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Nilpotent Lie groups and eigenfunction expansions of Schrödinger operators II *

by

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Abstract. Let $\mathscr{L} = -d^2/dx^2 + |P(x)|$, where P is a polynomial of degree $d+1$. Following the general pattern of [9] and using new estimates proved in [3] the following theorem is proved.

THEOREM. *Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ be the eigenvalues corresponding to the orthonormal basis $\varphi_1, \varphi_2, \ldots$ of eigenfunctions of $\mathscr{L}$ in $L^2(\boldsymbol{R})$. Let $K \in C^\infty(\boldsymbol{R})$, with $K(0) = 1$, be such that for some $\gamma > 1$ and $R > 0$*

$$\sup_{\lambda > 0} (1+\lambda)^{n(s+1)} |K^{(j)}(\lambda)| \leqslant R^n (n!)^\gamma, \quad j \leqslant n,\ n = 1, 2, \ldots,$$

where $s = [(2+d)(5+d)/4]+1$. Then for every $f \in L^p(\boldsymbol{R})$, $1 \leqslant p < \infty$, we have

$$\lim_{t \to 0} \Big\| \sum_{n=1}^{\infty} K(t\lambda_n)(f, \varphi_n)\varphi_n - f \Big\|_{L^p} = 0.$$

In our previous paper [9] we used nilpotent Lie groups to obtain results on the summability of eigenfunction expansions of Schrödinger operators on $\boldsymbol{R}^n$ whose potentials were sums of squares of polynomials. In an attempt to prove similar results for operators with more general potentials we investigate here the operator

$$\mathscr{L} = -\frac{d^2}{dx^2} + |P(x)|,$$

where P is a polynomial of degree $d+1$, say.

We believe that most of our present results are valid also in higher dimensions but the technique used here is restricted to dimension one. Also our summability results are weaker than those for operators considered in [9]. An application of the methods of the present paper gives the following theorem.

THEOREM. *Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ be the eigenvalues corresponding to the orthonormal basis $\varphi_1, \varphi_2, \ldots$ of eigenfunctions of $\mathscr{L}$ in $L^2(\boldsymbol{R})$. Let $K \in C^\infty(\boldsymbol{R})$,*

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with $K(0) = 1$, *be such that for some* $\gamma > 1$ *and* $R > 0$

(0.1) $$\sup_{\lambda>0}(1+\lambda)^{n(s+1)}|K^{(j)}(\lambda)| \leqslant R^n (n!)^\gamma, \quad j \leqslant n,$$

for all $n = 1, 2, \ldots$, *where*

$$s = [(2+d)(5+d)/4]+1.$$

Then for every $f \in L^p(\boldsymbol{R})$, $1 \leqslant p < \infty$, *we have*

(0.2) $$\lim_{t\to 0}\Big\|\sum_{n=1}^{\infty} K(t\lambda_n)(f, \varphi_n)\varphi_n - f\Big\|_{L^p} = 0.$$

We note that functions with compact support that belong to a Gevrey class satisfy (0.1).

As in [9] we use here an idea introduced first by W. Cupała to regard $\mathscr{L}$ as the image under a unitary representation of a left invariant operator L on a nilpotent Lie group G, suitably chosen for the operator $\mathscr{L}$. If the potential in $\mathscr{L}$ is a sum of squares of polynomials, the operator L on the group G is hypoelliptic, by Hörmander's theorem on sums of squares of vector fields [6], and so the densities of the probability measures in the semigroup generated by L are in $L^2(G)$ (in C^∞, as a matter of fact). To prove this in the present situation, where Hörmander's theorem is not applicable, is the main difficulty. Another obstacle is that these measures decay at infinity too slowly for the use of the functional calculus of [8], as applied in [9]. Therefore (0.2) can be proved only for K in a Gevrey class instead of K in $C^n(\boldsymbol{R}^+)$, by an application of Pytlik's functional calculus [13].

Most of this paper is devoted to the proof that the semigroup considered consists of measures with L^2 densities. The proof goes via representation theory and the Plancherel formula, which in the case of our nilpotent group is fairly simple, and relies heavily on a result proved in a recent paper by C. Fefferman [3].

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1. The group. Let $X, Y_0, \ldots, Y_{d+1}$ be the basis of a Lie algebra $\mathfrak{g}$ such that $[X, Y_j] = Y_{j+1}$ are the only nontrivial commutation relations. Let G be the corresponding simply connected nilpotent Lie group. The following facts can be verified by a routine application of the Kirillov theory [12].

