STUDIA MATHEMATICA, T. LXXXVIII. (1988)

On infinitely small orbits

by

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Abstract. In this paper we begin the study of infinitely small representations of a nilpotent Lie group G, i.e. the representations which cannot be Hausdorff-separated from the identity representation. We show that, in the case where G is a semidirect product of $\boldsymbol{R}$ with $\boldsymbol{R}^n$, the cortex, i.e. the totality of all those representations, considered as a subset of the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G, is the set of the common zeros of the G-invariant polynomials on $\mathfrak{g}^*$.

1. Infinitely small orbits and invariants. Let G be a locally compact group and Π a continuous representation of G on a finite-dimensional (real) vector space V.

We say that a point $v \in V$ (resp. its G-orbit $\omega = \Pi(G)v$) is *infinitely small* with respect to the action of G if there exist no disjoint G-invariant neighborhoods of $0 \in V$ and v. If we provide the orbit space $V/\Pi(G)$ of V under $\Pi(G)$ with the canonical quotient topology, then v is infinitely small if and only if $\omega = \Pi(G)v$ and $\omega_0 = \{0\}$ cannot be Hausdorff-separated in $V/\Pi(G)$.

The set of all infinitely small elements of V with respect to the action of G will be called the *Π-cortex of* 0 or simply the *Π-cortex*, and denoted by $C_V(\Pi)$.

Obviously, v is in $C_V(\Pi)$ if and only if there are sequences $\{v^{(k)}\}_k \subset V$ and $\{g_k\}_k \subset G$ such that $\lim v^{(k)} = 0$ and $\lim \Pi(g_k)v^{(k)} = v$.

In [4] the cortex of a general locally compact group G was defined as the set of points in the dual $\hat{G}$ of G which cannot be Hausdorff-separated from the identity representation. If G is a connected, simply connected nilpotent Lie group, then by the results of Kirillov [3] and Brown [1] $\hat{G}$ is homeomorphic to $\mathfrak{g}^*/\mathrm{Ad}^*(G)$, where Ad^* denotes the coadjoint representation of G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$. So in this case our definition of a cortex extends the definition from [4], whereas in the case of general groups the two definitions are not so easily related.

However, in this paper we shall only deal with nilpotent groups G, mainly with $G = \boldsymbol{R}$.

In the case that Π is a unipotent representation of a nilpotent Lie group G, we define the *cortex of invariants* of Π as

$$IC_V(\Pi) = \{v \in V\colon\ p(0) = p(v) \text{ for all } G\text{-invariant polynomials } p \text{ on } V\}.$$

Clearly one has the inclusion

$$C_V(\Pi) \subset IC_V(\Pi), \tag{1}$$

and the results which will follow support the conjecture that in fact $C_V(\Pi) = IC_V(\Pi)$. However, in the last example presented in this paper we were not able to prove that $C_V(\Pi) = IC_V(\Pi)$, especially we could not determine $IC_V(\Pi)$. This may be a test case for our conjecture.

Since in general it is troublesome to prove the G-invariance of a polynomial on V directly, it is better to use the derived representation $d\Pi$ of the Lie algebra $\mathfrak{g}$ of G.

Let Π^* denote the Π-contragredient representation of G on the dual space V^* of V, i.e. $\Pi^*(g) = \Pi(g^{-1})^*$, and let $\mathcal{T}(V^*)$ denote the tensor algebra of V^*. There is a canonical epimorphism γ from $\mathcal{T}(V^*)$ onto the algebra $\mathcal{P}(V)$ of polynomials on V, and γ factorizes to an isomorphism γ from $\mathcal{T}_s(V^*) = \mathcal{T}(V^*)/\ker\gamma$ onto $\mathcal{P}(V)$. $\mathcal{T}_s(V^*)$ is of course isomorphic under the symmetrization mapping to the subspace of symmetric tensors in $\mathcal{T}(V^*)$.

