infty \) coefficients such that

\[
\langle \langle Tp, \varphi \rangle \rangle \geq c \langle \langle \varphi \rangle \rangle^2 \quad \text{on} \quad W^m_2(D).
\]

Then

\[
\begin{align*}
(a) & \quad (\text{Harm}_m^s(D)) = B_p^s \circ \text{Harm}_m^s(D), \\
(b) & \quad (\text{Harm}_m^s(D)) = W_p^s(D) \cap \text{Ker} T, \\
& \quad B_p^s \circ \text{Harm}_m^s(D) = B_p^s(D) \cap \text{Ker} T, \quad -\infty < s < \infty, \quad 1 < p < \infty.
\end{align*}
\]

If \( T \) is in addition selfadjoint then the spaces \( A_p^s = B_p^s \circ \text{Harm}_m^s(D), \quad -\infty < s < \infty, \quad 1 < p \leq \infty \), have the same interpolation properties as in Theorem 3.

**Remark 3 will be proved later.**

Since \( B_p^m \circ \text{Harm}_m^s(D) = A_s(D) \cap \text{Ker} T, \) and \( B_p^m \circ \text{Harm}_m^s(D) = L^1 \circ \text{Harm}_m^s(D) \) by (a), we see that (b) gives the possibility of interpolation between Hölder and \( L^p \) spaces of functions from \( \text{Ker} T \) as in the case of \( m \)-polyharmonic functions. However, we have no explicit characterization of the spaces \( B_p^m \circ \text{Harm}_m^s(D) \) for \( s < 0 \) (see Problem 4 at the end of this paper).

We can also consider the more general scale of spaces \( F^m_p(D) \) described in [31]. Note that \( F^m_p(D) = W^m_p(D) \) and \( F^m_p(D) = B^m_p(D) \). It turns out that if \( T \) is such as in Remark 3 then \( F^m_p \circ \text{Harm}_m^s(D) = \text{Harm}_m^s(D) \) for all \( -\infty < s < \infty, \quad 0 < p < \infty, \quad 0 < q < \infty \).

It is also possible to use Theorem 1 to get an explicit characterization of the spaces \( B^m_p \circ \text{Harm}_m^s(D) \) for \( s < 0 \). We deal with those general Triebel–Lizorkin spaces and Besov spaces in Appendix 1 in the final part of this paper.

The duality theorem cannot be extended to the case \( p < 1 \) since Sobolev and Besov spaces are only quasi-Banach for \( p < 1 \). However, the recent results of Franke [10], [11] (also mentioned in the Russian edition of [31]) permit us to show that \( P_p \) maps continuously \( F^m_p \) into \( F^m_p \), for \( p < 1 \), \( s > n/(p-1) \), and that one can adjoin the spaces \( F^m_p \circ \text{Harm}_m^s(D) \) to \( L^1 \circ \text{Harm}_m^s(D) \), \( s < 0 < s < \infty, \quad p < 1 \), to the interpolation scale described in Remark 3. We deal with this case in Appendix 2. In particular, we show there that \( B^m_p \circ \text{Harm}_m^s(D) = L^1 \circ \text{Harm}_m^s(D) \) for \( s > 0 \).

Putting aside these generalizations, we return to our polyharmonic functions.

**Theorem 4.** Let \( k \geq m > 0 \) be integers. The projection \( P_k \) restricted to \( \text{Harm}(k)(D) \) is bounded in every \( L^t(D, |\cdot|^q) \) norm, \( 0 < t < \infty, \quad 1 \leq p < \infty, \quad \min(1/p, b) < \infty \).

Let now \( D \) be a strictly pseudoconvex domain in \( C^\infty \). We denote by \( B \) the orthogonal projection from \( L^2(D) \) onto the space of square-integrable holomorphic functions (the Bergman projection). We proved in [21] and [22] that \( B \) restricted to the space of harmonic functions is bounded in \( L^p(D, |\cdot|^q) \) norms if \( 1 < p < \infty, \quad t > 0 \). Thus Theorem 4 yields immediately

**Theorem 5.** For every integer \( m > 0 \), the projection \( B \) restricted to \( \text{Harm}(m)(D) \) is bounded in every \( L^t(D, |\cdot|^q) \) norm for \( 0 < t < \infty, \quad 1 \leq p < \infty \).

Much information about Sobolev spaces of holomorphic functions on the unit ball can be found in [3] (see also [8]). It should also be mentioned that Theorem 5 remains valid if we replace the Bergman projection \( B \) by the orthogonal projection \( Q \) onto the space of pluriharmonic functions or the projection \( S \), onto the space of the real parts of holomorphic functions (see [20]).
Let $D$ be a strictly pseudoconvex domain, and let $\varphi$ be a defining function of $D$ which is strictly plurisubharmonic in a neighborhood of $\partial D$. Let

$$L = \sum_{i,j} \partial \varphi / \partial z_i \partial \varphi / \partial z_j \wedge dz_i \wedge dz_j$$

be the Kähler metric on $D$ induced by the potential $\varphi$. Denote by $\langle \cdot, \cdot \rangle_k$ the scalar product induced by $L$.

Let $\Box_k = \Box_k^2 + \Box_k^3$ (the adjoint is taken with respect to $\langle \cdot, \cdot \rangle_k$). Let $N_k$ be the operator solving the $\delta$-Neumann problem $\Box_k \omega = \xi$, $\omega \wedge (0, q)$-differential forms, $\omega \in \text{Dom} \Box_k$ (see [9] for details). Lieb and Range in [15]-[17] constructed an integral representation for the operator $\Box_k^2 N_k$ and obtained the Hölder estimates for the operators $\Box_k^2 N_k$ and $N_k$. They proved that $\Box_k^2 N_k$ maps $A_p(0, q + 1)(D)$ into $A_{p+1}(0, q)(D)$ and $N$ maps $A_p(0, q)(D)$ into $A_{p+1}(0, q)(D)$ ($A_p(0, q)(D)$ denotes here the space of $(0, q)$-forms with coefficients from $A_p(D)$).

Remark 3 permits us to show that the Lieb-Range results imply the following

**Theorem 6.** The operator $N_k$ maps continuously $W^p_2(0, q)(D)$ into $W^{p+1}_{2q}(0, q)(D)$ for $0 < s < \infty$, $1 < p < \infty$, and the operator $\Box_k^2 N_k$ maps continuously $W^p_2(0, q + 1)(D)$ into $W^{p+1}_{2q+1}(0, q)(D)$, $0 < s < \infty$, $1 < p < \infty$.

$W^p_2(0, q)(D)$ denotes here the space of $(0, q)$-forms with coefficients from $W^p_2(D)$.

**Proof.** Let $D$ be equal to the unit ball then the usual Euclidean metric on $D$ is equal to $L$ since $\varphi = |z|^2 - 1$. Thus we have $\Box_k = -\Delta$ and we get the following application of Theorem 1:

**Corollary 2.** If $D = B(0, 1)$ then $N$ extends to a continuous mapping from the space of $(0, q)$-forms with coefficients in $L^s$ (m,1)(D, $|q|^{-l}$) into the space of $(0, q)$-forms with coefficients in $L^s$ (m+1,1)(D, $|q|^{-l}$) if $t < p - 1$ and in $\text{Harm}^{s-1}(m+1)(D)$ if $t \leq p - 1$, $1 < p < \infty$.

