The generalized skew product $T(x, y) = (h(x), T_x(y))$ has the 1-sided generator

$$\alpha = \{[0, \tfrac{1}{8}], (\tfrac{1}{8}, \tfrac{1}{4}], (\tfrac{1}{4}, \tfrac{3}{8}], (\tfrac{3}{8}, \tfrac{5}{8}], (\tfrac{5}{8}, \tfrac{3}{4}], (\tfrac{3}{4}, \tfrac{7}{8}], (\tfrac{7}{8}, 1]\} \times \{[0, \tfrac{1}{2}], (\tfrac{1}{2}, 1]\}$$

because $T \upharpoonright C$ is expanding for any $C \in \alpha$. By $J_{T_x^{-1}} \neq 1$, the transformations T_x do not preserve the measure m_0.

References

[1] L. M. Abramov and V. A. Rokhlin, *Entropy of the skew product of measure preserving transformations*, Vestnik Leningrad. Univ. 1962, no. 7, 5–13.

[2] V. A. Rokhlin, *On the fundamental ideas of measure theory*, Amer. Math. Soc. Transl. Ser. 1 10 (1962), 1–54; Mat. Sb. 25 (1949), 107–150.

INSTYTUT MATEMATYKI POLITECHNIKI WROCŁAWSKIEJ
INSTITUTE OF MATHEMATICS, WROCŁAW TECHNICAL UNIVERSITY
Wybrzeże St. Wyspiańskiego 27, 50-370 Wrocław, Poland

Received October 12, 1985
Revised version August 1, 1986 (2110)

STUDIA MATHEMATICA, T. LXXXVII. (1987)

On the space of Bloch harmonic functions and interpolation of spaces of harmonic and holomorphic functions

by

EWA LIGOCKA (Warszawa)

Abstract. We prove that the orthogonal projection P from $L^2(D)$ onto $L^2\operatorname{Harm}(D)$, the space of square-integrable harmonic functions, maps $L^\infty(D)$ onto the space $\operatorname{Bl}\operatorname{Harm}(D)$ of Bloch harmonic functions on D if D is a smooth bounded domain in $\mathbf{R}^n$. We prove an interpolation theorem which permits us to interpolate between Sobolev or Hölder spaces of harmonic functions and the space $L^p\operatorname{Harm}(D, |\varrho|^r)$ of harmonic functions from $L^p(D, |\varrho|^r)$, where ϱ is a defining function for D. We prove analogous results for spaces of holomorphic functions on strictly pseudoconvex domains.

I. Introduction and the statement of results. The present paper is the direct continuation of [14] and [15]. First, let us recall some notation from those papers.

For a bounded domain D in $\mathbf{R}^n$ we denote by P the orthogonal projection from $L^2(D)$ onto the space $L^2\operatorname{Harm}(D)$ of square-integrable harmonic functions. If D is a domain in $\mathbf{C}^n$, we denote by B the orthogonal projection from $L^2(D)$ onto the space $L^2\operatorname{Hol}(D)$ of square-integrable holomorphic functions (the Bergman projection). $\operatorname{Harm}_p^s(D)$ is the space of harmonic functions from the Sobolev space $W_p^s(D)$, $-\infty < s < +\infty$, $1 < p < \infty$, and $\Lambda_s\operatorname{Harm}(D)$ the space of harmonic functions from the Hölder space $\Lambda_s(D)$; analogously, $\operatorname{Hol}_p^s(D)$ denotes the space of holomorphic functions from $W_p^s(D)$ and $\Lambda_s\operatorname{Hol}(D)$ the space of holomorphic functions from $\Lambda_s(D)$. If D is a C^∞-smooth domain in $\mathbf{R}^n$ then a function $\varrho \in C^\infty(\mathbf{R}^n)$ is a *defining function* for D iff $D = \{x \in \mathbf{R}^n : \varrho(x) < 0\}$ and $\operatorname{grad}\varrho \neq 0$ on ∂D.

The space of *Bloch harmonic functions* on D consists of functions h harmonic on D such that

$$\|h\|_{\mathrm{Bl}} = \sup_{x \in D}\big(|\varrho(x)h(x)| + |\varrho(x)\operatorname{grad} h(x)|\big) < \infty$$

for a defining function ϱ. We denote it by $\operatorname{Bl}\operatorname{Harm}(D)$. If $D \subset \mathbf{C}^n$ then $\operatorname{Bl}\operatorname{Hol}(D)$ denotes the subspace of $\operatorname{Bl}\operatorname{Harm}(D)$ consisting of holomorphic functions.