If we let G operate on $\mathcal{P}(V)$ by

$$g\cdot p(v) = p(\Pi(g^{-1})v),$$

and on $\mathcal{T}_s(V^*)$ by extending each $\Pi^*(g)$ to an algebra homomorphism of $\mathcal{T}(V^*)$ and then factorizing by $\ker\gamma$, then γ commutes with the two actions of G. The representation of G on $\mathcal{T}_s(V^*)$ will for simplicity also be denoted by Π^*. However, an element $a \in \mathcal{T}_s(V^*)$ is G-invariant if and only if we have for the derived representation $d\Pi$ of the Lie algebra $\mathfrak{g}$ of G

$$d\Pi^*(\mathfrak{g})\,a = 0. \tag{2}$$

Note that $d\Pi^*(X)$ is a derivation of $\mathcal{T}_s(V^*)$ for every $X \in \mathfrak{g}$.

In the next sections we shall concentrate on the case $G = \mathbf{R}$. In this case we may write

$$\Pi(t) = e^{tA}, \quad t \in \mathbf{R}, \tag{3}$$

where A is a linear endomorphism of V. Sometimes we shall write $C_V(A)$ resp. $IC_V(A)$ for $C_V(\Pi)$ resp. $IC_V(\Pi)$, and if W is a Π-invariant subspace of V, we denote the cortices of the action of Π restricted to W also by $C_W(A)$ resp. $IC_W(A)$.

We start with a special unipotent action, which is in some sense the most interesting case since all other cases can be reduced to it.

2. The cortex of the "thread" operation. Let $A = N$ be a nilpotent endomorphism of V, and assume that $V = V_n$ has a basis $\{b_1, \ldots, b_n\}$ such that

$$N(b_j) = b_{j+1} \quad \text{for } j = 1, \ldots, n-1, \quad N(b_n) = 0.$$

For $\xi = \sum_{j=1}^n \xi_j b_j \in V$ we then have

$$\Pi(t)(\xi) = e^{tN}\xi = \sum_{j=1}^{n} \xi_j(t)\, b_j,$$

where

$$\xi_j(t) = \sum_{k=0}^{j-1} t^k \xi_{j-k}/k!.$$

Let $\{X_1, \ldots, X_n\}$ denote the basis of V^* dual to $\{b_1, \ldots, b_n\}$. Then clearly the adjoint operator N^* of N is given by

$$N^*(X_j) = X_{j-1} \quad \text{for } j = 2, \ldots, n, \quad N^*(X_1) = 0. \tag{4}$$

With regard to the considerations in the preceding section, we consider N^* as a derivation of $\mathcal{T}(V^*)$ resp. $\mathcal{T}_s(V^*)$. We have shown that a polynomial $p(\xi_1, \ldots, \xi_n)$ on V is Π-invariant if and only if the corresponding element $p(X_1, \ldots, X_n)$ of $\mathcal{T}_s(V^*)$ satisfies

$$N^*(p(X_1, \ldots, X_n)) = 0. \tag{5}$$

The next lemma provides us with "sufficiently many" invariants to determine the cortex of Π. In combination with the following technical lemma it permits the determination of the cortex of Π.

LEMMA 1. *For $n \geq 2$ and $k = 2, \ldots, [(n+1)/2]$ the polynomials*

$$p_k\Big(\sum_{j=1}^{n} \xi_j b_j\Big) = \tfrac{1}{2}\xi_k^2 + \sum_{j=1}^{k-1} (-1)^j \xi_{k-j}\xi_{k+j}$$

are Π-invariant.

Proof. Consider

$$P_k = \tfrac{1}{2} X_k X_k + \sum_{j=1}^{k-1} (-1)^j X_{k-j} X_{k+j}$$

as an element of $\mathcal{T}_s(V^*)$. Then we have

$$\begin{aligned} N^*(P_k) &= \tfrac{1}{2} X_k X_{k-1} + \tfrac{1}{2} X_{k-1} X_k + \sum_{j=1}^{k-2} (-1)^j X_{k-j-1} X_{k+j} \\ &\quad + \sum_{j=1}^{k-2} (-1)^j X_{k-j} X_{k+j-1} + (-1)^{k-1} X_1 X_{2k-2}, \end{aligned}$$

hence, since $X_k X_{k-1} = X_{k-1} X_k$ in $\mathcal{T}_s(V^*)$,

$$\begin{aligned} N^*(P_k) &= X_k X_{k-1} + \sum_{j=1}^{k-2} (-1)^j X_{k-j-1} X_{k+j} - \sum_{j=0}^{k-3} (-1)^j X_{k-j-1} X_{k+j} \\ &\quad + (-1)^{k-1} X_1 X_{2k-2} \\ &= X_k X_{k-1} + (-1)^{k-2} X_1 X_{2k-2} - X_{k-1} X_k + (-1)^{k-1} X_1 X_{2k-2} \\ &= 0. \ \blacksquare \end{aligned}$$