The operator $\Box_k^2 N$ extends to a mapping from the space of $(0, q + 1)$-forms with coefficients in $L^s$ (m,1)(D, $|q|^{-l}$) into the space of $(0, q + 1)$-forms with coefficients in $L^s$ (m+1,1)(D, $|q|^{-l}$) if $t > p - 1$ in $\text{Harm}^{s-1}(m+1)(D)$ if $t < p/2 - 1$ and in $\text{Harm}^{s-1/2}(m+1)(D)$ if $t \leq p/2 - 1$. Here $1 \leq p < \infty$. [For $p = 1$ we take $L^1$ (m,1)(D, $|q|^{-l}$), which is the closure of $L^2$ (m,1)(D, $|q|^{-l}$).]

The last corollary justifies our conjecture from [22] that the operators $N$ and $\Box_k^2 N$ should "behave better" on forms with harmonic coefficients. For $(0, 1)$-forms the Hölder and $W^p_2$ estimates were first proved by Greiner and Stein [12].

We end the present paper with a list of open problems.

**III. Proofs.**

1. Proof of Proposition 1. We begin with two lemmas:

**Lemma 1.** (a) For $h \in \text{Harm}^s(m)(D)$ we have

$$|D^k h(x)| \leq c \langle |h|_s \rangle^{k-\alpha} \quad \text{if } k > [\alpha].$$

(b) For $h \in L^s(\text{Harm}^s(m)(D))$ we have

$$|D^k h(x)| \leq c \langle |h|_s \rangle^{k-\alpha} \quad \text{if } k > [\alpha].$$

(c) For $h \in L^s(\text{Harm}^s(m)(D, |q|))$ we have

$$|D^k h(x)| \leq c \langle |h|_s \rangle^{k-\alpha} \quad \text{if } k > [\alpha].$$

**Proof.** Let $x \in D$, $\delta = \text{dist}(x, \partial D)$. We can assume that $x = 0$. Let $u \in \text{Harm}^s(m)(D)$. We have for $m > k > 0$

$$\int_{B_{0}(\delta/2)} A^{m-k}(x^2 - 2 \delta^2)^{2m-k-u} u(x) dV_x = \int_{B_{0}(\delta/2)} \int_{B_{0}(\delta/2)} A^{m-k} u(x) dV_x = \int_{B_{0}(\delta/2)} \int_{B_{0}(\delta/2)} A^{m-k} u(x) dV_x$$

$$= \frac{1}{\delta^2} \int_{B_{0}(\delta/2)} A^{m-k} u(x) dV_x + \frac{1}{\delta^2} \int_{B_{0}(\delta/2)} A^{m-k} u(x) dV_x = \int_{B_{0}(\delta/2)} A^{m-k} u(x) dV_x.$$

The above formula permits us to show after elementary calculations that $|D^{m-k} u(x)| \leq c \langle |u|_s \rangle^{m-k-\alpha}$ and next, by induction on $k$, that $|D^{m-k} u(x)| \leq c \langle |u|_s \rangle^{m-k-\alpha}$. In particular, $|u(x)| \leq c \langle |u|_s \rangle^{m-k-\alpha}$.

We now proceed as in the lemma in the proof of Theorem 2 in [19] and write $u_{B_{0}(\delta/2)} = u_{B_{0}(\delta/2)}$, where $h$ is harmonic, $u_{B_{0}(\delta/2)} = 0$ on $\partial B(0, \delta/2)$. We use the mean value theorem for $h$ and the fact that

$$u(x) = \int_{B_{0}(\delta/2)} G(x, y) u(y) dV_y,$$

where $G(x, y)$ is the Green function of $B(0, \delta/2)$ to get the desired estimates for all derivatives of $u$. The same procedure permits us to prove parts (b) and (c) of the lemma.

Similar estimates of the derivatives of a polyharmonic function can also be found in [2].

**Lemma 2.** Assume that Proposition 2(a1) holds for $(m - 1)$-polyharmonic functions, i.e., for each $k > 0$, $T f = \partial^k f$ maps continuously $\text{Harm}_s^p(m - 1)(D)$ into $W^p_2(D)$ for $-s < \infty$, $1 < p < \infty$. Then:
For all \( \varphi \in C^\infty (\bar{D}) \) and \( -\infty < s < \infty \), \( u \rightarrow A^s (\varphi^s u^s) \) maps continuously \( \operatorname{Harm}_s^m (D) \) into \( W_q^{s+k} (D) \).

(ii) \( T \) maps continuously \( \operatorname{Harm}_s^m (D) \) into \( W_q^{s+k} (D) \) for \( s \geq 0 \).

**Proof.** (i) Let \( u \in \operatorname{Harm}_s^m (D) \). For \( k = 1 \) we have

\[
A (\varphi u) = A (\varphi \Delta) u + \varphi \Delta u + 2 \sum_i \frac{\partial (\varphi \Delta) u}{\partial x_i} \frac{\partial u}{\partial x_i}.
\]

Now, \( \varphi \Delta u \in W_q^{s-1} (D) \) since \( \Delta u \) is \( (m-1) \)-polyharmonic. Thus \( A (\varphi u) \in W_q^{s-1} (D) \) and \( \| A (\varphi u) \|_{s-1} \leq c \| u \|_s \).

Assume now that (i) holds for all \( l \leq k \). Then

\[
A^{k+1} (\varphi^{k+1} u^k) = A^{k+1} (\varphi^{k+1} \varphi \Delta u) A^k (\varphi^{k+1} \varphi \Delta u) + (k+1) A^k (\varphi^k \varphi \Delta u)
\]

\[
+ 2 \sum_i \left( A^k \left( \varphi^{k+1} \varphi \Delta u \frac{\partial u}{\partial x_i} \right) \frac{\partial u}{\partial x_i} + (k+1) A^k \left( \varphi^k \varphi \Delta u \frac{\partial u}{\partial x_i} \right) \right) + \frac{1}{k} \Delta (A^{k-1} (\varphi^{k-1} \varphi \Delta u))
\]

\[
= c \left( \frac{\partial u}{\partial x_i} \right)^2 u_i \left( \frac{\partial u}{\partial x_i} \right)
\]

\[
\left( \frac{\partial \varphi}{\partial x_i} \right)
\]

by the assumptions of the lemma and the inductive assumption on \( k \). We have again

\[
\| A^{k+1} (\varphi^{k+1} u^k) \|_{s-1} \leq c \| u \|_s.
\]

(ii) If \( s \geq 0 \) then \( \varphi^s u = G_k (d^s \varphi^s u) \). The estimates from [26] (Theorem 5.4) yield that

\[
\| \varphi^s u \|_{s-k} \leq c \| A^s \varphi^s u \|_{s-k} \quad \text{(cf. [18], [22]).}
\]

We now prove parts (a) and (b) of Proposition 1. We proceed in exactly the same manner as in [19] and [22]. We have

\[
P_m u = u - A^s G_m d^m u = d^m (G_m u - G_m d^m G_m u).
\]

Note that the function in parentheses on the right is the \( 2m \)-polyharmonic extension of \( \operatorname{Tr} G_m u \) from \( \partial D \) to \( D \). (\( \operatorname{Tr} f \) is equal here to \( \int f \partial D, \ldots, \int f \partial D^* d_D \), \( \partial D^* d_D \), \( \partial D^* d_D \), \( \partial D^* d_D \), \( \partial D^* d_D \).)

\[
G_m u (x, y) = \int \frac{1}{|x-y|} G_m (x, y) u (y) dV_y = \int \frac{1}{|x-y|} V_m (x, y) u (y) dV_y = \int \frac{1}{|x-y|} G_m (x, y) dV_y
\]

where \( G_m (x, y) \) is a Green function for the Dirichlet problem \( A^s f = g, f \) vanishes on \( \partial D \) up to order \( m-1 \), \( V_m (x, y) \) is a fundamental solution of the equation \( A^s f = 0 \) and \( G_m (x, y) \) is \( m \)-polyharmonic with respect to \( x \).