In the present paper we also consider the spaces $L^\infty(D, |\varrho|^s)$ of functions f on D such that

$$\operatorname*{vrai\,max}_D |\varrho|^s |f| < \infty,$$

and the spaces $L^\infty \operatorname{Harm}(D, |\varrho|^s)$, $s > 0$ ($L^\infty \operatorname{Hol}(D, |\varrho|^s)$, $s > 0$), defined as the subspaces of $L^\infty(D, |\varrho|^s)$ consisting of harmonic (holomorphic) functions.

In [14] we proved that if P maps continuously $L^\infty(D)$ onto $\operatorname{Bl}\operatorname{Harm}(D)$ then the space $\operatorname{Bl}\operatorname{Harm}(D)$ is the "vertex" of the double interpolation scale formed by the spaces $\operatorname{Harm}_p^s(D)$, $\Lambda_s \operatorname{Harm}(D)$ (as the right column) and $L^p \operatorname{Harm}(D) = \operatorname{Harm}_p^0(D)$ (as the bottom row).

In [15] we proved that if D is the unit ball in $\boldsymbol{R}^n$ then P maps continuously $L^\infty(D)$ onto $\operatorname{Bl}\operatorname{Harm}(D)$. It was done by explicitly writing down the kernel of the operator P and estimating it. We are now going to prove the following general

THEOREM 1. *If D is a bounded domain with smooth boundary in $\boldsymbol{R}^n$, then P maps continuously $L^\infty(D)$ onto $\operatorname{Bl}\operatorname{Harm}(D)$.*

Theorem 1 is a consequence of the estimates from [14] and of the following

PROPOSITION 1. *Let D be as above and let $0 < s < 1$. Then P maps continuously $L^\infty(D, |\varrho|^s)$ onto $L^\infty \operatorname{Harm}(D, |\varrho|^s)$.*

Theorem 1 yields the following

COROLLARY 1. *$\operatorname{Bl}\operatorname{Harm}(D)$ represents the dual to the space $L^1 \operatorname{Harm}(D)$ of integrable harmonic functions via the pairing $\langle u, v\rangle_1 = \langle u, L^1 v\rangle_0$, where $\langle\ ,\ \rangle_0$ is the usual L^2 scalar product and*

$$L^1 v = v - \tfrac{1}{2}\Delta\left(\frac{\varrho^2 v\varphi}{|\operatorname{grad}\varrho|^2}\right)$$

is the operator introduced by S. Bell in [4] ($\langle u, v\rangle_1 = \langle u, v\rangle_0$ if both u, v belong to $L^2 \operatorname{Harm}(D)$).

Remark E. Straube observed that if $u \in L^1 \operatorname{Harm}(D)$, $v \in \operatorname{Bl}\operatorname{Harm}(D)$, then

$$\langle\langle u, v\rangle\rangle = \lim_{\varepsilon\to 0^+} \int_{D_\varepsilon} u(x)\overline{v(x)}\, dV_x = \langle u, v\rangle_1,$$

where $D_\varepsilon = \{x \in \boldsymbol{R}^n\colon \varrho(x)+\varepsilon < 0\} \Subset D$. The pairing $\langle\langle\ ,\ \rangle\rangle$ was introduced by E. Straube in [17] and used to study the duality problems (see [17], Th. 3.4 and the following remarks).

The next part of this paper will be devoted to the extension of the double interpolation scale described above. If E, F are Banach spaces then we denote by $[E, F]_{[\theta]}$ the value of the complex interpolation functor at θ, $0 < \theta < 1$, and by $[E, F]_{\widetilde{[\theta]}}$ the completion of $[E, F]_{[\theta]}$ with respect to $E+F$. (For the informations concerning the complex interpolation functor see [5] and [8].)

In [14] (in [10] for $p = 2$) it was proved that for every integer $k > 0$ the mapping $T_k u = \varrho^k u$ maps $\operatorname{Harm}_p^s(D)$ into $W_p^{s+k}(D)$, $-\infty < s < \infty$, and $\Lambda_s \operatorname{Harm}(D)$ into $\Lambda_{s+k}(D)$. Here we prove the following.