LEMMA 2. *The* $n\times n$*-matrix*

$$M=\begin{bmatrix}1 & 1/2! & \dots & 1/n!\\ 1/2! & 1/3! & \dots & 1/(n+1)!\\ \dots & \dots & \dots & \dots\\ 1/n! & 1/(n+1)! & \dots & 1/(2n-1)!\end{bmatrix}$$

is regular.

Proof. We multiply the jth row of M by $(n+j-1)!$ and obtain the matrix

$$M'=\begin{bmatrix}2\cdot 3\cdot\ldots\cdot n & 3\cdot 4\cdot\ldots\cdot n & \dots & 1\\ 3\cdot 4\cdot\ldots\cdot(n+1) & 4\cdot 5\cdot\ldots\cdot(n+1) & \dots & 1\\ \dots & \dots & \dots & \dots\\ (n+1)(n+2)\ldots(2n-1) & (n+2)(n+3)\ldots(2n-1) & \dots & 1\end{bmatrix}.$$

For $n\geqslant 2$ let $A(K,n)$ be the matrix

$$\begin{bmatrix}[K,K+n-2] & [K+1,K+n-2] & \dots & [K+n-2,K+n-2] & 1\\ [K+1,K+n-2+1] & [K+2,K+n-2+1] & \dots & [K+n-2+1,K+n-2+1] & 1\\ \dots & \dots & \dots & \dots & \dots\\ [K+n-1,K+n-2+n-1] & [K+1+n-1,K+n-2+n-1] & \dots & [K+n-2+n-1,K+n-2+n-1] & 1\end{bmatrix}$$

where $[K,l]=K\cdot(K+1)\cdot\ldots\cdot l$ for $l\geqslant K$. Then $M'=A(2,n)$. Therefore in order to prove the regularity of M it is sufficient to prove (by induction on n) the regularity of the matrices $A(K,n)$, $n\geqslant 2$.

For $n=2$, $A(K,2)=\begin{bmatrix}K & 1\\ K+1 & 1\end{bmatrix}$ is regular. Assume that $A(K,j)$ is regular for $2\leqslant j\leqslant n-1$. Consider the matrix $A(K,n)$. Subtraction of the $(i-1)$th row from the ith row for $i\geqslant 2$ gives the following matrix:

$$A'(n,K)=\begin{bmatrix}[K,K+n-2] & [K+1,K+n-2] & \dots & [K+n-2,K+n-2] & 1\\ & & & & 0\\ & & A^{\sim} & & \vdots\\ & & & & 0\end{bmatrix}$$

where $A^{\sim}=[(n-1)\tilde a_1\ (n-2)\tilde a_2\ \dots\ \tilde a_{n-1}]$ and the $\tilde a_i$ are column vectors in $\boldsymbol{R}^{n-1}$. Furthermore, $[\tilde a_1\ \tilde a_2\ \dots\ \tilde a_{n-1}]=A(K+1,n)$, which is regular by the induction hypothesis. Hence $A^{\sim}$ and then also $A(n,K)$ are regular. ■

If $v_1,\dots,v_k$ are vectors in V, then let $\langle v_1,\dots,v_k\rangle$ denote their linear span in V.

PROPOSITION 1. *If* $n=0$ *or* $n=1$ *then* $C_{V_n}(\Pi)=IC_{V_n}(\Pi)=\{0\}$, *and if* $n\geqslant 2$ *then*

$$C_{V_n}(\Pi)=IC_{V_n}(\Pi)=\langle b_{[(n+1)/2]+1},b_{[(n+1)/2]+2},\dots,b_n\rangle.$$

Proof. The cases $n=0$ and $n=1$ are trivial. Therefore assume $n\geqslant 2$. Set $M_n=\langle b_{[(n+1)/2]+1},\dots,b_n\rangle$. Lemma 1 easily implies

$$IC_{V_n}(\Pi)\subset\{v\in V_n\colon\ p_k(v)=0\text{ for }k=2,\dots,[(n+1)/2]\}=M_n,$$

so we have

$$C_{V_n}(\Pi)\subset IC_{V_n}(\Pi)\subset M_n$$

and it remains to prove that $M_n\subset C_{V_n}(\Pi)$.