\( (G_m (x, y) = G_m (y, x)) \) and such that for every \( y \in D \), \( G_m (x, y) \) vanishes on \( \partial D \) up to order \( m-1 \). The fundamental solution \( V_m (x, y) \) is

1. \( V_m (x, y) = c(n) |x-y|^{2m-s} \) if \( n \) is odd or if \( n \) is even and \( n > 2m \).
2. \( V_m (x, y) = c(n) |x-y|^{2m-s} \ln |x-y| \) if \( 2m \geq n \) and \( n \) is even.

Let

\[
w (x) = \int \frac{1}{|x-y|} V_m (x, y) u (y) dV_y
\]

Then

\[
P_m u = A^s (w (x) - G_m d^m w (x))
\]

is a \( 2m \)-polyharmonic extension of \( \operatorname{Tr} w (x) \) on \( \partial D \) in view of the estimates from [1] (Th. 12.10 and what follows). It remains to prove:

(a) If \( u = |q|^s m \), \( m \in L^m (D) \), \( 0 < s < \infty \), then

\[
w (x) \mid_{\partial D} \in A_{2m-s} (\partial D), \quad \frac{\partial w}{\partial n} (x) \mid_{\partial D} \in A_{2m-s-j} (\partial D), \quad 0 \leq j \leq 2m-1.
\]

(b) If \( u = m |q|^s \), \( 0 < \beta < 1 \), \( m \in L^m (D) \), then

\[
w (x) \mid_{\partial D} \in A_{2m-s} (\partial D), \quad \frac{\partial w}{\partial n} (x) \mid_{\partial D} \in A_{2m-s-j} (\partial D), \quad 0 \leq j \leq 2m-1.
\]

If \( x \in \partial D \) then

\[
w (x) = c(n) \int |x-y|^2 + c(x) q (y) \right)^{m-n} u (y) dV_y = w_0 (x)
\]

or

\[
w (x) = c(n) \int |x-y|^2 + c(x) q (y) \right)^{m-n} u (y) \ln |x-y|^2 + c(x) q (y) dV_y = w_0 (x)
\]

If \( u = |q|^s m \) we differentiate \( w_0 (x) \), \( k = 2m + |x| + 1 \) times in order to get gradient estimates and use the Hardy–Littlewood lemma. The standard calculations show that \( \| D^k w_0 (x) \| \leq c \| m \|_{L^m (\Omega)} \| q (x) \|^{m-n} \) if \( |x| = k \). Thus \( w_0 (x) \in A_{2m-s} (D) \) and \( w (x) \mid_{\partial D} \in A_{2m-s} (\partial D) \).

If \( u = m |q|^s \) we proceed as in [24], assuming first that \( \sup m \in D \) and differentiating \( w_0 (x) \) \( 2m \) times. We then have

\[
\| D^k w_0 (x) \| \leq c \| m \|_{L^m (\Omega)} \int \frac{1}{|x-y|^2 + c(x) q (y)} dV_y = \frac{1}{|x|^2} dV_x
\]
if $|\psi| = 2m$. The last integral can be written as

$$\sum \frac{(\psi / 2 \partial \psi)^2}{(\partial / \partial x)^2} \psi (y) dV_y \left(1 - \sum (\partial / \partial x)^2\right) \psi (y) dV_y$$

where $\psi (y) = \psi (y)\sum (\partial / \partial x)^2\psi (y) \psi (y) = C \in C^2 (\Omega)$, $\psi = 0$ in a neighborhood of the set $\{|\psi| = 0\}$, and $\psi = 1$ in a neighborhood of $\partial D$. The second integral is a bounded function. We can now integrate by parts in the first integral and prove that it is bounded by $c |\psi| (x)$, where $D^{1/2} w_0 (x) \leq c |\psi| (x)$ and $w_0 \in A_{2m - 1} (\partial D)$. For every $m \in L^p (D)$ there exists a sequence $\mu \to m$ in every $L^p (D)$, $p < \infty$, such that $|\mu|_{L^p} \leq |\mu|_{L^p}$. Thus we can extend the above estimates to arbitrary $m \in L^p (D)$. The same procedure can be applied to the functions $w_1 = \partial / \partial x^1 w_0$, which ends the proof of Proposition 1(a), (b).

Proposition 1(c) follows from interpolation. For all $i$ and $m$ the mapping $(\partial / \partial x_i)^* P_m$ maps continuously $L^{p} (D, |\psi|^i)$ into $L^{p} (D, |\psi|^i)$ and $L^{\infty} (D, |\psi|^{i+1})$ into $L^{\infty} (D, |\psi|^{i+1})$ by Lemma 1 ($0 < x < 1$). Thus $(\partial / \partial x_i)^* P_m$ maps continuously

$$L^{p} (D) = L^{p} (D, |\psi|^i), L^{\infty} (D, |\psi|^{i+1})$$

to

$$L^{p} (D, |\psi|^{i+1}), L^{\infty} (D, |\psi|^{i+1})$$

(see [24]). This means that $P_m$ maps continuously $L^{p} (D)$ into $B_l H^{m} (m) (D)$.

We now prove part (d) of Proposition 1 under the assumptions of Lemma 3.

As was said in the introduction, the projection $P_m$ maps continuously $L^{p} (D)$ into $L^{p} (D)$ for $1 < p < \infty$. It follows from Lemma 2 that for every $\beta$, $|\beta| = k$, the mapping $(\partial / \partial x) P_m$ maps continuously $L^{p} (D, |\psi|^k)$ into $L^{p} (D, |\psi|^k)$ since $\partial / \partial x P_m \in W^{k} (D)$ for each $1 < k$ and $u \in L^{p} (D)$ and thus $\partial / \partial x P_m \in W^{k} (D)$. Lemma 1 implies that if $k > |\beta| + 1$ then $(\partial / \partial x)^* P_m$ maps continuously $L^{p} (D, |\psi|^k)$ into $L^{p} (D, |\psi|^{k-1})$. In [22, Proposition 2] we have proved that

$$[L^{p} (D, |\psi|^k), L^{p} (D, |\psi|)]_{\theta} = L^{p} (D, |\psi|). \quad q = \frac{p}{\theta - \bar{\theta}}, \quad t = s + \frac{p \theta r}{1 - \bar{\theta}}.$$

Putting $s = 0$, $r = - \infty$ or $s = 0$, $r = - \infty$ we prove by interpolation that

$$L^{p} (D, |\psi|) \supseteq L^{p} (D, |\psi|) \supseteq L^{p} (D, |\psi|).$$

Let $s_1 = \theta_0 - \bar{\theta}_0 = 1 - \theta$. We have $L^{p} (D, |\psi|^k) \supseteq W^{k} (D)$, $p \geq 1$. Hence $P_m$ maps $L^{p} (D, |\psi|^k)$ into $W^{k} (D)$.

Thus we have proved Proposition 1(d) for $m$-polyharmonic functions under the condition that Proposition 2(a) is valid for $(m - 1)$-polyharmonic functions. Proposition 1(d) for $-1/p + 1 < s < 0$ follows immediately from Proposition 1(c) and interpolation.

