to infinity: either $\sigma_{n_k,m_k}(f, x)$ converges to $f(x)$ a.e. for all $f \in L([0, 2\pi]^2)$, or for any function $g(t)$ with $g(t)\downarrow 0$ as $t \to \infty$ there is an $f \in g(L)\,L\log^+ L([0, 2\pi]^2)$ such that

$$\limsup_{k\to\infty} \sigma_{n_k,m_k}(f, x) = +\infty \quad \text{a.e. on } [0, 2\pi]^2.$$

Analogous results hold for (C, α, β) summability $(0 < \alpha \leqslant 1,\ 0 < \beta \leqslant 1)$.

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On subspaces of H^1 isomorphic to H^1

by

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Abstract. We show that any subspace of H^1 which is isomorphic to H^1 contains a complemented copy of H^1. H^1 is proved to be primary.

Introduction. This work is best regarded as an appendix to the book *Symmetric Structures in Banach Spaces* by W. Johnson, B. Maurey, G. Schechtman and L. Tzafriri ([JMST]), where the result analogous to our Theorem 1 is proved for L^p spaces $(1 < p < \infty)$.

We use their notation and follow their arguments rather closely.

I feel obliged to indicate at which point the treatment of H^1 spaces requires different tools than that for L^p spaces $(1 < p < \infty)$:

In trying to find complemented subspaces in the range of embeddings on L^p, JMST rely on the following martingale inequality due to E. M. Stein: Given an increasing sequence of σ-fields $(\mathscr{F}_n)_{n\in\boldsymbol{N}}$ in $[0, 1]$ with corresponding conditional expectations $(E_n)_{n\in\boldsymbol{N}}$, for any $1 < p < \infty$ there exists $C_p \in \boldsymbol{R}^+$ such that for any sequence of measurable functions $(f_n)_{n\in\boldsymbol{N}}$ the following holds:

$$\int \Big(\sum_{n=1}^{\infty} |E_n f_n|^2\Big)^{p/2} \leqslant C_p \int \Big(\sum_{n=1}^{\infty} |f_n|^2\Big)^{p/2}.$$

There exist examples (cf. [St], p. 105) showing that this inequality does not hold for $p = 1$ or $p = \infty$.

Here we modify the selection process of [JMST] in such a way that projections can be constructed which are bounded on H^1. At this point the third component of the vector measure used below becomes crucial.

Definitions and notation. Recall that H^1 is the closed linear span of the L^∞-normalized Haar system

$$\{h_{ni}\colon (ni) \in \mathscr{A}\} \quad \text{where } \mathscr{A} = \{(ni)\colon n \in \boldsymbol{N},\ 0 \leqslant i \leqslant 2^n - 1\}$$

under the norm

$$\|f\|_{H^1} = \int S(f), \qquad S(f) = \Big(\sum a_{ni}^2 h_{ni}^2\Big)^{1/2},$$

with $f = \sum a_{ni} h_{ni}$.

BMO is the space of integrable functions f on $(0, 1]$ such that

$$\|f\|_{\mathrm{BMO}} = \sup_I \left(|I|^{-1} \int_I (f-f_I)^2\right)^{1/2} < \infty.$$

Here the supremum is taken over all dyadic intervals I, and $f_I = |I|^{-1} \int_I f$.

Given $f \in H^1$ and a collection of dyadic intervals $\mathscr{D}$, we write

$$f \cdot \chi_{\mathscr{D}} = \sum_{I \in \mathscr{D}} \langle f, h_I \rangle h_I/|I|.$$

We *identify* $(ni) \in \mathscr{A}$ with the dyadic interval $\left(2^{-n} i, 2^{-n}(i+1)\right]$. $\mathscr{E}_N$ denotes the σ-algebra generated by $\{(2^{-n} i, 2^{-n}(i+1)]: (ni) \in \mathscr{A}_N\}$, where $\mathscr{A}_N = \{(ni) \in \mathscr{A}: n < N\}$. Subsequently the letters I and J are reserved for dyadic intervals. Observe that for $f = \sum a_J h_J$ we obtain

$$\|f\|_{\mathrm{BMO}} = \sup_I \left(|I|^{-1} \sum_{J \subset I} |a_J|^2 |J|\right)^{1/2}.$$

THEOREM 1. *Let X be a subspace of H^1. Assume X is isomorphic to H^1. Then X contains a smaller subspace Y, complemented in H^1 and isomorphic to H^1.*

Proof. Let $T: H^1 \to H^1$ denote the embedding of X into H^1.