PROPOSITION 2. *Let $k > 0$ be an integer. The mapping $R_k u = P(\varrho^k u)$ is an isomorphism between $\operatorname{Harm}_p^s(D)$ and $\operatorname{Harm}_p^{s+k}(D)$ and between $\Lambda_s \operatorname{Harm}(D)$ and $\Lambda_{s+k} \operatorname{Harm}(D)$ for $s \geq 0$, and extends to an isomorphism between $\operatorname{Harm}_p^s(D)$ and $\operatorname{Harm}_p^{s+k}(D)$ for $s < 0$. Moreover, R_k is an isomorphism between $\operatorname{Bl}\operatorname{Harm}(D)$ and $\Lambda_k \operatorname{Harm}(D)$ and between $L^\infty \operatorname{Harm}(D, |\varrho|^s)$ and $\Lambda_{k-s} \operatorname{Harm}(D)$ for $0 < s < 1$.*

Proposition 2 has interesting consequences. The results proved in [14] yield that $\operatorname{Harm}_p^s(D)$ is equal to $L^p \operatorname{Harm}(D, |\varrho|^{-sp})$ with an equivalent norm if $s < 0$. In [15] it was proved that $\operatorname{Harm}_p^s(D) = L^p \operatorname{Harm}(D, |\varrho|^{-sp})$ for $0 \leq s < 1/p$ if D is the unit ball in $\boldsymbol{R}^n$. However, the proof from [15] (part c) of the proof of Theorem 3) remains valid for every smooth bounded domain in $\boldsymbol{R}^n$. On the other hand, the Poisson formula gives an isomorphism between the Besov spaces $B_{p,p}^{s-1/p}(\partial D)$ of the traces of functions from $W_p^s(\boldsymbol{R}^n)$ on ∂D and $\operatorname{Harm}_p^s(D)$ for $s > 1/p$, and between the Hölder spaces $\Lambda_s(\partial D)$ and $\Lambda_s \operatorname{Harm}(D)$ for $s > 0$. Thus we get the following

COROLLARY 2. *$B_{p,p}^{s-1/p}(\partial D)$ is isomorphic to $L^p \operatorname{Harm}(D, |\varrho|^{p(k-s)})$ for every integer $k > s > 1/p$.*

$\Lambda_s(\partial D)$ is isomorphic to $L^\infty \operatorname{Harm}(D, |\varrho|^{s-[s]})$ if $s-[s] > 0$ ($[s]$ denotes the integer part of s). If $k > 0$ is an integer then $\Lambda_k(\partial D)$ is isomorphic to $\operatorname{Bl}\operatorname{Harm}(D)$. It should be mentioned here that $\Lambda_k \operatorname{Harm}(D)$ consists exactly of those harmonic functions whose kth derivatives belong to $\operatorname{Bl}\operatorname{Harm}(D)$.

We shall also prove

PROPOSITION 3. *Let $t > 0$ and $t-[t] > 0$. Denote by R_t the mapping $R_t u = P(|\varrho|^t u)$. Then R_t maps continuously $\operatorname{Harm}_p^s(D)$ into $\operatorname{Harm}_p^{s+t}(D)$ and $\Lambda_s \operatorname{Harm}(D)$ into $\Lambda_{s+t} \operatorname{Harm}(D)$ for $s \geq 0$.*

COROLLARY 3. *The projection P maps the set $\{|\varrho|^t f\colon f \in W_p^s(D)\}$ into $W_p^{s+t}(D)$ for $t \leq 0$ and $s \geq 0$, and the set $\{|\varrho|^t f\colon f \in \Lambda_s(D)\}$ into $\Lambda_{s+t}(D)$ for $s > 0$ and $t \geq 0$.*

We do not know whether the mapping R_t from Proposition 3 is an isomorphism between $\operatorname{Harm}_p^s(D)$ and $\operatorname{Harm}_p^{s+t}(D)$ or between $\Lambda_s \operatorname{Harm}(D)$ and $\Lambda_{s+t} \operatorname{Harm}(D)$.