2. Proof of Proposition 2 and Theorem 1. Parts (a)–(d) of Proposition 2 follow immediately from Lemma 1. Parts (b) and (c) also follow from Lemma 1 and the construction of $P_m$, since $A_{2m}$ consists of terms of the type $\partial / \partial x u$ (smooth function), where $k \geq r$ and $|\beta| < k$. Thus the fact that $L^{1} (D, H^{m} (D, |\psi|^k)) \supseteq L^{1} (H^{m} (D, |\psi|^k))$ can be extended in the same manner as in [19], namely for every $u \in H^{m} (D, |\psi|^k)$ a continuous functional on $L^{1} (H^{m} (D, |\psi|^k))$, and every continuous functional on $L^{1} (H^{m} (D, |\psi|^k))$ can be extended to a continuous functional on $L^{1} (H^{m} (D, |\psi|^k))$ and hence represented by a function $u \in L^{1} (D, |\psi|^k)$. The function $u = u \in P_m u \in A_m H^{m} (D, |\psi|^k)$ represents $\phi$ on $L^{1} (H^{m} (D, |\psi|^k))$ via $\phi \in H^{m}$. Analogously we can prove that $B_l H^{m} (D, |\psi|^k)$ represents the dual of $L^{1} (H^{m} (D, |\psi|^k))$.

Hence it remains to prove Proposition 2(a), (b) and Theorem 1 for $1 < p < \infty$. We do it by induction on $m$.

In the case of $m = 1$, i.e. in the case of harmonic functions the needed facts were proved in [22]. Let us assume that they are valid for all $k \leq m - 1$. Then Proposition 2(a) yields that Proposition 1(d) is valid for $k = m$. Proposition 1(d) and the construction of $P_m$ imply that $L^{p} (D, |\psi|^k)$ maps $L^{p} (D, |\psi|)$ into $L^{p} (D, |\psi|)$ and $L^{p} (D, |\psi|)$ into $L^{p} (D, |\psi|)$, and thus complex interpolation implies that Proposition 2(b) holds for $k = m$.

We can now repeat the proof of the theorem in [22] in order to prove that $A_{2m} H^{m} (D, |\psi|^k)$ and $A_{2m + 1} H^{m} (D, |\psi|^k)$, $1 + \frac{p}{1 - \theta} = 1$, are mutually dual via the pairing $\langle \cdot, \cdot \rangle$, $r > s$. Briefly, Proposition 2(b) yields that every $u \in B_{m} H^{m} (D, |\psi|^k)$ represents a continuous functional of $H^{m} (D, |\psi|^k)$ via $\langle \cdot, \cdot \rangle$, $r > s$. If $\phi$ is a continuous functional on $H^{m} (D, |\psi|^k)$ then it can be extended to a continuous functional on $H^{m} (D, |\psi|^k)$ and then represented by $u \in W^{k} (D)$. The function $u = u \in P_m u \in H^{m} (D, |\psi|^k)$ represents $\phi$ on $H^{m} (D, |\psi|^k)$.

If $\phi$ is a continuous functional on $H^{m} (D, |\psi|^k)$ then it can be extended to $\phi \in W^{k} (D)$. Since $P_m$ maps $W^{k} (D) \supseteq W^{k} (D)$ into $W^{k} (D)$, $W^{k} (D)$, and self-adjoint, $P_m$ extends to a continuous mapping from $W^{k} (D)$ into $W^{k} (D)$, $W^{k} (D)$. Thus $P_m (\nu)$ satisfies the identity $\nu \in H^{m} (D, |\psi|^k)$. (The above proof is based on the ideas from [4], [5] and [8]).

On the other hand, $H^{m} (D, |\psi|^k)$ represents the dual of $L^{p} (D, |\psi|^k)$ via the same pairing. This implies that $H^{m} (D, |\psi|^k) = L^{p} (D, |\psi|^k)$. Thus Proposition 2(a) holds for $m = m$. In order to end the proof of Theorem 1 we must show that $H^{m} (D, |\psi|^k) = L^{p} (D, |\psi|^k)$ for $0 < s < 1/p$. By Proposition 1(b), $P_m$ maps $L^{p} (D, |\psi|^k)$ into itself if $0 < s < 1/p$. Since $P_m$ maps $L^{p} (D, |\psi|^k)$ into itself,

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1 < p < \infty$, we can use complex interpolation to show that $P_m$ maps $L^s(D,|\cdot|^{-m})$ into itself for $-1+1/q < s < 0$. Since $P_m$ is selfadjoint and $L^s(D,|\cdot|^{m}), 1/p + 1/q = 1$, represents the dual of $L^s(D,|\cdot|^{m})$ via \( \langle \cdot, \cdot \rangle_0 \), the operator $P_m$ must map $L^s(D,|\cdot|^{m})$ into itself. Thus $L^s(D,|\cdot|^{m}) = \text{Harm}^s(m)(D), -1/p < s < 0, 1 < p < \infty$, being the ranges of the same set under the projection $P_m$. The above duality relations show that the norms must also be equivalent.

3. Proof of Theorem 2. Theorem 2 is obviously a special case of Remark 3(a), but we wish to give here another proof since it can be used in extending our results to the case of domains with boundary of class $C^4$ (see [19, 22]).

First, assume that $s > 1/p - 1$. Since $P_m$ maps $W^s_p(D)$ into itself (for $1/p - 1 < s < 0$ this follows from $W^{s'-1}_p(D) = W^{s'-1}_p(D)$ and from the fact that $P_m$ is selfadjoint) and

$$W^s_p(D), W^{s'}_p(D)_{\mathbf{k}=j} = B^s_{p,j}(D), s = (1-\theta)s_1 + \theta s_2, 1 < s < 0,$$

$P_m$ maps $B^s_{p,j}(D)$ into itself if $s > 1/p - 1$. In particular, we have

$$\text{Harm}^s(m)(D), \text{Harm}^{s'}(m)(D)_{\mathbf{k}=j} = B^s_{p,j}(D)$$

if $\min(s_1, s_2) > 1/p - 1, s = (1-\theta)s_1 + \theta s_2$.

Thus $L^s, r \geq \max(s_1, s_2)$, maps $B^s_{p,1}(\text{Harm}(m)(D))$ into

$$(L^s(D,|\cdot|^{-m}), L^s(D,|\cdot|^{-m}))_{\mathbf{k}=j} = L^s(D,|\cdot|^{-m})$$

by Theorem 5.5.1 of [7]. Thus

$$B^s_{p,j}(\text{Harm}(m)(D)) = P_m(L^s(D,|\cdot|^{-m})) = \text{Harm}^s(m)(D) \quad \text{for } s > 1/p - 1.$$

The Sobolev and Besov norms are both equivalent to $\|L^s u\|_{L^p(D,|\cdot|^{-m})}^2$.