The representations in general position of G are parametrized by $\boldsymbol{R}^{d+1}$. For $\boldsymbol{c}$ in $\boldsymbol{R}^{d+1}$ we write $\boldsymbol{c} = (c_{d+1}, c_{d-1}, \ldots, c_0)$, or, in the other words, $c_d = 0$ whenever we have c_j, $j = 0, \ldots, d+1$. The representation π^c acts on functions φ on $\boldsymbol{R}$ and

(1.1) $$\pi^c(g)\varphi(x) = a^c(g, x)\varphi(x+\theta(g)),$$

where $|a^c(g, x)| = 1$ and θ is the natural homomorphism $\theta\colon G \to G/N = \boldsymbol{R}$ with $N = \exp\operatorname{lin}\{Y_0, \ldots, Y_{d+1}\}$.

The infinitesimal form of π^c is

(1.2) $$\pi^c(X) = \frac{d}{dx}, \quad \pi^c(Y_0) = M_{p^c},$$

where

$$M_{p^c}\varphi(x) = iP^c(x)\varphi(x), \quad P^c(x) = \sum_{j=0}^{d+1}\frac{1}{j!}c_j x^j$$

(note that $c_d = 0$).

The Plancherel measure on $\boldsymbol{R}^{d+1}$ is $|c_{d+1}|\,dc_{d+1}\,dc_{d-1}\ldots dc_0$, i.e. $f \in L^2(G)$ if and only if $\operatorname{Tr}\pi^c(f * f^*)$ is finite for almost all $\boldsymbol{c}$ and

(1.3) $$\int \operatorname{Tr}\pi^c(f * f^*)|c_{d+1}|\,dc_{d+1}\,dc_{d-1}\ldots dc_0 < \infty.$$

We introduce dilations in $\mathfrak{g}$ putting, for $r > 0$,

$$\delta_r X = rX, \quad \delta_r Y_j = r^{2+j} Y_j, \quad j = 0, \ldots, d+1.$$

Of course, δ_r is an automorphism of $\mathfrak{g}$ and so $\delta_r \exp Z = \exp\delta_r Z$, $Z \in \mathfrak{g}$, is an automorphism of G.

We have

(1.4) $$\pi^c(\delta_r g) = \pi^{\delta_r^* c}(g),$$

where $\delta_r^*(c_{d+1}, c_{d-1}, \ldots, c_0) = (r^{d+3}c_{d+1}, \ldots, r^{j+2}c_j, \ldots, r^2 c_0)$.

2. The semigroup. It follows immediately from G. Hunt's theory of convolution semigroups on a Lie group that for Y_0 in $\mathfrak{g}$ the operator Y_0^2 is the infinitesimal generator of a semigroup of probability measures ν_t, $t > 0$, on G (cf. e.g. [7]). For $f \in C_c^\infty(G)$ we define

(2.1) $$-|Y_0| f = -|Y_0^2|^{1/2} f = -\int_0^\infty t^{-(1+1/2)}(f - \nu_t * f)\,dt.$$

Since $\pi^c(Y_0^2) = (M_{p^c})^2$, $\pi^c(\nu_t)\varphi(x) = \exp[-tP^c(x)^2]\varphi(x)$ and so, by (2.1),

(2.2) $$\pi^c(|Y_0|)\varphi(x) = |P^c(x)|\varphi(x).$$

It is known that if

$$L = -X^2 + |Y_0|$$

then $-L$ is the infinitesimal generator of a semigroup of probability measures μ_t on G (cf. e.g. [7]). It follows from (1.2) and (2.2) that $\pi^c(L)$ is the Schrödinger operator

(2.3) $$\pi^c(L) = -\frac{d^2}{dx^2} + |P^c|$$

and, by (1.1), the operators

$$\pi^c(\mu_t) = \int \pi^c(g)\, d\mu_t(g)$$

form a semigroup of contractions on every $L^p(\mathbf{R})$, $1 \leqslant p < \infty$, whose infinitesimal generator is $\pi^c(L)$.

The dilations δ_r, $r > 0$, are defined in such a way that the operator L is homogeneous of degree 2, i.e.

(2.4) $$L(f \circ \delta_r) = r^2 Lf \circ \delta_r$$

and so

(2.5) *If λ is an eigenvalue of $\pi^c(L)$, then $r^2\lambda$ is an eigenvalue of $\pi^{\delta_r^* c}(L)$.*

This can also be deduced directly from (1.4) and (2.3).

