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An operator on a separable Hilbert space with many hypercyclic vectors

by

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Abstract. We construct a separable Hilbert space H and a linear continuous operator T on it with hypercyclic vector x_0 such that for any polynomial p with rational coefficients, $p(T)x_0$ is also hypercyclic.

Let H be a Hilbert space and T a (linear continuous) operator on it. A point x_0 is said to be *cyclic* (for T) if $H = \text{span}(x_0, Tx_0, T^2 x_0, \ldots)$. It is said to be *hypercyclic* if the orbit $(T^k x_0, k \geqslant 0)$ is dense in H. An intermediate notion, called quasi-hypercyclicity, was introduced by the author in [1].

The present construction provides the following:

THEOREM. *There is a separable complex Hilbert space H and an operator T on it with a cyclic vector x_0 such that every point of the form $p(T)x_0$, where p is a polynomial with complex coefficients having both real and imaginary parts rational, is hypercyclic.*

For this operator, the set of hypercyclic vectors, being a G_δ, is uncountable. But the operator might still have invariant subspaces: to have no nontrivial invariant subspaces requires all points to be cyclic.

It is well known (Rolewicz [6], 1969) that on l_p $(1 \leqslant p < \infty)$ or c_0 there is an operator with a hypercyclic vector x_0. Of course, the iterates of this point are also hypercyclic, so the set of hypercyclic points is infinite, and therefore, being a G_δ, is uncountable. Also, Rolewicz's construction can be modified in order to provide a finite number of polynomials to be hypercyclic. But our construction provides obviously many more independent hypercyclic vectors (i.e. not just iterates of one another).

The present construction uses some of the ideas introduced by P. Enflo to solve the invariant subspace problem in Banach spaces ([3], see also [1]). But there are of course significant differences. Also, C. Read has given an example of an operator on a Banach space with no nontrivial invariant subspaces ([4]), and then such an example in l_1 ([5]).

In an earlier version of this paper (with the same title), the notion of hypercyclicity was replaced by a weaker one: quasi-hypercyclicity, which is an important feature in our example [1]. The strengthening was made possible by the fact that we now use weighted l_2 norms, instead of the classical one. We now turn to the construction of the example.

1. Enumeration of the triples. We first enumerate all the triples $(q_j, q'_j, \varepsilon_j)_{j\geqslant 1}$, where:

– q_j, q'_j are polynomials with rational coefficients (i.e. both real and imaginary parts rational).

– ε_j are of the form $1/2^l$, $l \geqslant 1$.

We require that this enumeration should be done with the following restrictions:

(a) For all $j \geqslant 1$, $d^\circ q_j < j$, $d^\circ q'_j < j$.

(b) For all $j \geqslant 1$, $\varepsilon_j \geqslant 1/2^j$.

This enumeration being done, our construction is totally determined by only one sequence of integers, $(N_j)_{j\geqslant 1}$, fast growing, which will be chosen by induction. We put $l_j = x^{N_j}$.

In our final norm, each q_j will be moved close to q'_j by the multiplication by l_j, with an accuracy of ε_j, i.e.

$$\|l_j q_j - q'_j\| \leqslant \varepsilon_j .$$

2. Definition of the norm $|\cdot|_{(n)}$. We start with a weighted l_2 norm: if $p = \sum_{j\geqslant 0} a_j x^j$, we put

$$|p|_w = \Big(\sum_{j\geqslant 0} (j+1)|a_j|^2\Big)^{1/2}.$$

The space of sequences $\{(a_j)_{j\geqslant 0}\colon \sum (j+1)|a_j|^2 < +\infty\}$ is a Hilbert space, an algebra, and multiplication by x has norm $\sqrt{2}$.