Now we can use the real interpolation functor $(\cdot, \cdot)_{\mathbf{r}}$ to prove that $L^s, r \geq s$, maps $B^s_{p,j}(\text{Harm}(m)(D))$ into $B^s_{p,j}(D)$ if $s \neq k + 1/p$, and complex interpolation to prove that $L^s, r \geq s$, maps $B^s_{p,j}(\text{Harm}(m)(D))$ into $B^s_{p,j}(D)$ for $s = k + 1/p$. Then we use the same considerations as in the proof of Theorem 1 to show that $B^s_{p,j}(\text{Harm}(m)(D))$ and $B^s_{p,j}(\text{Harm}(m)(D))$ are mutually dual via $\langle \cdot, \cdot \rangle_{\mathbf{s},\mathbf{r}}$, $r \geq s$, $1/q + 1/p = 1$. Finally, we obtain

$$B^s_{p,j}(\text{Harm}(m)(D)) = L^s(\text{Harm}(m)(D),|\cdot|^{m}) = \text{Harm}^s(m)(D).$$

4. Proof of Remark 3 and Theorem 3. We begin with a proof of part (a) of Remark 3. We can assume that $T$ is selfadjoint (since we can always replace $T$ by $T^* T$). Thus for each $k$, the Dirichlet problem is uniquely solvable for $T^k$. Since $T^k$ is of order $2mk$, the estimates from [26] (Th. 5.7.2) show that $G_{k,m}$ maps continuously $W^s_{p+k}(D)$ onto $W^s_{p+k}(D), 1 < p < \infty$. It is well known that $G_{k,m}$ maps $W^s_{p+k}(D)$ into $W^s_{p+k}(D)$ for $s \geq 0$. Then complex interpolation shows that $G_{k,m}$ maps $W^s_{p+k}(D)$ into $W^s_{p+k}(D)$ for each $s \geq -mk$. If we use the real interpolation functor $(\cdot, \cdot)_{\mathbf{r}}$, we can prove that $G_{k,m}$ maps $B^s_{p,j}(D)$ into $B^s_{p,j}(D)$ for $s \geq -mk$.

We now prove that

$$\text{Harm}^s_{p,j}(D) = B^s_{p,j}(\text{Harm}(m)(D)) \quad \text{for } s > mk.$$
Theorem 3 is now an immediate consequence of Remark 3. The only thing that remains to be proved is that $B^0_m L^2(D) = B^0_m (m)(D)$ and $B^0_m L^2(M)(D) = L^2(M)(D), s > 0$. These facts will be proved in Appendix 2.

5. Proof of Theorem 4. We prove Theorem 4 for $1 < p < \infty$. The case $p = \infty$ will be considered separately in Appendix 2.

Let $1 < p < \infty$. We have by Theorem 1

$$L^p M(D) = B^p_m (M)(D).$$

Thus for every $u \in L^2 M(D)$ and $q = p/(p-1)$

$$\|P_m u\|_{L^q(D)} = \sup_{x \in M_0(D)} \langle P_m u, v \rangle.$$

Since $P_m$ is self-adjoint the last expression is bounded by

$$|\langle P_m u, v \rangle| \lesssim \|P_m u\|_{L^q(D)} \leq c \|u\|_{L^p(D)}.$$

Since the smooth functions are dense in $L^q M(D)$, $P_m$ extends to a continuous mapping of $L^q M(D)$, $|\langle P_m u, v \rangle|$ into itself.

If $p = 1$ then we must use the fact that each Banach space imbeds isomorphically into its second dual. Thus we again have for $u \in L^2 M(D)$

$$\|P_m u\|_{L^1(D)} = \sup_{v \in M_0(D)} |\langle P_m u, v \rangle| \leq c \|u\|_{L^2(D)}.$$

Thus $P_m$ extends to a continuous map of $L^q M(D)$, $|\langle P_m u, v \rangle|$ into itself.

Remark. In the same manner we can prove that the operator $G_m$ solving the Dirichlet problem extends to a continuous mapping from $L^q M(D)$ into $H^m_{-\alpha}(D)$ for every $1 < p < \infty$.

6. Proof of Theorem 6. The operator $T = \Box$ is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_\Box$, and this scalar product is equivalent to the Euclidean scalar product

$$\langle \omega, \zeta \rangle_\Box = \sum \omega_j \zeta_j, \quad (\omega, \zeta)_\Box.$$

We can treat $0, q$-forms as vector-valued functions and we have $\langle \omega, \zeta \rangle_\Box = \langle \omega, A(\xi) \rangle_\Box$, where $A$ is a matrix invertible on $D$ with $C^\infty(D)$ coefficients. The operator $\Box = T$ is a strongly elliptic operator of order 2 such that the Dirichlet problem is uniquely solvable for $T$. We shall consider the operator $N_{1-T}G_T$, which maps $B^m(D)$ into itself.

The operator $T$ fulfills the assumptions of Remark 3 and thus the spaces $B^m_{p,s} M(D), -\infty < s < \infty, 1 < p < \infty$, form an interpolation scale and $B^m_{p,s} B^m_{p,s}(D) = B^m_{p,s}(D)$ (Remark 3). Moreover, since $\langle \cdot, \cdot \rangle_\Box$ and $\langle \cdot, \cdot \rangle_{1-T}$ are equivalent in the above-explained sense, the spaces $W_{p,s}^{1T}(0, q)(D)$ and $\hat{W}_{s}^{p,1}(0, q)(D)$, $r = p/(p-1)$, are mutually dual via the pairing $\langle \cdot, \cdot \rangle_\Box$.

Consider the map $L^2_D$ constructed as in the introduction with $T' = T$. Then $L^2_D$ maps $B^m_{p,s}(D)$ into $B^m_{p,s}(D)$ if $0 < s < k$. (Since $T$ is of second order, the proof of this fact is the same as in the case of $T = A_1$; see [18], [22]). We also have $L^2_D u - u \perp B^m_{p,s}(D)$ with respect to $\langle \cdot, \cdot \rangle_\Box$. Hence we have the same situation as in Theorem 1 and obtain

(i) $(B^m_{p,s}(D))$ and $(B^m_{p,s}(D))$ are mutually dual via the pairing $\langle u, L_T v \rangle_D, k \geq s$.

Now the estimates from [17] yield that $N_{1-T}G_T$ maps $A_1 B^m_{p,s}(D)$ into $A_1 B^m_{p,s}(D)$. On the other hand, the classical Kohn estimate [9] shows that $N_{1-T}G_T$ maps $L^p B^m_{p,s}(D)$ into $(B^m_{p,s}(D))$. Then Remark 3 and (i) imply by the standard interpolation and duality argument that

(ii) $(B^m_{p,s}(D))$ into $(B^m_{p,s}(D))$ for all $0 < s < k, 1 < p < \infty$.

Recall that $N_{1-T}G_T$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{1-T}$.

In order to prove the needed estimates for $N_{1-T}G_T$, we must observe that (ii) implies that $N_{1-T}G_T$ maps continuously $(B^m_{p,s}(D))$ into $W_{p,s}^{s+1}(0, q)(D)$ for $-1 < s < k, 1 < p < \infty$.

Let now $\omega \in W_{0,s}^{s+1}(0, q)(D)$ and $P_{T,D}$ be the orthogonal projection on $L^2 M(D)$ (orthogonal with respect to $\langle \cdot, \cdot \rangle_\Box$). We have $\omega = P_{T,D} \omega + \Box g_{t2} G_{t2} \omega$. Since $G_{t2} \Box g_{t2} \Box G_{t2} \omega \in \text{Dom} \Box G_{t2}$, we obtain

$$N_{1-T}G_T \omega = N_{1-T}G_{T,D} \omega + G_{t2} \Box G_{t2} \omega.$$

Hence $N_{1-T}G_T$ maps $W_{0,s}^{s+1}(0, q)(D)$ into $W_{p,s}^{s+1}(0, q)(D)$ for $s \geq 0$ (this can even be proved for $0 > s > 1 + 1/p$ but we shall not do it here).