Also (2.4) implies that

(2.6) $$\langle f, \mu_t \rangle = \langle f \circ \delta_{t^{-1/2}}, \mu_1 \rangle.$$

Now we are ready to prove

(2.7) THEOREM. *For every $t > 0$ the measure μ_t is absolutely continuous with respect to the Haar measure on G, i.e.*

(2.8) $$d\mu_t(g) = p_t(g)\, dg,$$

and moreover,

(2.9) $$p_t \in L^2(G).$$

Remark. As a matter of fact, (2.8) implies (2.9). This has been shown to us by Paweł Głowacki. His proof (virtually contained in [5]) uses the homogeneity of L. However, we see no direct proof of (2.8). Thus we use the Plancherel formula (1.3), i.e. we show that

(2.10) $$\int \operatorname{Tr} \pi^c(\mu_1) |c_{d+1}|\, dc_{d+1}\, dc_{d-1} \ldots dc_0 < \infty,$$

which implies (2.9) for $t = 1/2$ and so, by (2.6), for all t.

Proof. Let $\lambda_1(c) \leqslant \lambda_2(c) \leqslant \ldots$ be the eigenvalues of (2.3). Thus (2.10) is equivalent to

(2.11) $$\int \sum_{n=1}^{\infty} \exp[-\lambda_n(c)] |c_{d+1}|\, dc_{d+1}\, dc_{d-1} \ldots dc_0 < \infty.$$

By an obvious change of variables this is equivalent to showing the same inequality where P^c in (2.3) is replaced by

$$P^c(x) = \sum_{j=0}^{d+1} c_j x^j, \quad c_d = 0.$$

Let

$$\Omega = \{c = (c_{d+1}, c_{d-1}, \ldots, c_0)\colon c_{d+1} = 1\}.$$

Then the left-hand side of (2.11) is equal to

(2.12) $$C \int_{\mathbf{R}} \int_{\Omega} \sum_{n=1}^{\infty} \exp[-\lambda_n(\delta_r^* c)] |r|^{(5+d)(2+d)/2}\, dc_{d-1} \ldots dc_0\, dr,$$

where here and in the following, C is a positive constant which may depend on d only and may vary from line to line.

By (2.5), $\lambda_n(\delta_r^* c) = r^2 \lambda_n(c)$. Hence (2.12) is equal to

(2.13) $$C \int_{\mathbf{R}} \int_{\Omega} e^{-r^2} |r|^{(5+d)(2+d)/2} \sum_{n=1}^{\infty} \lambda_n(c)^{-(5+d)(2+d)/4-1/2}\, dc_{d-1} \ldots dc_0\, dr$$
$$= C \int_{\Omega} \sum_{n=1}^{\infty} \lambda_n(c)^{-(5+d)(2+d)/4-1/2}\, dc_{d-1} \ldots dc_0$$
$$= C \int_{\Omega} \int_{\lambda \geqslant \lambda_1(c)} \lambda^{-(5+d)(2+d)/4-1/2}\, dN(\lambda, c)\, dc_{d-1} \ldots dc_0,$$

where

$$N(\lambda, c) = \#\{\lambda_n(c) \leqslant \lambda\}.$$

We shall use the following two estimates which are proved in [3], p. 144–150, for positive polynomial potentials but the proofs go through also for potentials of the form $|P^c|$.

(2.14) $$N(\lambda, c) \leqslant C \operatorname{vol}\{(x, \xi)\colon \xi^2 + |P^c(x)| < \lambda\},$$

(2.15) $$\lambda_1(c) \geqslant C \inf_{\delta > 0} \{\delta^{-2} + \inf_{x_0} \max_{|x - x_0| < \delta} |P^c(x)|\}.$$

The following fact is well known and easy to prove:

(2.16) LEMMA. *If P is a monic polynomial of degree $d+1$, then there is a constant C which depends only on d such that*

$$\inf_{x_0} \max_{|x_0 - x| < \delta} |P(x)| \geqslant C\delta^{d+1}.$$

Now we use (2.14) and (2.16) to obtain

(2.17) $$N(\lambda, c) \leqslant C\lambda^{1/2+1/(d+1)}, \quad c \in \Omega.$$

In fact, by (2.14) we have

$$N(\lambda, c) \leqslant 2\lambda^{1/2} \operatorname{vol}\{x\colon |P^c(x)| < \lambda\}.$$

But $\{x\colon |P^c(x)| < \lambda\}$ is the union of at most d intervals $I_1, \ldots, I_d$, and by (2.16), since P^c is monic, $\max_{x \in I_j} |P^c(x)| < \lambda$ implies $|I_j| \leqslant C\lambda^{1/(d+1)}$.

Thus (2.17) implies, for c in Ω,

$$\int\limits_{\lambda \geqslant \lambda_1(c)} \lambda^{-(5+d)(2+d)/4-1/2}\, dN(\lambda, c) \leqslant C\lambda_1(c)^{-\varrho},$$

where

$$\varrho = (5+d)(2+d)/4 - 1/(d+1). \tag{2.18}$$

Consequently, in virtue of (2.13), it is sufficient to prove

(2.19) LEMMA. *Let*

$$P^c(x) = x^{d+1} + \sum_{j=0}^{d-1} c_j x^j.$$

Let $\lambda(c)$ *be the smallest eigenvalue of the operator* $-d^2/dx^2 + |P^c|$. *Then*

$$\int \lambda(c)^{-\varrho}\, dc_{d-1} \ldots dc_0 < \infty,$$

where ϱ *is given by* (2.18).