Part a: Reduction. Without loss of generality we assume that

$$T(h_{ni}), \quad (ni) \in \mathscr{A}, \quad \text{is a block basis}$$

with respect to the Haar basis in H^1. (This is justified by standard arguments as given e.g. in [JMST], pp. 254–255.) Define

$$v_n = S\Big(\sum_{i=1}^{2^n-1} T(h_{ni})\Big).$$

Then there exist $\eta, R \in \mathbf{R}^+$ such that

$$\int v_n \chi_{\{v_n < R\}} > \eta, \qquad \forall n \in N$$

(cf. [JMST], pp. 265–266).

Put

$$\mathscr{B}_n = \Big\{I: \Big\langle \sum_{i=0}^{2^n-1} T(h_{ni}), h_I \Big\rangle \neq 0\Big\}.$$

A standard stopping time argument gives us a collection of dyadic intervals $\mathscr{G}_n \subset \mathscr{B}_n$ such that

$$v_n \wedge R = S\Big(\sum_{i=0}^{2^n-1} T(h_{ni}) \cdot \chi_{\mathscr{G}_n}\Big).$$

Next define

$$v_n(A) = S^2\Big(\sum_{(ni) \subset A} T(h_{ni})\Big) \wedge R^2$$

where A is taken from $\mathscr{E}_n$.

Due to the L^∞-boundedness of the range of v_n, $n \in N$, there exist (for given $A \in \mathscr{E}_n$) disjoint finite subsets N_j, $j \in N$, of the natural numbers and positive real numbers α_n with

$$\sum_{n \in N_j} \alpha_n = 1$$

such that

$$\sum_{n \in N_j} \alpha_n v_n(A) =: \Lambda_j(A)$$

converges in L^2 (hence in L^1 and almost everywhere). This limit is denoted by $\Lambda(A)$. Consequently, there exist $\varepsilon, c \in \mathbf{R}^+$ such that

$$\inf_n \int_0^1 \max_{1 \leqslant i \leqslant 2^n} \Lambda\big((ni)\big) \geqslant 2\varepsilon, \qquad \int_0^1 \Lambda(A) \leqslant c|A|$$

for any $A \in \mathscr{E}$ (for all that cf. [JMST], pp. 266–268). Subsequently we will use the following notation:

$$\sum_{n \in N_j} \alpha_n^{1/2} \Big(\sum_{(ni) \subset A} T(h_{ni})\Big) \chi_{\mathscr{G}_n} =: \lambda_j(A),$$

$$\sum_{n \in N_j} \alpha_n^{1/2} \Big(\sum_{(ni) \subset A} h_{ni}\Big) =: \gamma_j(A),$$

where (α_n) are the same as in the construction of $\Lambda_j(A)$.

Egorov's Theorem implies (cf. [JMST], p. 256) that there exists a measurable subset $G \subset [0, 1]$ such that for $(ni) \in \mathscr{A}$

$$(*) \qquad \lim_{j \to \infty} \sup_{t \in G} \big|\Lambda_j\big((ni)\big)(t) - \Lambda\big((ni)\big)(t)\big| = 0,$$

$$(**) \qquad \int_G \max_{0 \leqslant i \leqslant 2^n - 1} \Lambda\big((ni)\big) > \varepsilon.$$

Now define

$$v(A)(t) := \Lambda(A)(t)\, \chi_G(t)$$

for every measurable set $A \subset [0, 1]$. $(*)$ and $(**)$ imply (cf. [JMST], pp. 250, 251) that there exist $\eta \in \mathbf{R}^+$, a measurable set E $(\subset G)$ of positive measure and a measurable function $\varphi: E \to [0, 1]$ such that for

$$M(t) := \lim_{n \to \infty} \max_{0 \leqslant j \leqslant 2^n - 1} v\big((n, i)\big)(t)$$

the following holds:

1) $M(t) > \eta$ for $t \in E$.