We fix an integer $n \geqslant 1$, and we now look at all representations of a polynomial p of the form

$$p = r + \sum_{j=1}^{n} \sum_{\alpha} a_{j,\alpha} x^\alpha (l_j q_j - q'_j) \tag{1}$$

where r is a polynomial, $a_{j,\alpha}$ are complex numbers ($j, \alpha \in N$). We set:

$$|p|^2_{(n)} = \inf \Big\{|r|^2_w + \sum_{j=1}^{n} \sum_{\alpha} |a_{j,\alpha}|^2 4^\alpha \varepsilon_j^2\Big\} \tag{2}$$

where the infimum is taken over all representations of the form (1).

Then, on the space of polynomials, $|\cdot|_{(n)}$ is a norm and the following properties are obvious:

(a) $|\cdot|_{(n)} \leqslant |\cdot|_{(n-1)} \leqslant \ldots \leqslant |\cdot|_w$.

(b) $|l_j q_j - q'_j|_{(n)} \leqslant \varepsilon_j$, $j = 1, \ldots, n$.

(c) $|l_j p|_{(n)} \leqslant 2^{N_j} |p|_{(n)}$, $j = 1, \ldots, n$.

(d) $|xp|_{(n)} \leqslant 2|p|_{(n)}$.

(e) The norm $|\cdot|_{(n)}$ is equivalent (with constants depending on n) to the original norm $|\cdot|_w$.

(f) The norm $|\cdot|_{(n)}$ is hilbertian.

Indeed, for this last property, one checks immediately that for all polynomials p_1, p_2,

$$2(|p_1|^2_{(n)} + |p_2|^2_{(n)}) \geqslant |p_1 + p_2|^2_{(n)} + |p_1 - p_2|^2_{(n)},$$

and the converse inequality follows after the change of variables $u = p_1 + p_2$, $v = p_1 - p_2$.

3. Almost stationarity of the sequence of norms. All the data required to construct $|\cdot|_{(n)}$ have been completely described, except the sequence $(N_j)_{j\geqslant 1}$. It will be chosen by induction, according to the following

PROPOSITION 1. *Assume $N_1 < \ldots < N_{n-1}$ have been chosen. We can choose N_n so large that for every polynomial p with $d^\circ p \leqslant n$,*

$$|p|_{(n)} \geqslant (1 - 1/4^n)|p|_{(n-1)}.$$

(The choice of N_n could be made precise; it depends only on the data n, $N_1, \ldots, N_{n-1}$, $|q_j|_w$ $(j \leqslant n)$, $|q'_j|_w$ $(j \leqslant n)$.)

Proof. We need a very simple lemma, the proof of which is left to the reader:

LEMMA 2. *For any polynomials p_1, p_2 and every η, $0 < \eta < 1$, if $C \geqslant (1-\eta)/\eta$ then*

$$|p_1 + p_2|^2_w + C|p_2|^2_w \geqslant (1-\eta)|p_1|^2_w .$$

Take now p, $d^\circ p \leqslant n$. We write an expansion of the form (1) as

$$p = p - A + A, \quad \text{with} \quad A = \sum_{j=1}^{n} \sum_{\alpha} a_{j,\alpha} x^\alpha (l_j q_j - q'_j). \tag{3}$$

We use the following notation:

$$[\![A]\!]^2_{(n)} = \sum_j \sum_\alpha |a_{j,\alpha}|^2 4^\alpha \varepsilon_j^2, \tag{4}$$

so the estimate given by (3) is just

$$I_1^2 = |p - A|^2_w + [\![A]\!]^2_{(n)}.$$

We also put $s_j = \sum_\alpha a_{j,\alpha} x^\alpha$, and write the expansion

$$s_j = s_{0,j} + x^{N_n} s_{1,j} + \ldots + x^{kN_n} s_{k,j} + \ldots,$$

where for all $k \geq 0$, $d^\circ s_{k,j} < N_n$. We put

$$U_k = x^{kN_n} \sum_{j=1}^{n} s_{k,j}(l_j q_j - q_j'),$$

so (3) becomes

(5) $$p = p - \sum_{k \geq 0} U_k + A.$$

We wish first to construct a new representation from which all terms U_k, $k \geq 1$, have been removed.

