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A function space $C(K)$ not weakly homeomorphic to $C(K)\times C(K)$

by

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Abstract. We construct an infinite separable compact space K with the third derived set empty such that the space $C(K)$ of continuous real-valued functions on K endowed with the weak or pointwise topology is not homeomorphic to its own square $C(K)\times C(K)$.

1. Notation and terminology. Our terminology follows Engelking [6] and Semadeni [12]. The real line is denoted by $\boldsymbol{R}$ and ω is the set of natural numbers; $|A|$ is the cardinality of the set A.

Given a space X, we denote by $\boldsymbol{R}^{|X|}$ the Tikhonov product of the real line, X being the index set, and for every $Y\subset X$, $\pi_Y\colon \boldsymbol{R}^{|X|}\to\boldsymbol{R}^{|Y|}$ is the projection; we shall write π_x instead of $\pi_{\{x\}}$, for $x\in X$. Given a function $f\colon X\to Y$ we denote by $\operatorname{dom}(f)$ and $\operatorname{ran}(f)$ the domain and range of f, respectively.

We write $X \underset{\text{top}}{=} Y$ ($\underset{\text{top}}{\neq}$) if X is (is not) homeomorphic to Y.

Given a compact space K we denote by $C(K)$ the Banach space of real-valued continuous functions on K endowed with the sup norm.

The space $C(K)$ endowed with the weak topology will be denoted by $C_{\mathrm{w}}(K)$, while $C_{\mathrm{p}}(K)$ will denote the space $C(K)$ equipped with the pointwise topology.

2. Function spaces associated with almost disjoint families in ω. A family $\mathscr{A}$ of infinite subsets of ω is *almost disjoint* if any two distinct members of $\mathscr{A}$ have at most finite intersection. Let us recall that there are almost disjoint families of cardinality 2^ω of subsets of ω (cf. [9, p. 300 and p. 317]); in the sequel we shall consider a fixed (but arbitrary) almost disjoint family $\mathscr{T}$ which has cardinality 2^ω.

Given an almost disjoint family $\mathscr{A}$ in ω, let $L_{\mathscr{A}}$ be the compact space defined in the following way: the underlying set is $\omega\cup\{p_A\colon A\in\mathscr{A}\}\cup\{p\}$ (all indicated points are distinct), the points in ω are isolated, the basic neighbourhoods of the points p_A are of the form $\{p_A\}\cup(A\setminus F)$, F being finite, and p is the "point at infinity" of the locally compact space $\omega\cup\{p_A\colon A\in\mathscr{A}\}$ (cf. [6, Exercise 3.6.I]; this is a standard construction going back to Aleksandrov and Urysohn [2, Ch. V, § 1.3].

Let us point out that the class of spaces $L_{\mathscr{A}}$ coincides with the class of compact spaces K having the following structure:

(i) K is separable,

(ii) the second derived set of K consists of one point.

Let us make a few simple observations about the function spaces $C(L_{\mathscr{A}})$ we shall need in the sequel.

The spaces $L_{\mathscr{A}}$ are scattered – the second derived set of $L_{\mathscr{A}}$ is $\{p\}$ – and therefore any Radon measure μ on $L_{\mathscr{A}}$ is of the form $\mu = \sum\{a_x\delta_x\colon x\in L_{\mathscr{A}}\}$, where δ_x is the probability measure concentrated at the point x and $\sum\{|a_x|\colon x\in L_{\mathscr{A}}\} < \infty$ (see [12, Cor. 19.7.7]).

Let $M(L_{\mathscr{A}})$ denote the space of all Radon measures on $L_{\mathscr{A}}$ (i.e. $M(L_{\mathscr{A}}) = C(L_{\mathscr{A}})^*$). Since ω is dense in $L_{\mathscr{A}}$ the restriction $f\to f|\omega$ is injective on $C(L_{\mathscr{A}})$ and hence for every $S\subset M(L_{\mathscr{A}})$ containing all δ_n, the map $i_S\colon C(L_{\mathscr{A}}) \to \boldsymbol{R}^{|S|}$ defined by $i_S(f)(\mu) = \mu(f)$ is an injection.