Assume first that $n=2m$, $m\geqslant 1$, is even, and let $\lambda=\lambda_{m+1}b_{m+1}+\dots+\lambda_n b_n$ be an arbitrary element of M_n. We seek for a sequence $\{\xi^{(k)}\}_k$ in V_n and sequence $\{t_k\}_k$ of real numbers such that $\lim\xi^{(k)}=0$ and $\lim\Pi(t_k)\xi^{(k)}=\lambda$.

Actually, we may choose for $\{t_k\}_k$ any sequence with $\lim|t_k|=\infty$, for example $t_k=-k$. Assume $\{t_k\}_k$ is fixed. We then determine $\{\xi^{(k)}\}_k$ by the conditions

$$\xi^{(k)}=\xi_1^{(k)}b_1+\dots+\xi_m^{(k)}b_m\in\langle b_1,\dots,b_m\rangle,\tag{6}$$

$$\xi_j^{(k)}(t_k)=\lambda_j\quad\text{for }j=m+1,\dots,n.\tag{7}$$

For fixed $\xi=\xi^{(k)}$ and $t=t_k$, (7) means

$$t\xi_m+\frac{t^2}{2}\xi_{m-1}+\dots+\frac{t^m}{m!}\xi_1=\lambda_{m+1},$$

$$\frac{t^2}{2}\xi_m+\frac{t^3}{3!}\xi_{m-1}+\dots+\frac{t^{m+1}}{(m+1)!}\xi_1=\lambda_{m+2},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{t^m}{m!}\xi_m+\frac{t^{m+1}}{(m+1)!}\xi_{m-1}+\dots+\frac{t^{2m-1}}{(2m-1)!}\xi_1=\lambda_{2m}.$$

Since, by Lemma 2, the matrix $A=\{1/(i+j+1)!\}_{i,j=0}^{m-1}$ is regular, this may be written as

$$[\xi_m,t\xi_{m-1},\dots,t^{m-1}\xi_1]^T=A^{-1}[t^{-1}\lambda_{m+1},t^{-2}\lambda_{m+2},\dots,t^{-m}\lambda_{2m}]^T.\tag{8}$$

So it remains to prove that the sequence $\{\xi^{(k)}\}_k$ defined by (8) satisfies $\lim \xi^{(k)} = 0$ and $\lim \Pi(t_k)\xi^{(k)} = \lambda$. But the first equality is obvious from (8), and for the second it remains by (7) to show that

$$\lim_{k\to\infty} \xi_j^{(k)}(t_k) = 0 \quad \text{for } j = 1, \ldots, m.$$

But, if $j \in \{1, \ldots, m\}$, then

$$\xi_j^{(k)}(t_k) = \sum_{l=0}^{j-1} \frac{t_k^l \xi_{j-l}^{(k)}}{l!}, \quad \text{and} \quad |t_k^l \xi_{j-l}^{(k)}| = |t_k|^{j-m} |t_k^{m-(j-l)} \xi_{j-l}^{(k)}| \to 0$$

as $k \to \infty$, since (8) implies that for $r = 1, \ldots, m$

$$\lim_{k\to\infty} t_k^{m-r} \xi_r^{(k)} = 0,$$

and since $j - m \leqslant 0$.

So we are left with the case of odd n, $n \geqslant 3$. Consider the Π-invariant subspace $\tilde{V}_{n-1} = \langle b_2, \ldots, b_n \rangle$ of V_n. The action of Π on $\tilde{V}_{n-1}$ is a "thread" action on an even-dimensional vector space, so by the preceding result we have

$$C_{\tilde{V}_{n-1}}(\Pi) = \langle b_{[n/2]+2}, \ldots, b_n \rangle.$$

Since $[(n+1)/2]+1 = [n/2]+2$, this implies

$$M_n = C_{\tilde{V}_{n-1}}(\Pi) \subset C_{V_n}(\Pi). \ \blacksquare$$

3. General one-parameter actions. Proposition 1 admits the following easy extension:

Corollary 1. *Let N be a nilpotent endomorphism of an n-dimensional space V and let Π be the associated linear action. Then $C_V(\Pi) = IC_V(\Pi) = C_+(\Pi) = C_-(\Pi)$, where*