LEMMA 3. *If N_n has been chosen large enough, we have*

$$\sum_{k \geq 1} |U_k|_w \leq I_1/8^{2n}.$$

Proof. We have

$$U_k = \sum_{j=1}^{n} \sum_{kN_n \leq \alpha < (k+1)N_n} a_{j,\alpha} x^\alpha (l_j q_j - q_j').$$

Thus, with $W_j = |l_j q_j - q_j'|_w$,

$$\begin{aligned} |U_k|_w &\leq \sum_{j=1}^{n} W_j \sum_\alpha \sqrt{\alpha+1}\, |a_{j,\alpha}| \\ &\leq \sum_{j=1}^{n} W_j \Big(\sum_{\alpha \geq kN_n} \frac{\alpha+1}{4^\alpha}\Big)^{1/2} \Big(\sum_\alpha |a_{j,\alpha}|^2 4^\alpha\Big)^{1/2} \\ &\leq \sum_j W_j \Big(\sum_{\alpha \geq kN_n} 1/2^\alpha\Big)^{1/2} \Big(\sum_\alpha |a_{j,\alpha}|^2 4^\alpha \varepsilon_j^2\Big)^{1/2} \\ &\leq 2^{-kN_n/2} \sum_j W_j \Big(\sum_\alpha |a_{j,\alpha}|^2 4^\alpha \varepsilon_j^2\Big)^{1/2}/\varepsilon_j \\ &\leq 2^{-kN_n/2} \Big(\sum_j \varepsilon_j^{-2} W_j^2\Big)^{1/2} \Big(\sum_j \sum_\alpha |a_{j,\alpha}|^2 4^\alpha \varepsilon_j^2\Big)^{1/2}. \end{aligned}$$

But $\varepsilon_j \geq 1/2^j \geq 1/2^n$, and

$$W_j = |l_j q_j - q_j'|_w \leq N_n^{1/2} \theta_n^*,$$

with $\theta_j = \max(|q_j|_w, |q_j'|_w)$, and $\theta_n^* = \max_{j \leq n} \theta_j$. Finally, we get

$$|U_k|_w \leq 2^{-kN_n/2} (nN_n \theta_n^{*2} 4^n)^{1/2} I_1, \quad \text{and}$$

$$\sum_{k \geq 1} |U_k|_w \leq \Big(\sum_{k \geq 1} 2^{-kN_n/2}\Big) (nN_n \theta_n^{*2} 4^n)^{1/2} I_1 \leq 8^{-2n} I_1,$$

by a proper choice of N_n, and the lemma is proved.

We use Lemma 2 with $p_1 = p - U_0$, $p_2 = \sum_{k \geq 1} U_k$, $\eta = 8^{-2n}$, $C = 8^{2n}$. We obtain

(6) $$\Big|p - \sum_{k \geq 0} U_k\Big|_w^2 + 8^{2n} \Big|\sum_{k \geq 1} U_k\Big|_w^2 \geq (1 - 8^{-2n}) |p - U_0|_w^2.$$

Let I_2 be the estimate given by the representation

(7) $$p = p - U_0 + U_0,$$

i.e. $I_2^2 = |p - U_0|_w^2 + [\![U_0]\!]_{(n)}^2$.

Then, since $[\![A]\!]_{(n)} \geq [\![U_0]\!]_{(n)}$, we get from (6)

$$I_1^2 + 8^{-2n} I_1^2 \geq (1 - 8^{-2n}) I_2^2,$$

or

(8) $$I_2^2 \leq \frac{1 + 8^{-2n}}{1 - 8^{-2n}} I_1^2.$$

We now observe that (7) is not convenient for $|\cdot|_{(n-1)}$ because

$$U_0 = \sum_{j=1}^{n} s_{0,j}(l_j q_j - q_j')$$

contains the term $s_{0,n}(l_n q_n - q_n')$, which should not appear in $|\cdot|_{(n-1)}$. Therefore we have to remove it. This will be obtained by careful considerations about the degrees of the polynomials involved. First, we look at the $s_{0,j}$, $j \leq n-1$.