Given a subfamily $\mathscr{A}\subset\mathscr{T}$, the space $C(L_{\mathscr{A}})$ endowed with the norm or weak or pointwise topology can be identified with the subspace

$$E(\mathscr{A}) = \{f\in C(L_{\mathscr{T}})\colon f(p_A) = f(p) \text{ for all } A\in\mathscr{T}\setminus\mathscr{A}\}.$$

of the space $C(L_{\mathscr{T}})$ equipped with the corresponding topology. Indeed, $E(\mathscr{A})$ is the subspace of $C(L_{\mathscr{T}})$ consisting of functions constant on the closed subset $F = \{p\}\cup\{p_A\colon A\in\mathscr{T}\setminus\mathscr{A}\}$ of $L_{\mathscr{T}}$ and $L_{\mathscr{A}}$ can be obtained from $L_{\mathscr{T}}$ by matching the set F to the point p (cf. [12, Prop. 5.2.7]).

For any function $f\in C(L_{\mathscr{T}})$ there is a countable subfamily $\mathscr{A}\subset\mathscr{T}$ such that $f\in E(\mathscr{A})$, since for every $\varepsilon > 0$, the set $\{A\in\mathscr{T}\colon |f(p_A)-f(p)| > \varepsilon\}$ is finite.

Let us remark finally that by identifying each $f\in C(L_{\mathscr{A}})$ with $f|\omega$ and applying the Stone–Weierstrass theorem it follows that the spaces $C(L_{\mathscr{A}})$ can be regarded as closed subalgebras of l_∞ spanned by the characteristic functions of the sets $A\in\mathscr{A}$, the functions with finite supports and the unit of l_∞.

3. The example. The objective of this paper is to construct the following example:

Example. There exists a compact infinite separable space K with the 3rd derived set empty such that the space $C(K)$ of continuous real-valued functions on K endowed with the weak or pointwise topology is not homeomorphic to its own square, i.e.

$$C_w(K) \underset{\text{top}}{\neq} C_w(K)\times C_w(K) \quad\text{and}\quad C_p(K) \underset{\text{top}}{\neq} C_p(K)\times C_p(K).$$

Remark. The question of existence of a Banach space E which is not homeomorphic in the weak topology to its own square was asked by H. H. Corson [5, p. 12]; A. V. Arkhangel'skii [3, Problem 22] asked a similar question about the spaces $C_p(X)$.

After this paper had been completed, S. P. Gul'ko kindly informed us in a letter about his recent result to the effect that $C_p(\omega_1+1) \underset{\text{top}}{\neq} C_p(\omega_1+1) \times C_p(\omega_1+1)$, where ω_1+1 is the set of ordinal numbers not greater than ω_1 with the usual ordinal topology.

It seems that Gul'ko's result and our construction provide first examples (with rather different features) answering the problems of Corson and Arkhangel'skii.

Arkhangel'skii also asked the following question: does there necessarily exist a continuous map from $C_p(X)$ onto $C_p(X)\times C_p(X)$ whenever X is a compact space [4, Problem 4, p. 6]? One can show that for compact spaces with the 3rd derived set empty (hence for the spaces $L_{\mathscr{A}}$) such a map always exists.

Our construction is based on a diagonal argument, whose applications to some problems concerning continuous maps go back to Kuratowski [8]; some essential ideas in this construction have also been used by the author in [10]. Briefly, we shall construct the compactum K in the following way. The space K will be of the form $L_{\mathscr{X}}$ where $\mathscr{X}$ is an almost disjoint family of subsets of ω (see Sec. 2). Starting with an arbitrary almost disjoint family $\mathscr{T}$ of cardinality 2^ω of subsets of ω, we shall choose $\mathscr{X}\subset\mathscr{T}$ by transfinite induction, destroying on the way all possible homeomorphisms from $C_w(L_{\mathscr{X}})$ onto $C_w(L_{\mathscr{X}})\times C_w(L_{\mathscr{X}})$ (the construction for the pointwise topology is parallel). To perform this construction we shall first restrict the cardinality of the family of maps we have to deal with during this process; this is achieved by (somewhat technical) results on factorization of continuous maps on $C_w(L_{\mathscr{T}})$ which we prove in Sec. 4. Section 5 is devoted to the construction of the family $\mathscr{X}$.

Some modifications in our argument yield a compact space K with $C_w(K)^n \underset{\text{top}}{\neq} C_w(K)^m$, for all natural numbers $n\neq m$ (the same is true for the pointwise topology). This refinement is discussed in Sec. 6.