Proof of Lemma (2.19). For a $k = 0, \ldots, d+1$ we denote by Ξ_k the subset of $\boldsymbol{R}^d$ consisting of $c = (c_{d-1}, \ldots, c_0)$ such that the polynomial P^c has the following properties:

(a) All the roots $z_0, \ldots, z_d$ of P^c are distinct.

(b) k of the roots, $z_0, \ldots, z_{k-1}$, are real, z_k, $z_{k+1} = \bar{z}_k, \ldots, z_{d-1}$, $z_d = \bar{z}_{d-1}$ are complex.

(c) The numbers $z_0, \ldots, z_{k-1}$, $\operatorname{Re} z_k$, $\operatorname{Re} z_{k+2}, \ldots, \operatorname{Re} z_{d-1}$ are all distinct.

Moreover, since $c_d = 0$, we have

(d) $z_0 + \ldots + z_d = 0$.

Of course, to prove the lemma it is sufficient to prove

$$\int\limits_{\Xi_k} \lambda(c)^{-\varrho}\, dc_{d-1} \ldots dc_0 < \infty \tag{2.20}$$

for every $k = 0, \ldots, d+1$.

Let Ω_k be the subset of $\boldsymbol{R}^k \times \boldsymbol{C}^l$, $k+2l = d+1$, consisting of $(z_0, \ldots, z_d)$ for which (a)–(d) hold.

We consider the diffeomorphism taking roots to coefficients, i.e.

$$\theta\colon \Omega_k \ni (z_0, \ldots, z_d) \to (c_{d-1}, \ldots, c_0) \in \Xi_k,$$

which, of course, is given by the symmetric polynomials in $z_0, \ldots, z_d$. Consequently, the Jacobian of this map is

$$\prod_{i<j} |z_i - z_j| \tag{2.21}$$

restricted to Ω_k. The fact that the Jacobian of the map given by the symmetric polynomials is (2.21) is not difficult to prove and is, of course,

classical, cf. e.g. [1]. Thus

$$dc_{d-1} \ldots dc_0 = J(z_0, \ldots, z_d)\, dz_0 \ldots dz_{k-1}\, dz_k\, d\bar{z}_k \ldots dz_{d-1}\, d\bar{z}_{d-1}, \tag{2.22}$$

where

$$J(z_0, \ldots, z_d) = c\delta_0(z_0 + \ldots + z_d) \prod_{i<j} |z_i - z_j|.$$

Now we fix $z = (z_0, \ldots, z_d)$ in Ω_k and we write $P_z = P^{\theta^{-1}c}$, i.e.

$$|P_z(x)| = \prod_{j=0}^{d} |x - z_j|.$$

In virtue of (2.15) we have to estimate

$$\inf_{\delta>0} \{\delta^{-2} + \inf_{x_0} \max_{|x_0 - x| < \delta} |P_z(x)|\} = \sigma(z)$$

in terms of z.

First we note that

$$\inf_{\delta>0} \{\delta^{-2} + A\delta^m\} = [(m/2)^{2/(2+m)} + (m/2)^{-m/(2+m)}]\, A^{2/(2+m)}. \tag{2.23}$$

Hence, by Lemma (2.16), we have

$$\sigma(z) \geqslant C > 0. \tag{2.24}$$

(We recall that C is a constant which may depend only on d.) Thus let $\delta > 0$ and let an interval I with $|I| = 2\delta$ be such that

$$\sigma(z) \geqslant \tfrac{1}{2}\{\delta^{-2} + \max_{x \in I} |P_z(x)|\}. \tag{2.25}$$

(Of course δ and I depend on z.) Let $s_0 \in I$ be such that

$$|s_0 - z_j| \geqslant |s_0 - \operatorname{Re} z_j| \geqslant (d+1)^{-1}\delta, \quad j = 0, \ldots, d, \tag{2.26}$$

and let z_{j_0} be such that $a = \operatorname{Re} z_{j_0}$ is closest to s_0, i.e.

$$|s_0 - a| \leqslant |s_0 - \operatorname{Re} z_j| \quad \text{for all } j = 0, \ldots, d.$$

We then have

$$|s_0 - z_j| \geqslant \tfrac{1}{2}|a - z_j| \quad \text{for all } j = 0, \ldots, d. \tag{2.27}$$

Let

$$M_n = \{(z_0, \ldots, z_d) \in \Omega_k\colon j_0 = n\}.$$

Clearly, Ω_k is the union of the M_n's and we may restrict our z's to one M_n only.