$$C_+(\Pi) = \{v \in V\colon \text{there exist sequences } \{v^{(k)}\}, \{t_k\} \text{ such that } \lim_{k\to\infty} v^{(k)} = 0,\ \lim_{k\to\infty} t_k = +\infty \text{ and } \lim_{k\to\infty} \Pi(t_k) v^{(k)} = v\},$$

$$C_-(\Pi) = \{v \in V\colon \text{there exist sequences } \{v^{(k)}\}, \{t_k\} \text{ such that } \lim_{k\to\infty} v^{(k)} = 0,\ \lim_{k\to\infty} t_k = -\infty \text{ and } \lim \Pi(t_k) v^{(k)} = v\}.$$

Proof. Let $V = V_1 \oplus \ldots \oplus V_l$ be an N-invariant direct decomposition of V such that the V_i, $i = 1, \ldots, l$, are indecomposable with respect to N. Then the action of N on the V_i is as in Proposition 1, and we obtain

(9) $$C_{V_i}(N) = IC_{V_i}(N) = \{v_i \in V_i\colon \text{there exists a sequence } \{v_i^{(k)}\}_k \text{ such that } \lim_{k\to\infty} v_i^{(k)} = 0 \text{ and } \lim_{k\to\infty} \Pi(k) v_i^{(k)} = v_i\}$$

$$= \{v_i \in V_i\colon \text{there exists a sequence } \{w_i^{(k)}\}_k \text{ such that } \lim_{k\to\infty} w_i^{(k)} = 0 \text{ and } \lim_{k\to\infty} \Pi(-k) w_i^{(k)} = v_i\}$$

where the last two equalities follow from the proof of the theorem. Since it is clear that

$$C_V(N) = \{v_1 + \ldots + v_l\colon \text{there are sequences } \{v_i^{(k)}\}_k,\ i = 1, \ldots, l, \text{ and a sequence } \{t_k\}, \text{ independent of } i, \text{ such that } \lim_{k\to\infty} v_i^{(k)} = 0 \text{ and } \lim_{k\to\infty} \Pi(t_k) v_i^{(k)} = v_i \text{ for } i = 1, \ldots, l\},$$

we obtain

(10) $$C_V(N) = C_{V_1}(N) + \ldots + C_{V_l}(N) = IC_{V_1}(N) + \ldots + IC_{V_l}(N).$$

However, any invariant polynomial p_i on V_i extends to an invariant polynomial on V by $p(v) = p(v_1 + \ldots + v_k) = p_i(v_i)$. Therefore $IC_V(N) \subseteq IC_{V_1}(N) + \ldots + IC_{V_l}(N) = C_V(N)$. Because one has always $C_V(N) \subseteq IC_V(N)$, one obtains $C_V(N) = IC_V(N)$. This, together with the description of the $C_{V_i}(N)$ in (9), proves the equalities stated in the corollary. $\blacksquare$

The next lemma deals with skew-symmetric "perturbations" of a nilpotent endomorphism.

Lemma 3. *Let A be an endomorphism of an n-dimensional real vector space V such that all eigenvalues of A are purely imaginary. Let $A = S + N$ be the additive Jordan decomposition of A, where S is the semisimple and N the nilpotent part. Then $C_V(A) = C_V(N)$.*

Proof. Let $v \in C_V(A)$. Then there exist a sequence $\{v_n\}$ in V and a sequence $\{t_n\}$ in $\boldsymbol{R}$ such that $\lim v_n = 0$ and $\lim \Pi(t_n) v_n = v$. Put $w_n = e^{t_n S} v_n$. Then we have $\lim w_n = 0$ and $\lim e^{t_n N} w_n = \lim \Pi(t_n) v_n = v$. Therefore $v \in C_V(N)$.