Finally, let us point out that the structure of $C(K)$ described at the end of Sec. 2 yields, by a result of Aharoni and Lindenstrauss [1], that the Banach space $C(K)$ is Lipschitz homeomorphic to $C(K)\times C(K)$.

4. Some auxiliary factorization results. Throughout Sections 4–6, $\mathscr{T}$ is a fixed almost disjoint family of cardinality 2^ω of infinite subsets of ω and we adopt the notation introduced in Sec. 2. We shall write L instead of $L_{\mathscr{T}}$.

The following fact, which we shall need in the sequel, is close to some topics discussed by Isbell in [7, Proof of Theorem 1]; for the sake of completeness we include a short justification, which is a variation of Isbell's arguments.

4.1. LEMMA. *Let X be a set and let E be a linear subspace of the product $\boldsymbol{R}^{|X|}$. If $f\colon E\to\boldsymbol{R}^{\omega}$ is a continuous map then f depends on countably many coordinates, i.e. there is a countable set $Y\subset X$ and a continuous map $g\colon \pi_Y(E)\to\boldsymbol{R}^{\omega}$ such that $f=g\circ\pi_Y|E$.*

Proof. Given an $S\subset X$, let p_S be the restriction of the projection π_S to E. Let $Z\subset X$ be such that p_Z maps homeomorphically E onto a dense subset $p_Z(E)$ of $\boldsymbol{R}^{|Z|}$ (if $\{p_x\colon x\in Z\}$ is a maximal linearly independent subset of $\{p_x\colon x\in X\}$, considered in the dual space E^*, then Z has the required properties, cf. [11, Ch. II, § 3]). Now, by [7, Corollary, p. 222], the map $f\circ p_Z^{-1}\colon p_Z(E)\to\boldsymbol{R}^{\omega}$ can be factorized through $R^{|Y|}$ for some countable $Y\subset Z$, i.e. there is a continuous map $g\colon p_Y(E)\to\boldsymbol{R}^{\omega}$ with $g\circ p_Y=f\circ p_Z^{-1}$ and then also $g\circ p_Y=f$, since $p_Y\circ p_Z=p_Y$.

4.2. LEMMA. *Let $\mathscr{A}$ be a subfamily of $\mathscr{T}$ and let $\varphi\colon E(\mathscr{A})\to E(\mathscr{A})\times E(\mathscr{A})$ be a homeomorphism, where $E(\mathscr{A})$ is considered as a subspace of either $C_{\mathrm{w}}(L)$ or $C_{\mathrm{p}}(L)$. Then there exists a countable set $S\subset M(L)$ containing all functionals δ_n, $n\in\omega$, and such that the function $(i_S\times i_S)\circ\varphi\circ i_S^{-1}$ maps homeomorphically $i_S(E(\mathscr{A}))$ onto $i_S(E(\mathscr{A}))\times i_S(E(\mathscr{A}))$.*

Proof. Consider first $E(\mathscr{A})$ with the weak topology, so the map $i_{M(L)}\colon E(\mathscr{A})\to\boldsymbol{R}^{|M(L)|}$ is a homeomorphic embedding. Put $F=i_{M(L)}(E(\mathscr{A}))$ and let $\psi\colon F\to F\times F$ be the homeomorphism defined by

$$\psi=(i_{M(L)}\times i_{M(L)})\circ\varphi\circ i_{M(L)}^{-1}.$$

Using Lemma 4.1 we shall now define by a back-and-forth induction a sequence of countable subsets $S_0\subset S_1\subset S_2\subset\ldots\subset M(L)$ such that $S_0=\{\delta_n\colon n\in\omega\}$, for even i, the map

$$(\pi_{S_i}\times\pi_{S_i})\circ\psi\circ\pi_{S_{i+1}}^{-1}\colon \pi_{S_{i+1}}(F)\to\pi_{S_i}(F)\times\pi_{S_i}(F)$$

is continuous and, for odd i, the map

$$\pi_{S_i}\circ\psi^{-1}\circ(\pi_{S_{i+1}}^{-1}\times\pi_{S_{i+1}}^{-1})\colon \pi_{S_{i+1}}(F)\times\pi_{S_{i+1}}(F)\to\pi_{S_i}(F)$$

is continuous (notice that since $\{\delta_n\colon n\in\omega\}\subset S_i$ the projections π_{S_i} are one-to-one on F).