Thus for z in M_n we put $a = \operatorname{Re} z_n$ and we reorder $z_0, \ldots, z_d$ in such a way that $\{z_0, \ldots, z_d\} = \{a'_0, \ldots, a'_d\} = \boldsymbol{a}$,

$$|a-a'_0| \leqslant |a-a'_1| \leqslant \ldots \leqslant |a-a'_d|$$

and $a'_i = \bar{a}'_j$ implies $|i-j| \leqslant 1$. Let

$$a'_{i_1}, a'_{i_1+1}, \ldots, a'_{i_l}, a'_{i_l+1}, \quad i_1 < \ldots < i_l,$$

be the complex roots among $a'_0, \ldots, a'_d$. We drop $a'_{i_1+1}, \ldots, a'_{i_l+1}$ from the sequence $a'_0, \ldots, a'_d$, thus leaving just one from each pair of complex conjugate roots, and we obtain the sequence $a_0, \ldots, a_m$, where $m = k+l$, such that

$$|a-a_0| \leqslant \ldots \leqslant |a-a_m|.$$

We write

$$\varepsilon_j = \begin{cases} 1 & \text{if } a_j \text{ is real,} \\ 2 & \text{if } a_j \text{ is complex.} \end{cases}$$

We consider three cases:

1. z_n is real. Then, of course, $a_0 = a$.
2. z_n is complex and $z_n = a_0 = a+ib$.
3. z_n is complex and $z_n = a_p = a+ib$, $p \geqslant 1$.

Case 1. By (2.26) and (2.27) for every $j = 1, \ldots, m+l$ we have

$$\max_{x \in I} |P_z(x)| \geqslant |P_z(s_0)| \geqslant C\delta^{\varepsilon_0 + \ldots + \varepsilon_{j-1}} \prod_{t=j}^{m+1} |a-a_t|^{\varepsilon_t}$$

(where $\varepsilon_{m+1} = 0$). Hence, by (2.25) and (2.23),

$$\sigma(z) \geqslant C\delta^{-2} + c\delta^{\varepsilon_0 + \ldots + \varepsilon_{j-1}} \prod_{t=1}^{m+1} |a-a_t|^{\varepsilon_t}$$

$$\geqslant C\Big[\prod_{t=j}^{m+1} |a-a_t|^{\varepsilon_t}\Big]^{2/m_j},$$

where $m_j = \varepsilon_0 + \ldots + \varepsilon_{j-1}$, for all $j = 1, \ldots, m+1$. Consequently,

$$\sigma(z) \geqslant C\Big(\sum_{j=1}^{m} \Big[\prod_{t=j}^{m} |a-a_t|^{\varepsilon_t}\Big]^{2/m_j} + 1\Big). \tag{2.28}$$

Let us denote by $\Lambda(\boldsymbol{a})$, $\boldsymbol{a} = (a_0, \ldots, a_m)$, the right-hand side of (2.28).

Since $J(z)$ does not depend on the order of $z_0, \ldots, z_d$, we may write

$$J(z) = J(\boldsymbol{a}) = \delta_0(a'_0 + \ldots + a'_d) \prod_{i<j} |a'_i - a + a - a'_j| \tag{2.29}$$

$$\leqslant C\delta_0(a'_0 + \ldots + a'_d) \prod_{j=1}^{d} |a-a'_j|^j$$

$$= C\delta_0(a'_0 + \ldots + a'_d) \prod_{j=1}^{m} |a-a_j|^{\varepsilon_j j}.$$

Let us write

$$da_j^{\varepsilon_j} = \begin{cases} da_j & \text{if } \varepsilon_j = 1, \\ da_j\, d\bar{a}_j & \text{if } \varepsilon_j = 2. \end{cases}$$

Hence, in virtue of (2.22),

$$dc_{d-1} \ldots dc_0 = J(\boldsymbol{a})\, da_0^{\varepsilon_0} \ldots da_m^{\varepsilon_m} = J(\boldsymbol{a})\, d\boldsymbol{a}.$$

Thus to complete the proof of case 1 it suffices to show

$$\int \Lambda(\boldsymbol{a})^{-\varrho} J(\boldsymbol{a})\, d\boldsymbol{a} < \infty. \tag{2.30}$$