2) $\chi_{\varphi^{-1}(A)}(t) M(t) \leqslant \nu(A)(t)$ for every measurable set $A \subset [0, 1]$ and almost every $t \in [0, 1]$.

3) There exists $c \in \mathbf{R}^+$ such that for every measurable set $A \subset [0, 1]$ we have $|\varphi^{-1}(A)| < c|A|$.

Now we are prepared for

Part b: Selection process.

Step 0a. Fix $\varepsilon_0 > 0$ and choose $m_0 \in \mathbf{N}$ large enough that

$$\sup_{t\in G} |\Lambda((0, 1])(t) - \Lambda_{m_0}((0, 1])(t)| < \varepsilon_0.$$

Define

$$I_{00} = \{I\colon \langle \lambda_{m_0}((0, 1]), h_I\rangle \neq 0\}.$$

Observe that I_{00} has finite cardinality. Put $F_{00} = (0, 1]$. Consider the nonatomic vector measure

$$\mu_{00}\colon F_{00} \to \mathbf{R}^k, \quad F \to \left(|F|, |\varphi^{-1}(F)|, (|\varphi^{-1}(F) \cap I|)_{I \in I_{00}}\right).$$

As an application of Lyapunov's Theorem there exist (for ε_1 given) a natural number k_1 and disjoint subsets F_{10}, F_{11} of $[0, 1]$ lying in $\mathscr{E}_{k_1}$ such that

$$|F_{1j}|(1+\varepsilon_1)^{-1} \leqslant \tfrac{1}{2}|F_{00}| \leqslant |F_{1j}|(1+\varepsilon_1),$$
$$|\varphi^{-1}(F_{1j})|(1+\varepsilon_1)^{-1} \leqslant \tfrac{1}{2}|\varphi^{-1}(F_{00})| \leqslant |\varphi^{-1}(F_{1j})|(1+\varepsilon_1),$$
$$|\varphi^{-1}(F_{1j}) \cap I|(1+\varepsilon_1)^{-1} \leqslant \tfrac{1}{2}|\varphi^{-1}(F_{00}) \cap I| \leqslant |\varphi^{-1}(F_{1j}) \cap I|(1+\varepsilon_1)$$

for $j \in \{0, 1\}$ and $I \in I_{00}$.

Step 0b. Find $l_1 \in \mathbf{N}$ and disjoint sets $\bar{G}_{1j}$, $j \in \{0, 1\}$, in $\mathscr{E}_{l_1}$ such that

$$|G_{1j} \Delta \bar{G}_{1j}| < \varepsilon_1, \quad \text{where } G_{1j} := \varphi^{-1}(F_{1j}).$$

Next choose $m_1 > m_0$ large enough that the following holds:

(i) $\sup_{t\in G} |\Lambda_{m_1}(F_{1j})(t) - \Lambda(F_{1j})(t)| < \varepsilon_1$.

(ii) $\inf\{l\colon l \in N_{m_1}(F_{1j})\} > 2^{k_1}$.

(iii) For $I_{1j} := \{I\colon I \cap \bar{G}_{1j} \neq \emptyset$ and $\langle \lambda_{m_1}(F_{1j}), h_I\rangle \neq \emptyset\}$ we have

$$\sup\{|I|\colon I \in I_{1j}\} < 2^{-l_1}.$$

We continue and arrive at

Step na. Here we are given a nonatomic vector measure (with finite-dimensional range)

$$\mu_{ni}\colon F_{ni} \to \mathbf{R}^k \quad F \to \left(|F|, |\varphi^{-1}(F)|, (|\varphi^{-1}(F) \cap I|)_{I \in I_{ni}}\right).$$