The same methods applied to the operators $\Box G_{t2} \Box G_{t2}$ and $\Box G_{t2} \Box G_{t2} \Box G_{t2}$ permit us to prove the rest of Theorem 6.

IV. Appendix 1. We now consider the general spaces $F^m_{p,s}(D)$ and $B^m_{p,s}(D)$.

The definitions of these spaces for $0 < p < \infty$ and $0 < q < \infty$ can be found in [31]. Recall that for $1 < p < \infty$, we have $W^{1,p}_{s+1}(D)$ and $B^{p,s}_{p,s}(D) = W^{1,p}_{s+1}(D), -\infty < s < \infty$.

We shall use the above notation also for $0 < p < 1$. The following interpolation formulas hold:

1. $[B^{m}_{p,s}(D), W^{m}_{p,s}(D)]_{\theta,q} = F^{m}_{p,s}(D), 1 - \theta + \frac{1}{p} = 1 - \frac{2p_{1}}{2 - 2p_{1}} + \frac{p_{1}}{p_{1} - p_{2}}.

2. $W^{m}_{p,s}(D), W^{m}_{p,s}(D)]_{\theta,q} = B^{m}_{p,s}(D), 1 - \theta + \frac{1}{p} = 1 - \frac{2p_{1}}{2 - 2p_{1}} + \frac{p_{1}}{p_{1} - p_{2}}.$
Thus all spaces $B^p_m(D), 0 < p, q < \infty,$ and $F^p_m(D), 0 < p < \infty, 2p(p+2) < q < \infty,$ can be obtained by interpolation from the spaces $B^p_m(D)$ and $W^p_m(D), 0 < p, q < \infty.$ The formulas (1) and (2) together with the definitions of $B^p_m(D)$ and $W^p_m(D)$ given above for $1 < p < \infty$ could serve as equivalent definitions of the spaces $B^p_m, 1 \leq p < \infty, 1 \leq q \leq \infty,$ and $F^p_m(D), 1 \leq p < \infty, 2p(p+1) < q < 2p.$

We now prove the following

**Proposition A.** Let $T$ be a strongly elliptic operator as in Remark 3(a). Then

$$F^p_m \cap \text{Harm}_T(D) = B^p_m \cap \text{Harm}_T(D)$$

for $-\infty < s < \infty$ and $1 < q < \infty, 1 < p < \infty.$

*Proof.* The interpolation formula (1) yields that $G^s_\omega$ maps $F^p_m(D)$ into $F^{p+1}_m(D)$ for $s > -mk, 2p(p+1) < q < 2p.$ We also have $\text{Tr} F^p_m(D) = \text{Tr} W^p_m(D) = V(\Omega)$ for all $s > mk.$ Moreover, [31, 2.7.2] implies that the extension operator $\mathcal{E}$ maps $\text{Tr} W^p_m(D) = V(\Omega)$ into $\bigcup F^p_m(D).$

Thus we can now repeat word by word the proof of Remark 3(a) and prove our proposition for $2p(p+1) < q < 2p.$ In order to prove Proposition A for all $s, 1 < q < \infty,$ we fix $p$ and observe that the estimates from [31] yield that Proposition A is valid for all $1 < q < \infty, 2m < s < \infty (2m$ is the order of $T).$ Thus we can use complex interpolation between the spaces $F^p_m(D), -\infty < s < \infty, 2p(p+1) < q < 2p,$ and $F^p_m(D), 2m < s < \infty, 1 < q < \infty,$ and prove our proposition for all $1 < q < \infty.$

**Remark B.** It will follow from Appendix 2 (see below) that Proposition A is in fact valid for $0 < q < \infty$ and $0 < p < \infty.$

Roughly speaking, if we deal with kernels of strongly elliptic differential operators, we have only one interpolation scale to consider—the scale of Besov spaces $B^p_m(D).$

Let us now describe the interpolation scale $B^p_m \cap \text{Harm}(m)(D), 1 < p < \infty, 1 < q < \infty.$

If $s > 0$ and $s$ is not an integer then for $u \in B^p_m(D)$

$$||u||_{B^p_m(D)} = ||u||_{B^p_m(D)} + \sum_{|\alpha| + |\beta| + m < s} ||D^\alpha_{\mathcal{E}} |D^\beta_f(x+h)^{-\alpha} - D^\beta_f(x)^{-\alpha}|(D^\alpha f)(x)\, dv^2 + v^2 D^{\alpha}_v dv_v$$

where $|s| = -|s|, D_v = D \cap \{x \in R^m: x+h \in D\}.$ Thus in this case the Besov norm has an explicit form.

To describe the spaces $B^p_m \cap \text{Harm}(m)(D)$ for other $s$ we shall use Proposition 2, Theorem 1 and the interpolation formula (2). Let $s = -1+1/p$ and $s = (1-\eta)s_1 + s_2, 0 < s_2 < s_1, s_1 > -1+1/p.$ The formula (2) yields that $L^p_{\text{loc}}$ maps continuously $B^p_m \cap \text{Harm}(m)(D)$ into $(L^p(D, |\xi|^{s_2}))_\text{Harm}(m)(\eta), r > s_2,$ and $F^p_m$ maps the last space onto $B^p_m \cap \text{Harm}(m)(D).$ Since in addition $L^p_{\text{loc}}$ maps continuously $B^p_m \cap \text{Harm}(m)(D)$ into $B^p_m \cap \text{Harm}(m)(D)$ (defined in the same way as for $B^p_m)$, we can repeat the proof of Theorem 1 to obtain for $1/p + 1/q = 1, 1/q + 1/q = 1$

$$B^p_m \cap \text{Harm}(m)(D) = (L^p(D, |\xi|^{-s_1}))_\text{Harm}(m)(\eta) \cap \text{Harm}(m)(D).$$

The space on the right has the following norm (see [7, 57, Ex. 10]):

$$||u|| = \left( \int_0^\infty t^{s_1}|u(t)|^{p-1} |dV(t)|^{1/p} \right)^{1/p}.$$

We can now change the notation and write $-s = s_1, q_1 = p, q_1 = p.$ In addition we can choose $s_2, s_3$ in the above formula in such a way that $s_2 - s_3 = 1/p$, or if $s > 1/q_1 - 1/p_1$ we can take $s_2 - s_3 = 1, \theta = 1/q_1$ and put $t = t^{s_1}.$ After those operations we get the following

**Proposition C.** If $s < 1/p,$ then the following norms are equivalent to the $B^p_m \cap \text{Harm}(m)(D)$:

$$||u||_s = \left( \int_0^\infty \left( \int_0^\infty |u(t)|^{p-1} |dV(t)|^{1/p} \right)^{1/p} dt \right)^{1/q_1}$$

for any $0 < \theta < \min(1, 1 - p_1).$

If in addition $s < p-1/q_1$ then the above norms are equivalent to

$$||u||_s = \left( \int_0^\infty \int |u(t)|^{p-1} |dV(t)|^{1/p} dt \right)^{1/q_1}.$$

By considering the derivatives we can now get the explicit characterization of the spaces $B^p_m \cap \text{Harm}(m)(D)$ for all $s.$

In the case $m = 1,$ Proposition C implies the following characterization of the space $B^p_m(\text{rel} D), 1 < p, q < \infty, s > 0:

**Corollary D.** The following conditions are equivalent:

1) $u \in B^p_m(\text{rel} D), 1 < p, q < \infty, s > 0.$

2) $u \in L^p(D)$ and for some integer $k > s + 1/q$

$$k \leq k \leq k$$

$$s < 1/p$$

$$||u||_s = \left( \int_0^\infty \int |u(t)|^{p-1} |dV(t)|^{1/p} dt \right)^{1/p} < \infty$$

where $u$ denotes the harmonic extension of $u$ over $D.$

3) $u \in L^p(D)$ and (*) holds for every $k > s + 1/q.$

**V. Appendix 2.** Let $T$ be a strongly elliptic operator of order $2m$ for which the Dirichlet problem is uniquely solvable.
Proposition A. The operator $G_T$ solving the Dirichlet problem for $T$ maps continuously $B^p_{k,q}(D) \rightarrow B^{p^*}_{k',q'}(D)$ and $F^p_{k,q}(D) \rightarrow F^{p^*}_{k',q'}(D)$ for $-m+(1/p-1)n \leq s < \infty$, $0 < p \leq 1$, $0 < q < \infty$.