Conversely, let $v \in C_V(N)$. Then there exist a sequence $\{w_n\}$ in V and a sequence $\{t_n\}$ in $\boldsymbol{R}$ such that $\lim w_n = 0$ and $\lim e^{t_n N} w_n = v$. Let $\{t_m\}$ be a subsequence of $\{t_n\}$ such that $\lim e^{t_m S} = A_0$. Then for $v_m = A_0^{-1} w_m$ we have $\lim v_m = 0$ and

$$\lim_{m\to\infty} e^{t_m A} v_m = \lim_{m\to\infty} (e^{t_m S} A_0^{-1}) e^{t_m N} w_m = v.$$

Therefore $v \in C_V(A)$. $\blacksquare$

Let now A be an arbitrary endomorphism of V. Let $V = V_+ \oplus V_- \oplus V_0$ be the A-invariant direct decomposition of V such that

$$\operatorname{Re}\lambda > 0 \quad \text{for all } \lambda \in \sigma(A|_{V_+}),$$

$$\operatorname{Re}\lambda < 0 \quad \text{for all } \lambda\in\sigma(A|_{V_-}),$$

$$\operatorname{Re}\lambda = 0 \quad \text{for all } \lambda\in\sigma(A|_{V_0}),$$

where $\sigma(E)$ denotes the spectrum of the endomorphism E.

THEOREM 1.

$$C_V(\Pi) = C_V(A) = (V_+ \cup V_-)\oplus C_{V_0}(N),$$

where N is the nilpotent part in the additive Jordan decomposition $A|_{V_0} = S + N$ of $A|_{V_0}$.

Proof. First we prove $C_V(A) \subset (V_+ \cup V_-)\oplus C_{V_0}(A)$. In general, for every A-invariant decomposition $V = V_1 \oplus V_2$ of V we have $C_V(A) \subset C_{V_1}(A)\oplus C_{V_2}(A)$. Therefore $C_V(A) \subset C_{V_+\oplus V_-}(A)\oplus C_{V_0}(A)$.

Let $v = v_+ + v_- \in C_{V_+\oplus V_-}(A)$. Then there exist sequences $\{v_n\}$ in $V_+\oplus V_-$ and $\{t_n\}$ in $\boldsymbol{R}$ such that $\lim \Pi(t_n)v_n = v$ and $\lim v_n = 0$. Also $v_n = v_n^+ + v_n^-$, and $\lim v_n^+ = \lim v_n^- = 0$, $\lim \Pi(t_n)v_n^+ = v_+$, $\lim \Pi(t_n)v_n^- = v_-$. But for $t_n \to +\infty$ we have $\lim \Pi(t_n)v_n^- = 0$ implying $v_- = 0$, and for $t_n \to -\infty$, $\lim \Pi(t_n)v_n^+ = 0$ implying $v_+ = 0$. Therefore $v\in V_+ \cup V_-$.

Now assume $v\in (V_+\cup V_-)\oplus C_{V_0}(A)$. Then $v = v_+ + v_0$ or $v = v_- + v_0$. Choose a sequence $\{v_0^n\}$ in V_0 such that $\lim v_0^n = 0$ and $\{t_n\}$ in $\boldsymbol{R}$ such that $\lim \Pi(t_n)v_0^n = v_0$. Put

$$v_+^n = \Pi(-t_n)v_+, \quad v_-^n = \Pi(-t_n)v_-, \quad w_+^n = v_+^n + v_0^n, \quad w_-^n = v_-^n + v_0^n.$$

Then we have

$$\lim_{n\to\infty} \Pi(t_n)w_+^n = v_+ + \lim_{n\to\infty}\Pi(t_n)v_0^n = v_+ + v_0 = v,$$

$$\lim_{n\to\infty} \Pi(t_n)w_-^n = v_- + \lim_{n\to\infty}\Pi(t_n)v_0^n = v_- + v_0 = v.$$

According to Corollary 1 we can choose $t_n \to +\infty$ if $v = v_+ + v_0$ and $t_n \to -\infty$ if $v = v_- + v_0$ and obtain

$$\lim_{n\to\infty} w_+^n = \lim_{n\to\infty} v_+^n = \lim_{n\to\infty}\Pi(-t_n)v_+ = 0,$$

$$\lim_{n\to\infty} w_-^n = \lim_{n\to\infty} v_-^n = \lim_{n\to\infty}\Pi(-t_n)v_- = 0.$$

This proves that $v\in C_V(A)$.