LEMMA 4. *For $0 < \eta < 1$, set*

$$K_{j,\eta} = \mathrm{Log}_2 \frac{\eta W_j}{2\varepsilon_j \sqrt{1-\eta}}.$$

Then there is an expansion of p of the form (1) *or* (7) *with $a_{j,\alpha} = 0$ if $\alpha > K_{j,\eta}$ which gives the estimate*

$$|r|_w^2 + \sum_j \sum_{\alpha \leq K_{j,\eta}} |a_{j,\alpha}|^2 4^\alpha \varepsilon_j^2 \leq \frac{1}{1-\eta} I_2^2.$$

Proof. It follows the same lines as that of Lemma 3: we put

$$B_j = \Big|\sum_{\alpha > K_{j,\eta}} a_{j,\alpha} x^\alpha (l_j q_j - q_j')\Big|_w,$$

we find that

$$B_j \leq W_j 2^{-K_{j,\eta}/2} I_2/\varepsilon_j,$$

and the proof is concluded as before, by an application of Lemma 1.

Using Lemma 4, with $\eta = 8^{-n}$, $K_{n-1} = \max_{j \leq n-1} K_{j,\eta}$, we then get a new representation of p:

(9) $$p = p - \sum_{j=1}^{n-1} \sum_{\alpha \leq K_{n-1}} a_{j,\alpha} x^\alpha (l_j q_j - q_j') - s_{0,n}(l_n q_n - q_n') + U_0'$$

which gives the estimate

$$(10)\qquad I_3^2 = |p-\sum_{j=1}^{n}\sum_{\alpha\leqslant K_{n-1}} a_{j,\alpha}x^\alpha(l_j q_j-q_j')-s_{0,n}(l_n q_n-q_n')|_{\mathrm{w}}^2+[\![U_0']\!]_{(n)}^2$$
$$\leqslant (1-8^{-n})^{-1}I_2^2.$$

The important point is that K_{n-1} does not depend on N_n (but K_n does, because W_n does), so the degree of

$$p_0 = p-\sum_{j=1}^{n-1}\sum_{\alpha\leqslant K_{n-1}} a_{j,\alpha}x^\alpha(l_j q_j-q_j')$$

is bounded by a number which we call D_n, independent of N_n.

We wish to compare I_3^2 with the estimate I_4^2 given by the representation

$$(11)\qquad p = p-\sum_{j=1}^{n-1}\sum_{\alpha\leqslant K_{n-1}} a_{j,\alpha}x^\alpha(l_j q_j-q_j')+\sum_{j=1}^{n-1}\sum_{\alpha\leqslant K_{n-1}} a_{j,\alpha}x^\alpha(l_j q_j-q_j')$$

which is now convenient for the norm $|\cdot|_{(n-1)}$.

We wish to prove that if N_n is conveniently chosen, then

$$(12)\qquad I_4^2\leqslant (1-8^{-4n})^{-1}I_3^2.$$

In $[\![U_0']\!]_{(n)}^2$, we have the term $s_{0,n}$, which gives the contribution $\sum_\alpha |a_{n,\alpha}|^2 4^\alpha \varepsilon_n^2$. We may of course assume that

$$(13)\qquad \sum_\alpha |a_{n,\alpha}|^2 4^\alpha \varepsilon_n^2\leqslant I_4^2;$$

otherwise (12) is proved. So we get

$$(14)\qquad \sum_\alpha |a_{n,\alpha}|^2 4^\alpha\leqslant 4^n I_4^2.$$

For a polynomial $s=\sum_{j\geqslant 0}c_j x^j$, we put

$$s|^K = \sum_{j\leqslant K}c_j x^j,\qquad s|_K = \sum_{j>K}c_j x^j.$$

By (14) we can find an integer $D_n'>D_n$ such that

$$(15)\qquad |s_{0,n}|_{D_n'}|_{\mathrm{w}}\leqslant 8^{-2n}I_4/|q_n'|_{\mathrm{w}}.$$

We put $s_{0,n}' = s_{0,n}|^{D_n'}$. Again by Lemma 1, it follows from (10) that

$$(16)\qquad I_3^2\geqslant (1-8^{-2n})^{-1}|p_0+s_{0,n}'q_n'-s_{0,n}l_n q_n|_{\mathrm{w}}^2-8^{-2n}I_4^2+[\![U_0']\!]_{(n)}^2.$$

Now we have two cases:

Case 1: $|s_{0,n}'|_{\mathrm{w}}\leqslant 8^{-2n}I_4/|q_n'|_{\mathrm{w}}$.