First we make a linear nonsingular change of variables

$$b'_j = a'_j + \sum_{t=1}^{d} a'_t, \quad j = 1, \ldots, d.$$

Since $a'_0 = a$, and $a'_0 + \ldots + a'_d = 0$,

$$|a-a'_j| = |b'_j| \tag{2.31}$$

and

$$\bar{b}'_j = \bar{a}'_j + \sum_{t=1}^{d} a'_t, \quad j = 1, \ldots, d.$$

If $b_1, \ldots, b_m$ is a sequence obtained from $b'_1, \ldots, b'_d$ in the same way as $a_1, \ldots, a_m$ was obtained from $a'_1, \ldots, a'_d$, by (2.29) we obtain

$$J(\boldsymbol{a})\, da_0^{\varepsilon_0} \ldots da_m^{\varepsilon_m} \leqslant C \prod_{j=1}^{m} |b_j|^{\varepsilon_j j}\, db_1^{\varepsilon_1} \ldots db_m^{\varepsilon_m}.$$

Also

$$\Lambda(\boldsymbol{a}) = C\Big(\sum_{j=1}^{m} \Big[\prod_{t=j}^{m} |b_t|^{\varepsilon_t}\Big]^{2/m_j} + 1\Big).$$

Hence the integral in (2.30) is not greater than

$$C \int_{r_1>0} \ldots \int_{r_m>0} \Big(\sum_{j=1}^{m} \Big[\prod_{t=j}^{m} r_t\Big]^{2/m_j} + 1\Big)^{-\varrho} \prod_{j=1}^{m} r_j^j\, dr_1 \ldots dr_m. \tag{2.32}$$

But

$$\prod_{j=1}^{m} r_j^j = \prod_{j=1}^{m}\prod_{t=j}^{m} r_t.$$

Thus, if we put

$$s_j = \prod_{t=j}^{m} r_t, \qquad j = 1, \ldots, m,$$

we see that (2.32) is equal to

$$C \int\limits_{s_1>0} \ldots \int\limits_{s_m>0} \Big(\sum_{j=1}^{m} s_j^{2/m_j}+1\Big)^{-\varrho} s_1\, ds_1 \ldots ds_m$$

which is finite as a routine calculation shows.

Case 2. By (2.26) and (2.27) we have

$$\max_{x\in I} |P_z(x)| \geqslant C|a-a_0|^2 \prod_{t=1}^{m} |a-a_t|^{\varepsilon_t},$$

. . . .

$$\max_{x\in I} |P_z(x)| \geqslant C\delta^{2+\varepsilon_1+\ldots+\varepsilon_{j-1}} \prod_{t=j}^{m} |a-a_t|^{\varepsilon_t},$$

whence, as in case 1,

$$\sigma(z) = \sigma(a) \geqslant C\Big(b^2 \prod_{t=1}^{m} |a-a_t|^{\varepsilon_t} + \sum_{j=1}^{m} \Big[\prod_{t=j}^{m} |a-a_t|^{\varepsilon_t}\Big]^{2/m_j}+1\Big),$$

where $m_j = 2+\varepsilon_1+\ldots+\varepsilon_{j-1}$. Moreover,

$$J(a) \leqslant \delta_0(a_0'+\ldots+a_d')|b| \prod_{j=1}^{m} |a-a_j|^{\varepsilon_j j}$$

and so after a linear nonsingular change of variable,

$$\int \sigma(a)^{-\varrho} J(a)\, da \leqslant C\int \Big(b^2 \prod_{t=1}^{m} |b_t|^{\varepsilon_t} + \sum_{j=1}^{m}\Big[\prod_{t=j}^{m} |b_t|^{\varepsilon_t}\Big]^{2/m_j}+1\Big)^{-\varrho}$$

$$\times |b| \prod_{t=1}^{m} |b_t|^{\varepsilon_t}\, db \prod_{t=2}^{m} |b_t|^{\varepsilon_t}\, db_1^{\varepsilon_1} \ldots |b_t|^{\varepsilon_m}\, db_{m-1}^{\varepsilon_{m-1}}\, db_m^{\varepsilon_m}$$

$$= C \int\limits_{r>0}\int\limits_{r_1>0} \ldots \int\limits_{r_m>0} \Big(r\prod_{t=1}^{m} r_t + \Big[r_1 \prod_{t=2}^{m} r_t\Big]^{2/m_1} + \ldots + r_m^{2/m_m}\Big)^{-\varrho}$$

$$\times \prod_{t=1}^{m} r_t\, dr \prod_{t=2}^{m} dr_1 \ldots r_m\, dr_{m-1}\, dr_m$$

$$= C\int \Big(\sum_{j=0}^{m} s_j^{2/m_j}\Big)^{-\varrho} ds_0 \ldots ds_m,$$

which, as in case 1, is finite.

