An operator on a separable Hilbert space with
many hypercyclic vectors

by

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Abstract. We construct a separable Hilbert space $H$ and a linear continuous operator $T$ on it
with hypercyclic vector $x_0$ such that for any polynomial $p$ with rational coefficients, $p(T)x_0$ is
also hypercyclic.

Let $H$ be a Hilbert space and $T$ a (linear continuous) operator on it. A point $x_0$ is said to be cyclic
(for $T$) if $H = \text{span} (x_0, Tx_0, T^2x_0, \ldots)$. It is said to be hypercyclic if the orbit $(T^kx_0, k \geq 0)$ is
dense in $H$. An intermediate notion, called quasi-hypercyclicity, was introduced by the author in [1].

The present construction provides the following:

Theorem. There is a separable complex Hilbert space $H$ and an operator $T$ on it with a cyclic vector $x_0$ such that every point of the form $p(T)x_0$, where $p$ is a polynomial with complex coefficients having both real and imaginary parts rational, is hypercyclic.

For this operator, the set of hypercyclic vectors, being a $G_\delta$, is uncountable. But the operator might still have invariant subspaces: to have no nontrivial invariant subspaces requires all points to be cyclic.

It is well known (Rolewicz [6], 1969) that on $l_p$ ($1 \leq p < \infty$) or $c_0$ there is
an operator with a hypercyclic vector $x_0$. Of course, the iterates of this
point are also hypercyclic, so the set of hypercyclic points is infinite, and
therefore, being a $G_\delta$, is uncountable. Also, Rolewicz’s construction can be
modified in order to provide a finite number of polynomials to be
hypercyclic. But our construction provides obviously many more
independent hypercyclic vectors (i.e. not just iterates of one another).

The present construction uses some of the ideas introduced by P. Enflo
to solve the invariant subspace problem in Banach spaces ([3], see also [1]).
But there are of course significant differences. Also, C. Read has given an
example of an operator on a Banach space with no nontrivial invariant
subspaces ([4]), and then such an example in $l_1$ ([5]).

In an earlier version of this paper (with the same title), the notion of
hypercyclicity was replaced by a weaker one: quasi-hypercyclicity, which is
an important feature in our example [1]. The strengthening was made possible by the fact that we now use weighted $l_2$ norms, instead of the classical one. We now turn to the construction of the example.

1. Enumeration of the triples. We first enumerate all the triples $(q_j, q_j, e_j)_{j \geq 1}$, where:
   - $q_j, q_j$ are polynomials with rational coefficients (i.e. both real and imaginary parts rational).
   - $e_j$ are of the form $1/2^j$, $j \geq 1$.

   We require that this enumeration should be done with the following restrictions:
   (a) For all $j \geq 1$, $d^r q_j < j$, $d^s q_j < j$.
   (b) For all $j \geq 1$, $e_j \geq 1/2^j$.

   This enumeration being done, our construction is totally determined by only one sequence of integers, $(N_j)_{j \geq 1}$, fast growing, which will be chosen by induction. We put $l_j = x^{N_j}$.

   In our final norm, each $q_j$ will be moved close to $q_j$ by the multiplication by $l_j$, with an accuracy of $e_j$, i.e.
   \[ ||l_j q_j - q_j||_o \leq e_j. \]

2. Definition of the norm $|| \cdot ||_o$. We start with a weighted $l_2$ norm: if $p = \sum_{j=0}^{\infty} a_j x^j$, we put
   \[ |p|_o = \left( \sum_{j=0}^{\infty} (j+1) |a_j|^2 \right)^{1/2}. \]

   The space of sequences $\{a_j : \sum_{j=0}^{\infty} (j+1) |a_j|^2 < +\infty \}$ is a Hilbert space, an algebra, and multiplication by $x$ has norm $\sqrt{2}$.