Then $|s_{0,n}'q_n'|_{\mathrm{w}}\leqslant 8^{-2n}I_4$, and if $N_n\geqslant D_n'+n$ then

$$I_3^2\geqslant (1-8^{-2n})|p_0+s_{0,n}'q_n'|_{\mathrm{w}}^2-8^{-2n}I_4^2+[\![U_0']\!]_{(n)}^2$$
$$\geqslant (1-8^{-2n})^2|p_0|_{\mathrm{w}}^2-8^{2n}|s_{0,n}q_n'|_{\mathrm{w}}^2-8^{-2n}I_4^2+[\![U_0']\!]_{(n)}^2$$
$$\geqslant (1-8^{-2n})^2|p_0|_{\mathrm{w}}^2-2\cdot 8^{-2n}I_4^2+[\![U_0']\!]_{(n)}^2,$$

and so

$$(17)\qquad I_3^2\geqslant ((1-8^{-2n})^2-2\cdot 8^{-2n})I_4^2.$$

Case 2:

$$(18)\qquad |s_{0,n}'|_{\mathrm{w}}\geqslant 8^{-2n}I_4/|q_n'|_{\mathrm{w}}\geqslant 8^{-2n}I_4/\theta_n.$$

Again we have

$$I_3^2\geqslant (1-8^{-2n})(|p_0+s_{0,n}'q_n'|_{\mathrm{w}}^2+|s_{0,n}l_n q_n|_{\mathrm{w}}^2)-8^{-2n}I_4^2+[\![U_0']\!]_{(n)}^2,$$

since $N_n>D_n'+n$.

By (15) and (18), there exists an integer $D_n''>D_n'$, depending only on n, θ_n, such that

$$|s_{0,n}|^{D_n''}|_2\geqslant |s_{0,n}|_2/2,$$

where $|\cdot|_2$ is the usual l_2 norm.

Therefore $s_{0,n}$ has concentration 1/2 at low degrees. Since $d^\circ q_n<n$, we deduce from [2], Corollaire 7, that there exists a constant λ_n, depending only on n, θ_n, such that

$$|s_{0,n}q_n|^{D_n''+n}|_2\geqslant \lambda_n|s_{0,n}|_2|q_n|_2.$$

Put $d_n=D_n''+n$, $\lambda_n'=\lambda_n|q_n|_2\cdot 8^{-2n}/\theta_n$. We then get

$$(19)\qquad |s_{0,n}q_n|^{d_n}|_2\geqslant \lambda_n' I_4.$$

We write $s_{0,n}q_n=\sum_{j\geqslant 0}g_j x^j$. Then

$$|l_n s_{0,n}q_n|_{\mathrm{w}} = |\sum_j g_j x^{j+n}|_{\mathrm{w}} = (\sum_j |g_j|^2(j+N_n+1))^{1/2}$$
$$\geqslant (\sum_{j=0}^{d_n}|g_j|^2(j+N_n+1))^{1/2}$$
$$\geqslant N_n^{1/4}(\sum_{j=0}^{d_n}(j+1)|d_j|^2)^{1/2}\qquad \text{if } N_n/(N_n^{1/2}-1)>d_n$$
$$\geqslant N_n^{1/4}|s_{0,n}q_n|^{d_n}|_{\mathrm{w}}\geqslant N_n^{1/4}\lambda_n' I_4>I_4$$

if N_n has been chosen large enough.