Case 3. We proceed as in case 2. First, by (2.26) and (2.27) we obtain

$$\max_{x\in I} |P_z(x)| \geqslant C\delta^{\varepsilon_0+\ldots+\varepsilon_{j-1}} |a-a_j|^{\varepsilon_j} \ldots |a-a_m|^{\varepsilon_m} \qquad \text{for } j \neq p,$$

$$\max_{x\in I} |P_z(x)| \geqslant C\delta^{\varepsilon_0+\ldots+\varepsilon_{p-1}+1} |a-a_p|\, |a-a_{p+1}|^{\varepsilon_{p+1}} \ldots |a-a_m|^{\varepsilon_m},$$

whence

$$\sigma(z) \geqslant C\Big(1+\sum_{\substack{j=0\\ j\neq p}}^{m} \Big[\prod_{t=j}^{m} |a-a_t|^{\varepsilon_t}\Big]^{2/m_j} + |b| \sum_{t=p+1}^{m} |a-a_t|^{\varepsilon_t}\Big)^{2/m_p} \tag{2.31}$$

where $m_j = \varepsilon_0+\ldots+\varepsilon_{j-1}$ for $j\neq p$ and $m_p = \varepsilon_0+\ldots+\varepsilon_{p-1}+1$.

Now we use estimate (2.31) and we proceed as in the previous cases. This completes the proof of the lemma and of Theorem (2.7) at the same time.

3. An application. Let G be the group described in Section 1 and let

$$L = -X^2+|Y_0|.$$

Then L is essentially selfadjoint on $\mathscr{S}(G)$. Let

$$Lf = \int_0^\infty \lambda\, dE(\lambda) f$$

be its spectral resolution on $L^2(G)$. By Theorem (2.7) we know that the semigroup of operators

$$T^t f = \int_0^\infty e^{-\lambda t}\, dE(\lambda) f$$

is of the form $T^t f = f * p_t$, where

$$p_t \in L^2(G) \cap L^1(G). \tag{3.1}$$

It follows from [7] that if τ is a function on G defined by

$$\tau(g) = \min\{n\colon\ g\in A^n\},$$

where $A = A^{-1}$ is a fixed open neighbourhood of the identity with compact closure, then for a positive a we have

$$\int \tau(g)^a p_t(g)\, dg < \infty. \tag{3.2}$$

Also, clearly,

$$p_t(g^{-1}) = p_t(g). \tag{3.3}$$

For the operator

$$Rf = \int_0^\infty (1+\lambda)^{-1}\, dE(\lambda) f$$

we have $Rf = f * m$, where

$$m = \int_0^\infty e^{-t} p_t\,dt \in L^1(G),$$

and consequently

$$\int_0^\infty (1+\lambda)^{-s} dE(\lambda) f = f * m^{*s}.$$

The following general proposition holds.

(3.4) Proposition. *Let G be a homogeneous group, and let $\{p_t\}_{t>0}$ be a convolution semigroup of $L^1 \cap L^2$ functions such that*

$$p_t(g) = t^{-Q/2} p(\delta_{t^{-1/2}} g),$$

where $\{\delta_t\}$ are dilations and Q is the homogeneous dimension. Then for $s = [Q/2]+1$ and

$$m(g) = \int_0^\infty e^{-t} p_t(g)\,dt,$$

$m^{*s} \in L^2(G)$.

Moreover, if G is stratified and for some $a > 0$

$$\int \tau(g)^a p_t(g)\,dg < +\infty,$$

then

$$\int m(g) \tau(g)^a dg < +\infty.$$

Proof.

$$\|m^{*s}\|_2^2 = \int \Big(\int e^{-(t_1+\ldots+t_s)} p_{t_1+\ldots+t_s}(g)\,dt_1 \ldots dt_s\Big)^2 dg$$
$$\leqslant \int e^{-(t_1+\ldots+t_s)} p^2_{t_1+\ldots+t_s}(g)\,dt_1 \ldots dt_s\,dg$$
$$\leqslant \int e^{-(t_1+\ldots+t_s)} (t_1+\ldots+t_s)^{-Q/2} dt_1 \ldots dt_s \int p_1^2(g)\,dg < +\infty.$$

To prove the second statement use the fact that for some, $c, C > 0$,

$$c|g| \leqslant \tau(g) \leqslant C(|g|+1)$$

holds for all $g \in G$ provided G is stratified (cf. [11]), and note that

$$\int m(g)|g|^a dg = \int e^{-t} \int p_1(g) t^a |g|\,dg\,dt < +\infty.$$

We say that a function F, on T, belongs to the *Gevrey class* G_γ if there is an R such that

$$\|F^{(n)}\|_{L^\infty} \leqslant R^n (n!)^\gamma, \quad n = 1, 2, \ldots$$

We apply Proposition (3.4) and we see that for $s = [Q/2]+1$

(a) m^{*s} is hermitian.

(b) $m^{*s} \in L^2(G)$.

(c) $\int |m^{*s}| \tau(g)^a dg < +\infty$ for some $a > 0$.