Suppose that we have chosen the sets $S_0, S_1,\ldots,S_i$. If i is even then the continuous function

$$(\pi_{S_i}\times\pi_{S_i})\circ\psi\colon F\to\pi_{S_i}(F)\times\pi_{S_i}(F)$$

maps the linear subspace F of $\boldsymbol{R}^{|M(L)|}$ to the space $\boldsymbol{R}^{|S_i|}\times\boldsymbol{R}^{|S_i|}$. Hence Lemma 4.1 guarantees the existence of a countable set $S_i'\subset M(L)$ and a continuous map $\psi_i\colon \pi_{S_i'}(F)\to\pi_{S_i}(F)\times\pi_{S_i}(F)$ such that $(\pi_{S_i}\times\pi_{S_i})\circ\psi=\psi_i\circ\pi_{S_i'}$. We put $S_{i+1}=S_i\cup S_i'$.

In the case of odd i the argument is the same, but the spaces involved interchange now their roles.

Finally, we put $S=\bigcup_{i=0}^{\infty}S_i$ and, the basic neighbourhoods in $\boldsymbol{R}^{|M(L)|}$ being determined by finitely many coordinates, a simple verification shows that S has the required properties.

If one replaces in the above reasoning the map $i_{M(L)}$ by the map $i_{\{\delta_x\colon x\in L\}}$, one obtains the desired result for the pointwise topology.

4.3. LEMMA. *There is a family of functions $\mathscr{F}=\{\varphi_\alpha\colon \alpha<2^{\omega}\}$ such that:*

(a) *$\operatorname{dom}(\varphi_\alpha)\subset C(L)$, $\operatorname{ran}(\varphi_\alpha)\subset C(L)\times C(L)$ and φ_α is one-to-one, for $\alpha<2^{\omega}$.*

(b) *For every subfamily $\mathscr{A}$ of $\mathscr{T}$ and every homeomorphism ψ from $E(\mathscr{A})$ onto $E(\mathscr{A})\times E(\mathscr{A})$, where $E(\mathscr{A})$ is considered as a subspace of either $C_{\mathrm{w}}(L)$ or $C_{\mathrm{p}}(L)$, there exists an $\alpha<2^{\omega}$ such that $\psi=\varphi_\alpha|E(\mathscr{A})$.*

(We do not require φ_α to be continuous functions.)

Proof. Let $\mathscr{F}$ be the family of all functions φ which satisfy the following condition: there is a countable set $S\subset M(L)$, a G_δ-set X in $\boldsymbol{R}^{|S|}$ and a homeomorphic embedding h of X into $\boldsymbol{R}^{|S|}\times\boldsymbol{R}^{|S|}$ such that:

(i) $\{\delta_n\colon n\in\omega\}\subset S$.
(ii) $\operatorname{dom}(\varphi)=i_S^{-1}\big(X\cap h^{-1}(i_S(C(L))\times i_S(C(L)))\big)$.
(iii) $\varphi(f)=(i_S^{-1}\times i_S^{-1})\circ h\circ i_S(f)$, for $f\in\operatorname{dom}(\varphi)$.

To begin with, observe that the cardinality of the family $\mathscr{F}$ is 2^{ω}. Indeed, the space $M(L)$ is isomorphic to $l_1(L)$ (see Sec. 2), hence it has cardinality 2^{ω}. Given a countable set $S\subset M(L)$, there are at most 2^{ω} sets of type G_δ in $\boldsymbol{R}^{|S|}$, and each has at most 2^{ω} homeomorphic embeddings into $\boldsymbol{R}^{|S|}\times\boldsymbol{R}^{|S|}$. Since each φ in our family $\mathscr{F}$ corresponds to such an embedding, it follows that $|\mathscr{F}|\leqslant 2^{\omega}$.