   We fix an integer $n \geq 1$, and we now look at all representations of a polynomial $p$ of the form
   \[ p = r + \sum_{j=1}^{n} \sum_{x \in N} a_{j,x} x^r (l_j q_j - q_j) \]
   where $r$ is a polynomial, $a_{j,x}$ are complex numbers ($j, x \in N$). We set:
   \[ |p|_o = \inf \{ |r|_o^2 + \sum_{j=1}^{n} \sum_{x \in N} |a_{j,x}|^2 4^s e_j \} \]

   where the infimum is taken over all representations of the form (1).

   Then, on the space of polynomials, $|| \cdot ||_o$ is a norm and the following properties are obvious:
   (a) $|| \cdot ||_o \leq || \cdot ||_o$.
   (b) $||l_j q_j - q_j||_o \leq e_j$, $j = 1, \ldots, n$.

   (c) $||l_j p||_o \leq 2^j \|p\|_o$, $j = 1, \ldots, n$.
   (d) $||x||_o \leq 2 \|p\|_o$.
   (e) The norm $|| \cdot ||_o$ is equivalent (with constants depending on $n$) to the original norm $|| \cdot ||_o$.
   (f) The norm $|| \cdot ||_o$ is hilbertian.

   Indeed, for this last property, one checks immediately that for all polynomials $p_1, p_2$,
   \[ 2(||p_1||_o^2 + ||p_2||_o^2) \geq ||p_1 + p_2||_o^2 + ||p_1 - p_2||_o^2, \]

   and the converse inequality follows after the change of variables $u = p_1 + p_2$, $v = p_1 - p_2$.

3. Almost stationarity of the sequence of norms. All the data required to construct $|| \cdot ||_o$ have been completely described, except the sequence $(N_j)_{j \geq 1}$. It will be chosen by induction, according to the following

   Proposition 1. Assume $N_1 < \ldots < N_{n-1}$ have been chosen. We can choose $N_n$ so large that for every polynomial $p$ with $d^r p \leq n$,
   \[ ||p||_o \geq (1 - 1/4^n) ||p||_{o-1}. \]

   (The choice of $N_n$ could be made precise; it depends only on the data $n, N_1, \ldots, N_{n-1}, |q_j|_o (j \leq n), |q_j|_o (j \leq n)$.)

   Proof. We need a very simple lemma, the proof of which is left to the reader:

   Lemma 2. For any polynomials $p_1, p_2$ and every $\eta$, $0 < \eta < 1$, if $\eta C \geq (1-\eta)/\eta$ then
   \[ ||p_1 + p_2||_o^2 + C ||p_1||_o^2 \geq (1-\eta) ||p_1||_o^2. \]

   Take now $r$, $d^r p \leq n$. We write an expansion of the form (1) as
   \[ p = p - A + A, \quad \text{with} \quad A = \sum_{j=1}^{n} \sum_{x \in N} a_{j,x} x^r (l_j q_j - q_j). \]

   We use the following notation:
   \[ \|A\|_o^2 = \sum_{j=1}^{n} \sum_{x \in N} |a_{j,x}|^2 4^s e_j, \]

   so the estimate given by (3) is just
   \[ ||p||_o^2 = ||p - A||_o^2 + ||A||_o^2. \]

   We also put $s_j = \sum_{x \in N} a_{j,x} x^s$, and write the expansion
   \[ s_j = s_{0,j} + x^{n_j} s_{1,j} + \ldots + x^{n_j} s_{k,j} + \ldots, \]
where for all $k > 0$, $d^k s_{j_k} < N_n$. We put
\[
U_k = x^{k^2} \sum_{j_1=1}^{d^k} s_{j_1} (l_j q_j - q_j),
\]
so (3) becomes
\[
(5) \quad p = p - \sum_{k > 0} U_k + A.
\]

We wish first to construct a new representation from which all terms $U_k$, $k \geq 1$, have been removed.