We apply Lyapunov's theorem and obtain, for $\varepsilon_{n+1} > 0$ given, a natural number $k_{n+1} > k_n$ and disjoint subsets of F_{ni}, $F_{n+1,2i}$ and $F_{n+1,2i+1}$ in $\mathscr{E}_{k_{n+1}}$ such that

$$|F_{n+1,2i+j}|(1+\varepsilon_{n+1})^{-1} \leqslant \tfrac{1}{2}|F_{ni}| \leqslant |F_{n+1,2i+j}|(1+\varepsilon_{n+1}),$$
$$|\varphi^{-1}(F_{n+1,2i+j})|(1+\varepsilon_{n+1})^{-1} \leqslant \tfrac{1}{2}|\varphi^{-1}(F_{ni})| \leqslant |\varphi^{-1}(F_{n+1,2i+j})|(1+\varepsilon_{n+1}),$$
$$|\varphi^{-1}(F_{n+1,2i+j}) \cap I|(1+\varepsilon_{n+1})^{-1} \leqslant \tfrac{1}{2}|\varphi^{-1}(F_{ni}) \cap I| \leqslant |\varphi^{-1}(F_{n+1,2i+j}) \cap I|(1+\varepsilon_{n+1})$$

for $I \in I_{ni}$ and $j \in \{0, 1\}$.

Step nb. Find $l_{n+1} \in \mathbf{N}$ and disjoint subsets $\bar{G}_{n+1,2i+j}$ of $\bar{G}_{ni}$, lying in $\mathscr{E}_{l_{n+1}}$, such that

$$|G_{n+1,2i+j} \Delta \bar{G}_{n+1,2i+j}| < \varepsilon_{n+1} 2^{-l_n}, \quad \text{where } G_{n+1,2i+j} := \varphi^{-1}(F_{n+1,2i+j}).$$

Next choose m_{n+1} large enough that the following holds:

(i) $\sup_{t\in G} |\Lambda_{m_{n+1}}(F_{n+1,2i+j})(t) - \Lambda(F_{n+1,2i+j})(t)| < \varepsilon_{n+1}$.

(ii) $\inf\{l\colon l \in N_{m_{n+1}}(F_{n+1,2i+j})\} > 2^{k_{n+1}}$.

(iii) For $I_{n+1,2i+j} := \{I\colon \bar{G}_{n+1,2i+j} \cap I \neq 0$ and $\langle \lambda_{m_{n+1}}(F_{n+1,2i+j}), h_I\rangle \neq 0\}$ we have

$$\sup\{|I|\colon I \in I_{n+1,2i+j}\} \leqslant 2^{-l_{n+1}}.$$

Finally, we put

$$\gamma_{m_n}(F_{ni}) =: g_{ni},$$
$$\chi_{I_{ni}} \cdot \lambda_{m_n}(F_{ni}) =: k_{ni} \quad \text{for all } (ni) \in \mathscr{A}.$$

Part c: Projecting onto span$\{Tg_{ni}\colon (ni) \in \mathscr{A}\}$. In this section we will verify the following statements:

A) $i\colon H^1 \to H^1$, $h_{ni} \to g_{ni}$ *extends to an isomorphism onto* span$\{g_{ni}\colon (ni) \in \mathscr{A}\}$.

B) $j\colon H^1 \to H^1$, $h_{ni} \to k_{ni}$ *extends to an isomorphism onto* span$\{k_{ni}\colon (ni) \in \mathscr{A}\}$.

C) *There exists a projection P bounded on H^1 onto* span$\{k_{ni}\colon (ni) \in \mathscr{A}\}$ *such that* $P(Tg_{ni} - k_{ni}) \equiv 0$, $(ni) \in \mathscr{A}$.

A, B and C imply that $\tilde{P} := Tij^{-1}P$ is a bounded idempotent operator onto span$\{Tg_{ni}\colon (ni) \in \mathscr{A}\}$.

Ad A. This is proved in [JMST], p. 129.

Ad B. Define $B_{ni} := G_{ni} \cap \bar{G}_{ni}$, $(ni) \in \mathscr{A}$. Observe that:

(a) $B_{n+1,2i+1} \cup B_{n+1,2i} \subset B_{ni}$.