Corollary B(a) The projection $P_T$ maps continuously $B^p_{k,q}(D)$ and $F^p_{k,q}(D)$ into itself for $(1/p-1)n \leq s < \infty$, $0 < p \leq 1$, $0 < q < \infty$.

(b) $F^p_{k,q}(D) = B^p_{k,q}(D)$ for $0 < p \leq 1$, $-\infty < s < \infty$, $0 < q < \infty$.

(c) The scale of spaces $B^p_{k,q}(\Omega, H^s(D))$, $0 < p \leq \infty$, $-\infty < s < \infty$, has the same interpolation properties as in Remark 3(b) provided that $T$ is selfadjoint.

Proof. In order to prove Proposition A we must use the results of Frankel [11]. His estimates imply that $G_T$ maps continuously $F^p_{k,q}(D)$ into $F^{p^*}_{k',q'}(D)$ and $B^p_{k,q}(D)$ into $B^{p^*}_{k',q'}(D)$ for $s > m(1/p-1)$, $0 < p \leq 1$. The estimates from [11] also yield that $G_T$ maps continuously $B^p_{k,q}(D)$ into $B^{p^*}_{k',q'}(D)$ and $F^p_{k,q}(D)$ into $F^{p^*}_{k',q'}(D)$ if $-m < q < \infty$, $1 < p \leq 1$, $0 < q < \infty$. This can be easily proved by using the estimates for $W^s(D)$ and the interpolation formulas from Appendix 1.

It follows from [31, 3.3.4, Remark 1] that $G_T$ has the interpolation property with respect to the Calderon-Tochensky construction (see [31] for details) and hence we can use it to interpolate between $F^p_{p,q}(D)$ and $F^{p^*}_{p',q'}(D)$ for $s > s_1 = m(1/p-1)n$, $0 < p_1 \leq 1$, $0 < q_1 < \infty$ and $s_2 > -m_1 < 1 < p_2 < \infty$, $0 < q_2 < \infty$ and get the required results for the spaces $F^p_{k,q}(D)$.

In order to prove Proposition A for the spaces $B^p_{k,q}(D)$ it suffices to use the real interpolation functor $(\cdot, \cdot)_s$.

Now, Corollary B(a) follows immediately from the definition of $P_T$. Corollary B(b) follows from the fact that $\text{Tr} \ F^p_{k,q}(D) = B^{p^*}_{k',q'}(D)$ for $s > \max(1/p, 1/(p-1))n$ in the same way as in Remark 3 and in Proposition A from Appendix 1. The proof of (c) is a corollary of that of Remark 3(b).

Corollary B(a) yields in particular that for every $m$, $p^*_{m}(D)$ maps continuously $W^s(D)$ into itself and $B^r_{m}(D)$ if only if $s > r$. Corollary B(b) implies that $\text{Harm}^m(D) = B^r_{m}(D)$ for $-\infty < s < \infty$.

We now prove

Proposition C.

(a) $\text{Harm}^m(D) = \text{Harm}^r(D)$ for $s < 1$.

(b) $B^p_{k,q}(\Omega, H^s(D)) = B^{p^*}_{k',q'}(\Omega, H^{s-1}(D))$ for $0 < p \leq 1$.

(c) $B^p_{k,q}(\Omega, H^s(D)) = B^{p^*}_{k',q'}(\Omega, H^{s+1}(D))$ for $0 < p \leq 1$.
because of the regularity of $P_{k,n}$ (see Corollary B(a)). Thus part (a) of Proposition C is proved for $s < 0$, $s 
eq [x]$. Thus part (a) of the above proof remains valid for $s = [x]$ and thus $L^1 \text{Harm}(m)(D), |q|^r) = B_{1}^1 \text{Harm}(m)(D)$. The opposite inclusion follows from interpolation since

$$B_{1}^1 \text{Harm}(m)(D) = \{ B_{1}^1 \text{Harm}(m)(D), B_{1}^1 \text{Harm}(m)(D) \}_{[1, \infty]} \subset \{ L^1(D), |q|^r \}_{2, [1, \infty]} = L^1(D), |q|^r \}.$$

Thus (a) holds for every $s < 0$.

We have $B_{1}^1 \text{Harm}(m)(D) \subset L^1(D)$ since $B_{1}^2 \text{Harm}(m)(D) = \text{Harm}(m)(D) = W_{2}^1(D) = L^1(D)$ for $s > 0$. This implies that $L^1(D)$ maps continuously $B_{1}^1 \text{Harm}(m)(D)$ into $L^1(D), |q|^r$ just as for Hölder spaces, $r \geq [x] + 1$, $s > 0$.

Let us now prove (1) and (b). Take $s > 0$, $s - [x] > 0$. The space $B_{1}^1 \text{Harm}(m)(D)$ is dual of $L^1(D)$. The mapping $L^1(D)$ maps continuously $B_{1}^1 \text{Harm}(m)(D)$ into $L^1(D)$. In view of Proposition A this fact can be proved in exactly the same way as for $p > 1$, using the already proven part of (a).

Now, $B_{1}^1 \text{Harm}(m)(D)$ represents the dual of $B_{1}^1 \text{Harm}(m)(D)$ via $\langle \varphi, \psi \rangle = \langle \varphi, \psi \rangle$. Indeed, it is obvious that every element of $B_{1}^1 \text{Harm}(m)(D)$ determines a continuous functional on $B_{1}^1 \text{Harm}(m)(D)$. Let now $\varphi$ be such a functional. We extend $\varphi$ to an element $\varphi \in (B_{1}^1(D))^\ast$. Since $P_{n}$ maps $B_{1}^1(D)$ into $B_{1}^2(D)$ and is selfadjoint, $P_{n} \varphi$ maps continuously $B_{1}^1(D)$ into $B_{1}^1(D)$. Thus $P_{n} \varphi$ is the element of $B_{1}^1 \text{Harm}(m)(D)$ representing $\varphi$.

The mapping $L_{n}(D)$ maps $B_{1}^1 \text{Harm}(m)(D)$ into $L^1(D), |q|^r$ for every function from $L^1(D), |q|^r$ represents a functional on $B_{1}^1 \text{Harm}(m)(D)$. Thus $L^1(D), |q|^r \subset B_{1}^1 \text{Harm}(m)(D)$. The opposite inclusion follows from Lemma 1 applied to the $(k + m)$-polyharmonic functions, $k > s$, since $B_{1}^1 \text{Harm}(m)(D)$ is the retract of $A_{k,m} \text{Harm}(m+k)(D)$ under $A^4$. Thus (b) is proved for noninteger $s$. Complex interpolation permits us to prove (b) for all $s$.