Let us verify condition (b). Let $\mathscr{A}$ be a subfamily of $\mathscr{T}$ and let $\psi\colon E(\mathscr{A})\to E(\mathscr{A})\times E(\mathscr{A})$ be a homeomorphism (in the weak or pointwise topology). From Lemma 4.2 there is a countable set $S\subset M(L)$ with $\{\delta_n\colon n\in\omega\}\subset S$ such that $g=(i_S\times i_S)\circ\psi\circ i_S^{-1}$ is a homeomorphism from $i_S(E(\mathscr{A}))$ onto $i_S(E(\mathscr{A}))\times i_S(E(\mathscr{A}))$. Applying Lavrent'ev's theorem [6, Th. 4.3.21] one can find a G_δ-set X in $\boldsymbol{R}^{|S|}$ containing $i_S(E(\mathscr{A}))$ and a homeomorphic embedding h of X into $\boldsymbol{R}^{|S|}\times\boldsymbol{R}^{|S|}$ which extends g. Now, the map φ defined by (ii) and (iii) belongs to $\mathscr{F}$ and the restriction of φ to $E(\mathscr{A})$ coincides with ψ.

5. The construction. Before we start the construction of a subset $\mathscr{X}$ of the family $\mathscr{T}$ with the required properties (recall that our notation was set up in Sec. 2), let us make one more observation about the space $C(L)$.

5.1. LEMMA. *Let $\mathscr{A}$ be a subfamily of $\mathscr{T}$ and let $\varphi\colon E(\mathscr{A})\to E(\mathscr{A})\times E(\mathscr{A})$ be a homeomorphism with respect to the weak or pointwise topology. Then for* *each* $A\in\mathscr{A}$ *one of the following two conditions holds, where* $\pi_i\colon E(\mathscr{A})\times E(\mathscr{A})\to E(\mathscr{A})$ *is the projection onto the i-th coordinate* $(i=1,2)$:

1° *There is an* $f\in E(\mathscr{A})$ *such that* $f(p_A)=f(p)$ *and*

$$\pi_i\circ\varphi(f)(p_A)\neq\pi_i\circ\varphi(f)(p)\quad \text{for some } i\in\{1,2\}.$$

2° *There is an* $f\in E(\mathscr{A})$ *such that* $f(p_A)\neq f(p)$ *and*

$$\pi_i\circ\varphi(f)(p_A)=\pi_i\circ\varphi(f)(p)\quad \text{for}\quad i=1,2.$$

Proof. Let

$$U=\{f\in E(\mathscr{A})\colon f(p_A)\neq f(p)\},$$
$$V=\{(f,g)\in E(\mathscr{A})\times E(\mathscr{A})\colon f(p_A)\neq f(p) \text{ or } g(p_A)\neq g(p)\}.$$

If neither of conditions 1° and 2° were satisfied, then we would have $\varphi(U)=V$, but this is impossible since

$$U=\{f\in E(\mathscr{A})\colon f(p_A)<f(p)\}\cup\{f\in E(\mathscr{A})\colon f(p_A)>f(p)\}$$

splits into two disjoint nonempty open sets, while

$$V=(U\times E(\mathscr{A}))\cup(E(\mathscr{A})\times U)$$

is connected.

Let $\mathscr{F}$ be the family of functions from Lemma 4.3. We shall choose by transfinite induction subfamilies $\mathscr{X}_\alpha\subset\mathscr{T}$ and sets $A_\alpha\in\mathscr{T}$ for $\alpha<2^\omega$ in such a way that the following conditions will be satisfied for each $\alpha<2^\omega$:

(a) $1\leqslant|\mathscr{X}_\alpha|\leqslant\omega$.

(b) $\mathscr{X}_\beta\cap\mathscr{X}_\alpha=\emptyset$ for $\beta<\alpha$.

(c) $A_\beta\notin\mathscr{X}_\gamma$ for $\beta,\gamma\leqslant\alpha$.

(d) If there exists a subfamily $\mathscr{A}$ of $\mathscr{T}$ such that $|\mathscr{A}|=2^\omega$, $A_\beta\notin\mathscr{A}$ for $\beta<\alpha$ and φ_α (restricted to $E(\mathscr{A})$) maps $E(\mathscr{A})$ homeomorphically onto $E(\mathscr{A})\times E(\mathscr{A})$, with respect to either the weak or the pointwise topology, then one of the following conditions is satisfied:

1° There is an $f\in E(\bigcup_{\beta\leqslant\alpha}\mathscr{X}_\beta)$ such that

$$\pi_i\circ\varphi_\alpha(f)(p_{A_\alpha})\neq\pi_i\circ\varphi_\alpha(f)(p)\quad \text{for some } i\in\{1,2\}.$$

2° There are $f,g\in E(\bigcup_{\beta\leqslant\alpha}\mathscr{X}_\beta)$ such that

$$\varphi_\alpha^{-1}((f,g))(p_{A_\alpha})\neq\varphi_\alpha^{-1}((f,g))(p)$$

(recall that φ_α is one-to-one).