Putting everything together, we get in both cases

$$I_4^2\leqslant ((1-8^{-2n})^2-2\cdot 8^{-2n})^{-1}(1-8^{-n})^{-1}\cdot(1+8^{-2n})(1-8^{-2n})^{-1}I_1^2,$$

from which Proposition 1 follows.

4. The final norm. Let now

$$\|p\| = \operatorname*{Lim}_{n\to+\infty} |p|_{(n)},$$

for any polynomial p. We have the following properties of the limit norm $\|\cdot\|$:

PROPOSITION 5.

(a) $\|l_j q_j - q_j'\| \leqslant \varepsilon_j, \quad j \geqslant 1.$

(b) $\|l_j p\| \leqslant 2^{N_j}\|p\|.$

(c) $\|xp\| \leqslant 2\|p\|.$

(d) *The norm* $\|\cdot\|$ *is hilbertian.*

(e) *For any* $n \geqslant 1$ *and any* p *with* $d^\circ p \leqslant n$,

$$\|p\| \geqslant \prod_{k\geqslant n} (1-4^{-k})\,|p|_{(n-1)}.$$

This last property ensures of course that the limit norm is nonzero. Therefore the completion of the polynomials for $\|\cdot\|$ is a Hilbert space on which the multiplication by x is continuous. Every polynomial q with rational coefficients is hypercyclic. Indeed, let $q' \neq 0$ in H, and let $\varepsilon > 0$. We can find in the enumeration an integer j such that

$$q_j = q, \quad \varepsilon_j < \varepsilon/2, \quad |q_j' - q|_w < \varepsilon/2.$$

Then $\|q_j' - q\| < \varepsilon/2$, and

$$\|x^{N_j} q_j - q\| \leqslant \|x^{N_j} q_j - q_j'\| + \|q_j' - q_j\| < \varepsilon,$$

which proves our claim.

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Some remarks on Triebel spaces

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Abstract. Some extensions of results in the recent monograph by Triebel [13] about Triebel spaces F^s_{pq} are given. This concerns multiplication properties, dual spaces and some remarks on the spaces $F^s_{\infty q}$.

0. Introduction. Triebel spaces are a natural generalization of Sobolev–Hardy spaces. The characterization of these spaces by decompositions of Littlewood–Paley type provides a useful tool for the study of multiplication properties, dual spaces, etc.

The plan of this paper is as follows. Chapter 1 is used to fix the notation and to recall some results on Besov and Triebel spaces. In Chapter 2 multiplication properties of Triebel spaces are studied: multiplication by functions belonging to Hölder–Zygmund spaces, multiplication algebras and multiplication by the characteristic function of an interval.

Chapter 3 is devoted to some complementary results in the determination of dual spaces. The main result can be phrased as follows. Let us denote by $\mathring{F}^s_{pq}$ the closure of the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$ in F^s_{pq}. Then for $1 \leqslant p, q \leqslant \infty$ the dual of $\mathring{F}^s_{pq}$ is isomorphic to $F^{-s}_{p'q'}$, $1/p+1/p' = 1/q+1/q' = 1$. Also some extensions to weighted spaces are given. The weight may belong to the Muckenhoupt class A_∞.

Finally, Chapter 4 contains some remarks on $F^s_{\infty q}$, $1 \leqslant q \leqslant \infty$. In particular, the trace problem is solved.

1. Besov and Triebel spaces. All functions and distributions are assumed to be defined on the n-dimensional Euclidean space $\boldsymbol{R}^n$. $\mathscr{S}(\boldsymbol{R}^n)$ is the Schwartz space of rapidly decreasing functions and $\mathscr{S}'(\boldsymbol{R}^n)$ its dual, the space of tempered distributions.

The Fourier transform is defined by

$$\hat{f}(\xi) := \int e^{-ix\cdot\xi} f(x)\,dx, \quad f \in \mathscr{S}(\boldsymbol{R}^n),$$

and extended to $\mathscr{S}'(\boldsymbol{R}^n)$ by duality. The inverse Fourier transform is

$$\check{f}(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} f(\xi)\,d\xi.$$