(b) There exists $c \in \mathbf{R}^+$ such that

$$c^{-1}2^{-n} \leqslant |B_{ni}| \leqslant c2^{-n} \quad \text{(with } c \text{ independent of } n).$$

Moreover, for $t \in B_{ni}$ we obtain

$$\eta \leqslant M(t) \leqslant v(F_{ni})(t) \leqslant \Lambda_{m_n}(F_{ni})(t)+\varepsilon_n \leqslant S^2(k_{ni})(t)+\varepsilon_n.$$

Hence for given (a_{ni}), $(ni) \in \mathscr{A}$, we obtain

$$\begin{aligned}\|\sum a_{ni} h_{ni}\|_{H^1} &\geqslant c_1 \|\sum a_{ni} g_{ni}\|_{H^1} \geqslant c_2 \|\sum a_{ni} Tg_{ni}\|_{H^1} \\ &\geqslant c_3 \int (\sum a_{ni}^2 S^2(k_{ni}))^{1/2} \geqslant c_4 \int (\sum a_{ni}^2 \chi_{B_{ni}} \eta^2)^{1/2} \\ &\geqslant c_5 \eta \int (\sum a_{ni}^2 h_{ni}^2)^{1/2}.\end{aligned}$$

Ad C. Observe first that $\{I\colon \langle k_{ni}, h_I\rangle \neq 0\}$ coincides with I_{ni}. Fix $I \in I_{ni}$ and $(mj) \in \mathscr{A}$; then we get

$$k_{mi}|_I = \text{const} \quad \text{for } m \leqslant n, \qquad \int_I k_{mi} = 0 \quad \text{for } m < n,$$

$$\int_I k_{mj}^2 \leqslant c|G_{mj} \cap I| \leqslant 2^{-m+n}|I| \cdot c \qquad \text{for } m \geqslant n.$$

Moreover, (k_{ni}), $(ni) \in \mathscr{A}$, is a block basis w.r.t. the Haar functions. Hence

$$P\colon H^1 \to H^1, \qquad f \to \sum \langle f, k_{ni}\rangle k_{ni}/\|k_{ni}\|_2^2$$

is bounded iff

$$(j^{-1}P)^*\colon \text{BMO} \to \text{BMO}, \qquad h_{ni} \to k_{ni}$$

extends to a linear map on BMO. Let us check that this is just the case: Fix (a_{mj}), $(mj) \in \mathscr{A}$, fix $(ni) \in \mathscr{A}$ and $I \in I_{ni}$. The preceding discussion implies now

$$\begin{aligned}|I|^{-1}\int_I \big(\sum a_{mj} k_{mj} - (\sum a_{mj} k_{mj})_I\big)^2\,dt &\leqslant \sum_{(mj) \subset (ni)} |I|^{-1} \int_I k_{mj}^2 a_{mj}^2 \\ &\leqslant \sum_{(mj) \subset (ni)} |I|^{-1}|I|\,2^{-m+n} a_{mj}^2 \leqslant \|\sum_{(mj)} h_{mj} a_{mj}\|_{\text{BMO}}^2.\end{aligned}$$

A glance at the definition of (k_{ni}) shows that P actually sends $k_{ni} - Tg_{ni}$ to zero.

THEOREM 2. *For any bounded operator T on H^1, either $T(H^1)$ or $(\text{Id}-T)(H^1)$ contains a complemented copy of H^1.*

Proof. Assume once again that (Th_{ni}), $(ni) \in \mathscr{A}$, is a block basis w.r.t. the Haar basis in H^1. Put

$$v_n^1 = S\Big(\sum_{i=0}^{2^n-1} Th_{ni}\Big),$$

$$v_n^2 = S\Big(\sum_{i=0}^{2^n-1} (\text{Id}-T)h_{ni}\Big), \qquad n \in \mathbf{N}.$$

Then there exist $j \in \{1, 2\}$ and $\delta \in \mathbf{R}$ such that $\int v_n^j > \delta$ for infinitely many $n \in \mathbf{N}$. Now we can repeat the whole argument given above.

Remark. Taking into account that H^1 is isomorphic to $(\sum H^1)_1$ we can use Theorem 2 to deduce that H^1 is primary. For a more elementary proof of this fact see [M].

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