COROLLARY 2. *Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$ such that $G = \boldsymbol{R}\ltimes \boldsymbol{R}^k$. The cortex of G is then equal to the image under the Kirillov homeomorphism of the cortex of the* Ad*-*invariant polynomials.*

Proof. Let us write $\mathfrak{g} = \boldsymbol{R}x\oplus\mathfrak{h}$ where $\mathfrak{h}$ is a k-dimensional abelian ideal of $\mathfrak{g}$. Let $0\neq v\in IC_{\mathfrak{g}^*}(\mathrm{Ad}^*)$. We must show that $v\in C_{\mathfrak{g}^*}(\mathrm{Ad}^*)$.

$v_1 = v|_{\mathfrak{h}}$ is then an element of $IC_{\mathfrak{h}^*}(A)$, where $A = (\mathrm{ad}(x)|_{\mathfrak{h}})^*$. Hence, if $v_1 \neq 0$, there exists by Corollary 1 a sequence $\{w_n\}_n$ in $\mathfrak{g}^*$ such that

$$\langle w_n, x\rangle = 0, \quad \lim_{n\to\infty} w_n = 0,$$

and a sequence $\{t_n\}_n \subset \boldsymbol{R}$ with

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} w_n'|_{\mathfrak{h}} = v_1, \quad w_n' = \exp\big(\mathrm{ad}^*(t_n x)\big)w_n.$$

Now, as $v_1\neq 0$, w_n is not G-fixed for (almost) all n. Thus

$$\mathrm{ad}^*(\mathfrak{h})w_n = w_n + \mathfrak{h}^\perp \quad \text{for all } n$$

and so we find for every n a $u_n\in H = \exp\mathfrak{h}$ with

$$\lim_{n\to\infty} \mathrm{Ad}^*(u_n)w_n' = v.$$

If $v|_{\mathfrak{h}} = 0$, then $\mathfrak{g}$ is not abelian (as $0\neq v\in IC_{\mathfrak{g}^*}(\mathrm{Ad}^*)$). Hence there exists $w\in\mathfrak{g}^*$ with $\mathrm{Ad}^*(H)w = w+\mathfrak{h}^\perp$, whence $\mathrm{Ad}^*(H)(tw) = tw+\mathfrak{h}^\perp$, $t\in\boldsymbol{R}$, and so

$$v\in\mathfrak{h}^\perp\subset C_{\mathfrak{g}^*}(\mathrm{Ad}^*). \quad \blacksquare$$

From Corollary 2 we can deduce the following characterization of the cortex of $G = \boldsymbol{R}\ltimes\boldsymbol{R}^n$ in terms of its C^*-algebra $C^*(G)$. We denote by $Z(\mathfrak{g})$ the center of the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G.

Considering the elements $X\in\mathfrak{g}$ as left- (or right-) invariant vector fields on G we form the subspace

$$Z*D(G) = \{x*f \mid x\in Z(\mathfrak{g}), f\in D(G)\}$$

of the space $D(G)$ of test functions on G.

It is clear that $Z*D(G)$ is right- and left-translation-invariant. Hence the closure of $Z*D(G)$ in $C^*(G)$, denoted by $Z\cdot C^*(G)$, is a two-sided ideal in $C^*(G)$.

THEOREM 2. *Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$ such that $G = \boldsymbol{R}\ltimes\boldsymbol{R}^k$. The cortex of G is then equal to the hull in $\hat{G}$ of the ideal $Z\cdot C^*(G)$ of $C^*(G)$.*

Proof. There exists an algebra isomorphism φ of $Z(\mathfrak{g})$ onto the space IP of G-invariant polynomials on $\mathfrak{g}^*$ such that

$$d\Pi(x) = \varphi(x)(l)\,\mathrm{Id}_\Pi$$

for $x\in Z(\mathfrak{g})$, every $\Pi\in\hat{G}$ and every l in the Kirillov orbit corresponding to Π (see [2], 10.4.5). Hence the hull in $\hat{G}$ of $Z\cdot C^*(G)$ corresponds to the subset $IC_{\mathfrak{g}^*}(\mathrm{Ad}^*)$ of $\mathfrak{g}^*$, and the theorem follows from Corollary 2.