Conditions (a)–(c) were used by T. Pytlik [13] to show that if $F \in G_{\gamma'}$ for some $\gamma' > 1$ and $F(0) = 0$, then the operator

$$f \to \int_0^\infty F\big((1+\lambda)^{-s}\big) dE(\lambda) f \tag{3.5}$$

is given by convolution $f \to f * k$, where $k \in L^2(G) \cap L^1(G)$ and $\int |k(g)| \tau(g)^a dg < +\infty$.

Suppose a function K on $\mathbf{R}^+$ satisfies

$$\max_{\lambda>0} (1+\lambda)^{n(s+1)} |K^{(j)}(\lambda)| \leqslant R^n (n!)^\gamma, \quad j \leqslant n,$$

for some $R > 0$ and $\gamma' > 1$ for all $n = 1, 2, \ldots$ Let

$$F(\xi) = K(\xi^{-1/s} - 1)$$

for $\xi \in (0, 1)$. It is easy to verify that

$$\max_{\xi\in(0,1)} |F^{(n)}(\xi)| \leqslant r^n (n!)^{\gamma'}$$

for some $r > 1$ and $\gamma' > 1$. We extend F to a function in $G_{\gamma'}(T)$ and we see that $F(0) = 0$. Consequently, the operator

$$f \to \int_0^\infty K(\lambda)\,dE(\lambda) f = \int_0^\infty F\big((1+\lambda)^{-s}\big) dE(\lambda) f$$

is of the form $f \to f * k$, where $k \in L^1(G)$.

Now, since L is homogeneous of degree 2,

$$\int_0^\infty K(t\lambda)\,dE(\lambda) f = f * k_t,$$

where

$$k_t(g) = t^{-Q/2} K(\delta_{t^{-1/2}} g),$$

with $Q = (2+d)(5+d)/2$, and so k_t is an approximate identity in $L^1(G)$, as $t \to 0$. Hence, by (1.1),

$$\lim_{t\to 0} \|\pi^c(k_t)\varphi - \varphi\|_{L^p(\mathbf{R})} = 0 \quad \text{for } \varphi \in L^p(\mathbf{R}). \tag{3.6}$$

Now we argue as in [9] (cf. also [2] for further applications) to derive the theorem mentioned in the introduction.

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Two weighted estimates for oscillating kernels I

by

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Abstract. In this paper we wish to determine those nonnegative weights w, v for which $\|Tf\|_{q,w} \leqslant c\|f\|_{p,v}$ where $\|\cdot\|_{s,u} = (\int|\cdot|^s u(t)\,dt)^{1/s}$. The operator $Tf(x)$ is a convolution transform with kernel

$$K_{a,b}(t) = (1+|t|^n)^{-b} e^{i|t|^a}, \quad a > 1.$$

Here, we study the cases where $b \leqslant 1-a/2$. Thus, we solve certain two weight problems for a wide class of transforms which includes the Fourier transform. The results agree with our earlier results on the Fourier transform.

§ 0. Introduction. In this paper we solve certain two weight problems for the kernels

$$K_{a,b}(t) = (1+|t|^n)^{-b} e^{i|t|^a}, \quad a > 1, \tag{0.1}$$

where n (a positive integer) coincides with the dimension of the variable t, i.e. $t = (t_1, t_2, \dots, t_n)$, $|t| = (t_1^2+t_2^2+\dots+t_n^2)^{1/2}$. Also, let $\|t\| = \max_{1\leqslant j\leqslant n}|t_j|$. We set

$$Tf(x) = \int K(x-t) f(t)\,dt$$

and we wish to determine those weights w, v for which

$$\|Tf\|_{q,w} \leqslant c\|f\|_{p,v}, \quad \text{where } \|g\|_{s,u} = (\int\|g\|^s u(t)\,dt)^{1/s}.$$

In this paper, we shall study the cases where $b \leqslant 1-a/2$. The arguments here work for a class of kernels more general than those defined through (0.1). This class is stated explicitly in Remark 1.6. The case where $a = 2$ and $b = 0$ in (0.1), which is identical to the Fourier kernel, is included among our results here (see e.g. Corollary 4.12). Hence this argument will furnish another way to solve a two weight problem for the Fourier transform, and agrees with our results in [4], but is general enough so that it works for a wider class of transforms.

Here, for the most part we shall just discuss the cases where $n = 1$ or 2.

We say a function $u(t)$ is *radial* if $u(t) = u(|t|)$. Furthermore, we say the radial function $u(t)$ is *essentially decreasing* over some region Ω if

$$u(t_1) \geqslant cu(t_2) \quad \text{where } |t_1| \leqslant |t_2|,\ t_1, t_2 \in \Omega,$$