**Lemma 3.** If $N_n$ has been chosen large enough, we have
\[
\sum_{k > 1} |U_k|_x \leq I_1 x^{-8^{2^n}}.
\]

**Proof.** We have
\[
U_k = \sum_{j_1=1}^{d^k} \sum_{x \equiv j_1} a_{j_1} x^2 (l_j q_j - q_j).
\]

Thus, with $W_j = |l_j q_j - q_j|_x$,
\[
|U_k|_x \leq \sum_{j=1}^{d^k} W_j \sum_{x \equiv j} |a_{j_1}| \sqrt{x+1} |a_{j_2}|,
\]
\[
\leq \sum_{j=1}^{d^k} W_j \left( \sum_{x \equiv j} |a_{j_1}|^2 4^j \right)^{1/2} \left( \sum_{x \equiv j} |a_{j_1}|^2 4^j \right)^{1/2},
\]
\[
\leq \sum_{j=1}^{d^k} W_j \left( \sum_{x \equiv j} |a_{j_1}|^2 4^j \right)^{1/2} \left( \sum_{x \equiv j} |a_{j_1}|^2 4^j \right)^{1/2},
\]
\[
\leq 2^{-k x^{k^2}} \sum_{j=1}^{d^k} W_j \left( \sum_{x \equiv j} |a_{j_1}|^2 4^j \right)^{1/2} |l_j|\text{,}
\]
\[
2^{-k x^{k^2}} \sum_{j=1}^{d^k} W_j \left( \sum_{x \equiv j} |a_{j_1}|^2 4^j \right)^{1/2} |l_j|\text{.}
\]

But $q_j \gtrsim 1/2^{j^2} \gtrsim 1/2^n$, and
\[
W_j = |l_j q_j - q_j|_x \lesssim N_n^{1/2} 2^{\star q},
\]
with $\theta_j = \max(l_j q_j, |q_j|_x)$, and $2^{\star q} = \max_{j \equiv x} \theta_j$. Finally, we get
\[
|U_k|_x \leq 2^{-k x^{k^2}} (n N_n 2^{\star q} 2^n 4^j)^{1/2} I_1\text{,}
\]
and
\[
\sum_{k > 1} |U_k|_x \leq 2^{-k x^{k^2}} (n N_n 2^{\star q} 2^n 4^j)^{1/2} I_1 \lesssim 8^{-2^n} I_1,
\]
by a proper choice of $N_n$, and the lemma is proved.

We use Lemma 2 with $p_1 = p - U_0$, $p_2 = \sum_{k > 1} U_k$, $\eta = 8^{-2^n}$, $C = 8^{2^n}$.

We obtain
\[
(6) \quad |p - \sum_{k > 0} U_k|_x \gtrsim (1 - 8^{-2^n}) |p - U_0|_x.
\]

Let $I_2$ be the estimate given by the representation
\[
(7) \quad p = p - U_0 + U_0,
\]
i.e.
\[
I_2^2 = |p - U_0|_x^2 + |U_0|_x^2.
\]

Then, since $|A|_x \lesssim |U_0|_x$, we get from (6)
\[
I_2^2 + 8^{-2^n} I_2^2 \gtrsim (1 - 8^{-2^n}) I_2^2,
\]
or
\[
(8) \quad I_2^2 \lesssim \frac{1 + 8^{-2^n}}{1 - 8^{-2^n}} I_2^2.
\]

We now observe that (7) is not convenient for $| \cdot |_{x^{-1}}$ because
\[
U_0 = \sum_{j=1}^{d^k} s_{j_1} (l_j q_j - q_j)
\]
contains the term $s_{j_1} (l_j q_j - q_j)$, which should not appear in $| \cdot |_{x^{-1}}$. Therefore we have to remove it. This will be obtained by careful considerations about the degrees of the polynomials involved. First, we look at the $n_{j_2, j} \leq n - 1$.