Suppose that we have chosen $\mathscr{X}_\beta$ and A_β for $\beta<\alpha<2^\omega$.

First, assume that there exists a subfamily $\mathscr{A}$ of $\mathscr{T}$ such as in condition (d).

We take an arbitrary $A_\alpha\in\mathscr{A}\setminus\bigcup_{\beta<\alpha}\mathscr{X}_\beta$ (notice that $|\bigcup_{\beta<\alpha}\mathscr{X}_\beta|\leqslant|\alpha|\cdot\omega<2^\omega$). Now, Lemma 5.1 applied to the family $\mathscr{A}$ and the element A_α of $\mathscr{A}$ yields the existence of an $f\in E(\mathscr{A})$ for which either

1° $f(p_{A_\alpha})=f(p)$ and there is an $i\in\{1,2\}$ such that

$$\pi_i\circ\varphi_\alpha(f)(p_{A_\alpha})\neq\pi_i\circ\varphi_\alpha(f)(p)$$

or

2° $f(p_{A_\alpha})\neq f(p)$ and $\pi_i\circ\varphi_\alpha(f)(p_{A_\alpha})=\pi_i\circ\varphi_\alpha(f)(p)$ for $i=1,2$.

In the first case there is a countable subfamily $\mathscr{Y}$ of $\mathscr{A}$ such that $f\in E(\mathscr{Y})$ (cf. Sec. 2). Since $f(p_{A_\alpha})=f(p)$ we can assume that $A_\alpha\notin\mathscr{Y}$. If $\mathscr{Y}\setminus\bigcup_{\beta<\alpha}\mathscr{X}_\beta\neq\emptyset$ we take $\mathscr{X}_\alpha=\mathscr{Y}\setminus\bigcup_{\beta<\alpha}\mathscr{X}_\beta$. Otherwise we choose an arbitrary $B\in\mathscr{A}\setminus(\bigcup_{\beta<\alpha}\mathscr{X}_\beta\cup\{A_\alpha\})$ and put $\mathscr{X}_\alpha=\{B\}$.

In case 2° we choose a countable $\mathscr{Y}\subset\mathscr{A}$ such that $\pi_i\circ\varphi_\alpha(f)\in E(\mathscr{Y})$ for $i=1,2$ and $A_\alpha\notin\mathscr{Y}$. The family $\mathscr{X}_\alpha$ is defined as in the preceding case.

If no subfamily $\mathscr{A}$ of $\mathscr{T}$ satisfying (d) exists, one chooses arbitrary $\mathscr{X}_\alpha$ and A_α satisfying conditions (a), (b) and (c). This completes the inductive step of our construction.

Put $\mathscr{X}=\bigcup_{\alpha<2^\omega}\mathscr{X}_\alpha$. We shall verify that $K=L_{\mathscr{X}}$ has the required properties.

As was pointed out in Sec. 2 one can identify $C(K)$ with $E(\mathscr{X})$ and hence we have to show that there is no homeomorphism $\psi\colon E(\mathscr{X})\to E(\mathscr{X})\times E(\mathscr{X})$, where $E(\mathscr{X})$ is equipped with either the weak or the pointwise topology. But, if ψ were such a homeomorphism, then by Lemma 4.3 we would have $\psi=\varphi_\alpha|E(\mathscr{X})$ for some $\alpha<2^\omega$ and then by (d) (where $\mathscr{A}=\mathscr{X}$) one of the conditions 1° and 2° would be satisfied.

However, both possibilities yield a contradiction: in the first case we would have an $f\in E(\bigcup_{\beta\leqslant\alpha}\mathscr{X}_\beta)\subset E(\mathscr{X})$ such that $\pi_i\circ\psi(f)\notin E(\mathscr{X})$, since $A_\alpha\notin\mathscr{X}$; and in case 2° we would have $f,g\in E(\mathscr{X})$ such that $\psi^{-1}((f,g))\notin E(\mathscr{X})$. This completes the construction of the compact space K described in the example.