Let us now prove the rest of (a). Let $0 < s < 1$. Since $L_{n}(D)$ maps $B_{1}^1 \text{Harm}(m)(D)$ into $L^1(D), |q|^r$ and $P_{n}$ maps $L^1(D), |q|^r$ into itself, we have $B_{1}^1 \text{Harm}(m)(D) \subset L^1(D), |q|^r$. For every $m$-polyharmonic $u$

$$||u||^{1, p}_{1, \infty} \approx \sup_{q \in \Omega, \sum_{\lambda \leq m} a_{\lambda}} ||u||^{1, p}_{1, \infty} \leq c ||u||^{1, p}_{1, \infty}.$$ 

Hence $L^1(D), |q|^r = B_{1}^1 \text{Harm}(m)(D)$.

Now, $B_{1}^1 \text{Harm}(m)(D)$ represents the dual of $L^1(D), |q|^r$ via the pairing $\langle u, L^1(D), |q|^r \rangle$. We shall prove in the sequel that $B_{1}^1 \text{Harm}(m)(D)$ represents the dual of $B_{1}^1 \text{Harm}(m)(D)$ via the pairing $\langle u, L^1(D), |q|^r \rangle$. Thus $||u||^{1, p}_{1, \infty} \approx c ||u||^{1, p}_{1, \infty}$ if $u$ is $m$-polyharmonic. Hence $B_{1}^1 \text{Harm}(m)(D)$

It now remains to prove (c2) and (c3).

Let us prove (c2). From the above-determined duality between $A_{k}^s(D)$ and $B_{1}^1(D)$ it follows that every $u \in B_{1}^1 \text{Harm}(m)(D)$ determines a continuous functional on $A_{k}^s \text{Harm}(m)(D)$, since $u \in A_{k}^s \text{Harm}(m)(D)$ if $L_{n} u \in A_{k}^s(D)$. We can now proceed in the standard way, extending $\varphi \in (A_{k}^s \text{Harm}(m)(D))$ to $\tilde{\varphi} \in (A_{k}^s(D))$ and taking $\tilde{u} = P_{n} \tilde{\varphi} \in (A_{k}^s(D))^\ast$, which is the closure of $C_{k}^s(D)$ in $B_{1}^1(D)$ (see [31, 2.1.1.2, Remark 2]). The operator $P_{n}$ maps the space $B_{1}^1 \text{Harm}(m)(D)$ into (this follows from Th. 12.10 of [1] and interpolation). The mapping $L_{n}(D)$ maps $B_{1}^1 \text{Harm}(m)(D)$ into $B_{1}^1(D)$, since $L^1(D), |q|^r \subset B_{1}^1(D)$. Thus $L_{n}(D)$ maps $B_{1}^1 \text{Harm}(m)(D)$ to $B_{1}^1(D)$. Hence as before each $v \in B_{1}^1 \text{Harm}(m)(D)$ determines a continuous functional on $B_{1}^1 \text{Harm}(m)(D)$ and $B_{1}^1 \text{Harm}(m)(D) = (P_{n}(B_{1}^1(D)))^\ast$. Thus $B_{1}^1 \text{Harm}(m)(D)$ and $L^1(D), |q|^r$ represent the dual of $B_{1}^1 \text{Harm}(m)(D)$.

In the above proof we have used implicitly the fact that the smooth functions $D$ are dense in $L^1(D, |q|^r)$. This fact needs a special proof, which is the same as the proof for $m = 1$ given in [21].

Remark D. It can be easily proved using the fact that $C_{k}^s(D)$ is dense in $C_{0}(D) = \{ f \in C_{k}(D); f_{|_{0}} = 0 \}$ that if $m = 1$ then

$$B_{1}^1 \text{Harm}(m(D)) = \{ u \in B_{1}^1 \text{Harm}(m(D)) \}; \varphi \psi u \rightarrow 0 \text{ if } \varphi \rightarrow 0\},$$

$$A_{k}^s \text{Harm}(m(D)) = \{ u \in A_{k}^s \text{Harm}(m(D)) \}; |q|^r \psi u \rightarrow 0 \text{ if } \varphi \rightarrow 0\}. \text{ } 0 < s < 1.$$ 

Remark E. Let us make a trivial but useful observation. Since for $D = C$ the holomorphic functions form a closed subspace of the space of harmonic functions with respect to any norm considered here, we have now an explicit description of Sobolev and Besov spaces of holomorphic functions for every smooth bounded domain in $C$. The additional assumptions on $D$ are needed only to establish duality and interpolation relations between these spaces.

Proposition C (c2) (c3) remains valid if we replace the spaces of harmonic functions with the corresponding spaces of holomorphic functions on strictly pseudoconvex domains (cf. [19], [21]). If $D$ is the unit disc in $C$ then $B_{1}^1 \text{Hol}(D)$ is the classical Bloch class $B_{1}^1$.

Part (a) of Proposition C permits us to get Corollary 1 for $p = 1$, and part (c1) proves Theorem 4 for $p = \infty$.

VI. Open problems. 1. It was proved in [24] that if $m = 1$ then the mapping $u \rightarrow P_{n}(\varphi \psi u)$ is an isomorphism between $B_{1}^1 \text{Harm}(m(D)$ and $B_{1}^1 \text{Harm}(m(D)$,
Is this true for $m > 1$, i.e. is $u \rightarrow P_m(\varphi u)$ an isomorphism? What can be said about the mapping $u \rightarrow P_m(\varphi u)$, where $s > 0$ is noninteger?

2. Find an explicit characterization of the spaces $\text{Harm}^p_m(D)$ for $p < 1$. Is it in particular true that $\text{Harm}^p_m(D) = L^p \text{Harm}(m)(D)$ for $p < 1$?

3. Are the smooth functions on $D$ dense in $L^1 \text{Harm}(m)(D)$, $|\varphi| > 0$? For which strongly elliptic second order operators $T$ Lemma 2 and Proposition 1(a), (b) hold? If these facts are true for $T$ then all results of the present paper remain valid if we replace the spaces $\text{Harm}(m)(D)$ with the spaces $\text{Harm}^p_m(D)$. It may also be interesting to try to extend these estimates to the case of pseudodifferential operators, since for every positive-definite elliptic operator $T$ of order $2m$ the operator $T^{1/2m}$ is a well-defined pseudodifferential operator (see [30]).

5. Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}$. Is the Bergman projection $B$ continuous from $W^p_2(D)$ into itself if $p < 1$ and $s > n(1/p - 1)$?

6. Let $D$ be a smooth bounded domain in $\mathbb{C}$. Is the operator $N$ solving the $\bar{z}$-Neumann problem considered with respect to the Euclidean metric regular in H"older norms? Note that in view of Corollary B in Appendix 2 the H"older estimates for such an $N$ yield automatically the estimates for $N$ in every $W^p_2(D)$ norm, $0 < s < \infty$, without any additional assumption on the domain $D$. The same is obviously true for the Bergman projection $B$: if, for a smooth bounded domain $D$, $B$ is regular in H"older norms then it must be regular in all norms $W^p_2(D)$, $1 < p < \infty$, $s \geq 0$. In particular, $B$ regular in H"older norms if $D$ is a pseudoconvex domain with real-analytic boundary.

References