**Lemma 4.** For $0 < \eta < 1$, set
\[
K_{j,n} = \log_2 \frac{\eta W_j}{2^{2j} \sqrt{1 - \eta}}.
\]

Then there is an expansion of $p$ of the form (1) or (7) with $a_{j,n} = 0$ if $x > K_{j,n}$ which gives the estimate
\[
|p|_x^2 + \sum_{j \leq K_{j,n}} |a_{j,n}|^2 2^n 4^j \lesssim \frac{1}{1 - \eta} I_2^2.
\]

**Proof.** It follows the same lines as that of Lemma 3: we put
\[
B_j = \sum_{j \leq K_{j,n}} a_{j,n} x^2 (l_j q_j - q_j),
\]
we find that
\[
B_j \lesssim W_j \cdot 2^{-K_{j,n}} I_2 / 2^{2j},
\]
and the proof is concluded as before, by an application of Lemma 1.

Using Lemma 4, with $\eta = 8^{-n}$, $K_{j,n} = \max_{j \leq K_{j,n}} K_{j,n}$, we then get a new representation of $p$:
\[
(9) \quad p = p - \sum_{j=1}^{n-1} \sum_{j \leq K_{j,n}} a_{j,n} x^2 (l_j q_j - q_j) - s_{j_1} (l_j q_j - q_j) + U_0.
\]
which gives the estimate

$$I_3 = |p - \sum_{j=1}^{N_x} \sum_{1 \leq k \leq q_{\alpha_{j}}} a_{jk} x^k |q_{\alpha_{j}} - q_{\alpha_{j}}|^{2} + |U_0|_{00}^2$$

$$\leq (1 - 8^{-3\eta})^{-1} I_2^3.$$  

The important point is that $K_{x-1}$ does not depend on $N_x$ (but $K_x$ does, because $K_{x}$ does), so the degree of

$$p_0 = p - \sum_{j=1}^{N_x} \sum_{1 \leq k \leq q_{\alpha_{j}}} a_{jk} x^k |q_{\alpha_{j}} - q_{\alpha_{j}}|$$

is bounded by a number which we call $D_n$, independent of $N_x$.

We wish to compare $I_3^3$ with the estimate $I_2^3$ given by the representation

$$I_3 = |p - \sum_{j=1}^{N_x} \sum_{1 \leq k \leq q_{\alpha_{j}}} a_{jk} x^k |q_{\alpha_{j}} - q_{\alpha_{j}}| + \sum_{j=1}^{N_x} \sum_{1 \leq k \leq q_{\alpha_{j}}} a_{jk} x^k |q_{\alpha_{j}} - q_{\alpha_{j}}|$$

which is now convenient for the norm $|.|_{I_2}$. We wish to prove that if $I_2$ is conveniently chosen, then

$$I_2^3 \leq (1 - 8^{-3\eta})^{-1} I_3^3.$$  

In $\|U_0\|_{00}^2$, we have the term $s_{0,n}$, which gives the contribution $\sum_{a} |a_{\alpha_{a}}|^2 4^a z_2^a$. We may of course assume that

$$\sum_{a} |a_{\alpha_{a}}|^2 4^a z_2^a \leq I_2^3;$$

otherwise (12) is proved. So we get

$$\sum_{a} |a_{\alpha_{a}}|^2 4^a z_2^a \leq I_2^3.$$  

For a polynomial $s = \sum_{j} c_j x^j$, we put

$$s_{K} = \sum_{j \geq K} c_j x^j, \quad s_{k} = \sum_{j \geq k} c_j x^j.$$  

By (14) we can find an integer $D_n > D_n$ such that

$$|s_{0,n} q_{\alpha_{a}}|_{a} \leq 8^{-2n} I_2 |q_{\alpha_{a}}|.$$  

We put $s_{0,a} = s_{0,a} D_n$. Again by Lemma 1, it follows from (10) that

$$I_3 \geq (1 - 8^{-2\eta})^{-1} |p_0 + s_{0,a} q_{\alpha_{a}} - s_{0,a} q_{\alpha_{a}}| + 8^{-2\eta} I_2^3 + |U_0|_{00}^2.$$  

Now we have two cases:

Case 1: $|s_{0,a} q_{\alpha_{a}}|_{a} \leq 8^{-2n} I_2 |q_{\alpha_{a}}|.$

Then $|s_{0,a} q_{\alpha_{a}}|_{a} \leq 8^{-2n} I_2$, and if $N_x \geq D_n + n$ then

$$I_3 \geq (1 - 8^{-2\eta}) |p_0 + s_{0,a} q_{\alpha_{a}} - s_{0,a} q_{\alpha_{a}}| + |U_0|_{00}^2$$

$$\geq (1 - 8^{-2\eta}) (|p_0|_{a} - 2^n |s_{0,a} q_{\alpha_{a}}|^2 - 8^{-2n} I_2^3 + |U_0|_{00}^2)$$

$$\geq (1 - 8^{-2\eta}) |p_0|_{a} - 2 - 8^{-2n} I_2^3 + |U_0|_{00}^2,$$

and so

$$I_3 \geq (1 - 8^{-2\eta})^{-1} I_2^3.$$  

Case 2:

$$|s_{0,a} q_{\alpha_{a}}|_{a} \leq 8^{-2n} I_2 |q_{\alpha_{a}}|.$$  

Again we have

$$I_3 \geq (1 - 8^{-2\eta}) (|p_0 + s_{0,a} q_{\alpha_{a}}|_{a} + |s_{0,a} q_{\alpha_{a}}|^2 - 8^{-2n} I_2^3 + |U_0|_{00}^2)$$

since $N_x > D_n + n$.

By (15) and (18), there exists an integer $D_n > D_n$, depending only on $n$, $q_{\alpha_{a}}$, such that

$$|s_{0,a} q_{\alpha_{a}}|_{a} \geq \lambda_n |q_{\alpha_{a}}|^2 I_4.$$  

Put $d_n = D_n + n$, $\lambda_n = |q_{\alpha_{a}}|^2 8^{-2\eta} |q_{\alpha_{a}}|$. We then get

$$|s_{0,a} q_{\alpha_{a}}|_{a} \geq \lambda_n I_4.$$  

We write $s_{0,a} q_{\alpha_{a}} = \sum_{j} c_j x^j$. Then

$$|s_{0,a} q_{\alpha_{a}}|_{a} = \sum_{j} |c_j| x^j I_1.$$  

By (17) and (18), we get

$$|s_{0,a} q_{\alpha_{a}}|_{a} \geq \lambda_n I_4.$$  

if $N_x$ has been chosen large enough.

Putting everything together, we get in both cases

$$I_2^3 \geq (1 - 8^{-2\eta})^{-1} (1 - 8^{-2\eta})^{-1} (1 + 8^{-2\eta}) (1 - 8^{-2\eta})^{-1} I_2,$$

from which Proposition 1 follows.
4. The final norm. Let now

\[ \|p\| = \lim_{n \to \infty} |p|_{Q_n} \]

for any polynomial \( p \). We have the following properties of the limit norm \( \| \cdot \| \):

**Proposition 5.**

(a) \( \|q_j - q\| \leq \varepsilon_j \), \( j \geq 1 \).

(b) \( \|q_j\| \leq 2^j \|p\| \).

(c) \( \|x\| \leq 2 \|p\| \).

(d) The norm \( \| \cdot \| \) is Hilbertian.

(e) For any \( n \geq 1 \) and any \( p \) with \( d^p \leq n \),

\[ \|p\| \geq \prod_{k=0}^{n-1} (1 - 4^{-k}) \|p|_{Q_{n-1}} \].

This last property ensures of course that the limit norm is nonzero.

Therefore the completion of the polynomials for \( \| \cdot \| \) is a Hilbert space on which the multiplication by \( x \) is continuous. Every polynomial \( q \) with rational coefficients is hypercyclic. Indeed, let \( q \neq 0 \) in \( H \), and let \( \varepsilon > 0 \). We can find in the enumeration an integer \( j \) such that

\[ q_j = q, \quad \varepsilon_j < \varepsilon/2, \quad |q_j - q| < \varepsilon/2. \]

Then \( |q_j - q| < \varepsilon/2 \), and

\[ \|x^n q_j - q\| < \|x^n q_j - q_j\| + |q_j - q| < \varepsilon, \]

which proves our claim.

**References**


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