6. Some refinements of the construction. By a modification of our construction one can obtain an example of an infinite compact space K such that $C_w(K)^n\underset{\text{top}}{\neq}C_w(K)^m$ and $C_p(K)^n\underset{\text{top}}{\neq}C_p(K)^m$ for all natural numbers $n\neq m$.

Let us indicate two most essential changes in our reasoning needed for such a construction.

First, instead of the family of functions $\mathscr{F}$ from Lemma 4.3, we should consider a family $\mathscr{F}'=\{\varphi'_\alpha\colon\alpha<2^\omega\}$ which satisfies the following conditions:

(a) For each $\alpha<2^\omega$, there are distinct $n,m\in\omega$ such that $\mathrm{dom}(\varphi'_\alpha)\subset C(L)^n$ and $\mathrm{ran}(\varphi'_\alpha)\subset C(L)^m$.

(b) φ'_α is one-to-one for $\alpha<2^\omega$.

(c) For every subfamily $\mathscr{A}$ of $\mathscr{T}$, any $m,n\in\omega$, $m\neq n$, and every homeomorphism ψ from $E(\mathscr{A})^n$ onto $E(\mathscr{A})^m$, with respect to the weak or pointwise topology, there exists an $\alpha < 2^\omega$ such that $\psi = \varphi'_\alpha | E(\mathscr{A})^n$.

The existence of the family $\mathscr{F}'$ can be derived in the same way as for the family $\mathscr{F}$.

Secondly, Lemma 5.1 in the inductive step should be replaced in our construction by the following:

6.1. Lemma. *Let $\mathscr{A}$ be a subfamily of $\mathscr{T}$ and let $\varphi\colon E(\mathscr{A})^n \to E(\mathscr{A})^m$ be a weak or pointwise homeomorphism, for some $m, n \in \omega$, $m \neq n$. Then for each $A \in \mathscr{A}$ one of the following two conditions holds:*

1° *There are $f_1, \ldots, f_n \in E(\mathscr{A})$ such that $f_i(p_A) = f_i(p)$ for $i = 1, \ldots, n$ and*

$$\pi_j \circ \varphi((f_1, \ldots, f_n))(p_A) \neq \pi_j \circ \varphi((f_1, \ldots, f_n))(p) \quad \text{for some } j \in \{1, \ldots, m\}.$$

2° *There are $f_1, \ldots, f_n \in E(\mathscr{A})$ such that $f_i(p_A) \neq f_i(p)$ for some $i \in \{1, \ldots, n\}$ and*

$$\pi_j \circ \varphi((f_1, \ldots, f_n))(p_A) = \pi_j \circ \varphi((f_1, \ldots, f_n))(p) \quad \text{for } j = 1, \ldots, m.$$

Proof. Define, as in the proof of Lemma 5.1, $U = \{f \in E(\mathscr{A})\colon f(p_A) \neq f(p)\}$. For each natural number n put

$$U_n = \{(f_1, \ldots, f_n) \in E(\mathscr{A})^n\colon f_i \in U \text{ for some } i \in \{1, \ldots, n\}\}.$$

It can be easily verified that the subspace U_n of $E(\mathscr{A})^n$ equipped with the weak or pointwise topology has the homotopy type of the sphere S^{n-1}. Indeed, for given $(f_1, \ldots, f_n) \in U_n$ and $t \in [0, 1]$ define the functions $f_i^t \in E(\mathscr{A})$, for $i = 1, \ldots, n$, by the formula

$$f_i^t(x) = \begin{cases} (1-t)f_i(x) + t(f_i(p_A) - f_i(p))\Big(\sum_{j=1}^{n} (f_j(p_A) - f_j(p))^2\Big)^{-1/2} & \text{for } x \in \{p_A\} \cup A, \\ (1-t) f_i(x) & \text{for } x \in L \setminus (\{p_A\} \cup A). \end{cases}$$

Now, the homotopy $H\colon U_n \times [0, 1] \to U_n$ defined by $H((f_1, \ldots, f_n), t) = (f_1^t, \ldots, f_n^t)$ is a deformation of U_n onto its subset $H(U_n \times \{1\})$ homeomorphic to S^{n-1}.

Therefore we have $\varphi(U_n) \neq U_m$, i.e. one of the conditions 1° or 2° holds true.

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