Two further examples. Let $\mathfrak{g}$ be the nilpotent Lie algebra spanned by the basis

$$B = \{X_1, X_2, X_3, H_1, H_2, H_3, H_4, Z\}$$

with the following commutators:

$$[X_1, X_3] = H_2, \quad [X_1, H_4] = Z,$$
$$[X_3, X_2] = H_1, \quad [X_2, X_3] = H_3,$$
$$[X_3, H_1] = H_4, \quad [H_2, H_1] = Z, \quad [H_3, X_2] = Z.$$

In this example no element $v \in \mathfrak{g}^*$ with $\langle v, Z\rangle \neq 0$ has a polarization which is an ideal in $\mathfrak{g}$.

Denote by $\{X_1^*, X_2^*, X_3^*, H_1^*, H_2^*, H_3^*, H_4^*, Z^*\} = B^*$ the basis dual to B in $\mathfrak{g}^*$.

The Pukánszky parametrization of the orbit space $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ gives us the following two G-invariant polynomials q_1 and q_2:

$$q_1(x_1 X_1^* + x_2 X_2^* + x_3 X_3^* + h_1 H_1^* + h_2 H_2^* + h_3 H_3^* + h_4 H_4^* + zZ^*) := z,$$
$$q_2(x_1 X_1^* + x_2 X_2^* + x_3 X_3^* + h_1 H_1^* + h_2 H_2^* + h_3 H_3^* + h_4 H_4^* + zZ^*) := zx_3 - h_3 h_1 - h_4 h_2.$$

An easy computation shows that in this case we also have

$$C_{\mathfrak{g}^*}(\mathrm{Ad}^*) = IC_{\mathfrak{g}^*}(\mathrm{Ad}^*) = \{v \in \mathfrak{g}^* \mid q_1(v) = q_2(v) = 0\}$$
$$= \{x_1 X_1^* + x_2 X_2^* + x_3 X_3^* + h_1 H_1^* + h_2 H_2^* + h_3 H_3^* + h_4 H_4^* + zZ^* \mid z = 0,\ h_3 h_1 + h_4 h_2 = 0\}.$$

Let us finish this article with a problem.

Consider the real 6-dimensional vector space

$$V = \langle b_1, \ldots, b_6\rangle \cong \boldsymbol{R}^6,$$

the nilpotent endomorphisms S and T of V where

$$S(b_i) = b_{i-1}, \quad i = 2, \ldots, 6, \quad S(b_1) = 0, \quad T = S^2,$$

and the unipotent group $G = \exp(\boldsymbol{R}S + \boldsymbol{R}T)$.

It is possible here to determine the cortex $C_V(G)$ with respect to the natural action of G on V. In fact, $C_V(G) = \langle b_4, b_5, b_6\rangle$. But we were not able to determine the cortex $IC_V(G)$ of invariants. The Pukánszky parametrization of V gives us 4 invariant polynomials:

$$q_1 = \lambda_4 \lambda_1^2 - \lambda_3 \lambda_2 \lambda_1 + \tfrac{1}{3}\lambda_2^3,$$
$$q_2 = 4\lambda_5 \lambda_1^3 - 4\lambda_1^2 \lambda_2 \lambda_4 + 4\lambda_1 \lambda_2^2 \lambda_3 - 2\lambda_1^2 \lambda_3^2 - \lambda_2^4,$$
$$q_3 = -\lambda_1^4 \lambda_6 + \lambda_5 \lambda_2 \lambda_1^3 - \lambda_4 \lambda_2^2 \lambda_1^2 + \lambda_1 \lambda_2^3 \lambda_3 - \lambda_1^2 \lambda_2 \lambda_3^2 + \lambda_4 \lambda_3 \lambda_1^3 - \tfrac{1}{5}\lambda_2^5,$$
$$q_4 = \lambda_1.$$

It follows immediately that $IC_V(G) \subset \langle b_3, b_4, b_5, b_6\rangle$.

In order to show that $IC_V(G) = C_V(G)$ we need an invariant polynomial of the form

$$\lambda_3^k + \sum_j \lambda_1^j p_j(\lambda_2, \ldots, \lambda_6) + \sum_m \lambda_2^m p_m(\lambda_1, \lambda_3, \ldots, \lambda_6)$$

for some $k \neq 0$. There does not exist such a polynomial of degree $\leqslant 8$. So we may have here an example of a group where the conjecture $C_{\mathfrak{g}^*}(\mathrm{Ad}^*) = IC_{\mathfrak{g}^*}(\mathrm{Ad}^*)$ fails.

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Received September 8, 1985 (2090)