Proposition 2 yields the following interpolation theorem which extends Theorem 3 of [14]:

THEOREM 2. *Let D be a smooth bounded domain in $\boldsymbol{R}^n$. Then:*

1) $[\mathrm{Harm}_{p_1}^{s_1}(D), \mathrm{Harm}_{p_2}^{s_2}(D)]_{[\theta]} = \mathrm{Harm}_q^t(D)$, *where* $0<\theta<1$,

$$1<p_1, p_2<\infty,\ -\infty<s_1, s_2<\infty,\ t=(1-\theta)s_1+\theta s_2,\ \frac{1-\theta}{p_1}+\frac{\theta}{p_2}=\frac{1}{q}.$$

2) $[\mathrm{Harm}_p^s(D), \Lambda_r \mathrm{Harm}(D)]_{[\theta]} = \mathrm{Harm}_q^t(D)$, *where* $r>0$,

$$0<\theta<1,\ 1<p<\infty,\ -\infty<s<\infty,\ q=\frac{p}{1-\theta},\ t=(1-\theta)s+\theta r.$$

3) $[\mathrm{Harm}_p^s(D), \mathrm{Bl\,Harm}(D)]_{[\theta]} = \mathrm{Harm}_q^t(D)$, *where*

$$\theta,\ p,\ s \text{ are as above},\ t=(1-\theta)s,\ q=\frac{p}{1-\theta}.$$

4) $[\mathrm{Harm}_p^s(D), L^\infty \mathrm{Harm}(D, |\varrho|^r)]_{[\theta]} = \mathrm{Harm}_q^t(D)$, *where*

$$\theta,\ p,\ s \text{ are as above},\ 0<r<1,\ t=(1-\theta)s-\theta r,\ q=\frac{p}{1-\theta}.$$

5) $[\Lambda_s \mathrm{Harm}(D), L^\infty \mathrm{Harm}(D, |\varrho|^r)]_{[\tilde{\theta}]}$

$$\begin{aligned} &= \Lambda_{(1-\theta)s-\theta r}\mathrm{Harm}(D) && \text{if } (1-\theta)s>\theta r,\\ &= \mathrm{Bl\,Harm}(D) && \text{if } (1-\theta)s=\theta r,\\ &= L^\infty \mathrm{Harm}(D, |\varrho|^t),\ t=\theta r-(1-\theta)s, && \text{if } (1-\theta)s<\theta r. \end{aligned}$$

Since $L^p\mathrm{Harm}(D, |\varrho|^r) = \mathrm{Harm}_p^{-r/p}(D)$ for $1<p<\infty$ and $r>-1$, the above theorem permits us to interpolate between $L^p\mathrm{Harm}(D, |\varrho|^r)$, $r>-1$, and the Hölder and Bloch spaces of harmonic functions. All the above results have their counterparts for the Bergman projection B and for spaces of holomorphic functions on smooth bounded strictly pseudoconvex domains in $\boldsymbol{C}^n$. The following facts hold:

PROPOSITION 4. *Let D be a bounded strictly pseudoconvex domain with C^4-smooth boundary in $\boldsymbol{C}^n$. Then the Bergman projection B maps continuously $L^\infty(D, |\varrho|^s)$ onto $L^\infty\mathrm{Hol}(D, |\varrho|^s)$ for $0<s<1$.*

THEOREM 3. *Let D be a smooth bounded strictly pseudoconvex domain in $\boldsymbol{C}^n$. Then:*

1) $[\mathrm{Hol}_{p_1}^{s_1}(D), \mathrm{Hol}_{p_2}^{s_2}(D)]_{[\theta]} = \mathrm{Hol}_q^t(D)$, *where* $0<\theta<1$,

$$1<p_1, p_2<\infty,\ -\infty<s_1, s_2<\infty,\ t=(1-\theta)s_1+\theta s_2,\ \frac{1-\theta}{p_1}+\frac{\theta}{p_2}=\frac{1}{q}.$$

2) $[\mathrm{Hol}_p^s(D), \Lambda_r \mathrm{Hol}(D)]_{[\theta]} = \mathrm{Hol}_q^t(D)$, *where* $0<\theta<1$,

$$1<p<\infty,\ -\infty<s<\infty,\ r>0,\ t=(1-\theta)s+\theta r,\ q=\frac{p}{1-\theta}.$$

3) $[\mathrm{Hol}_p^s(D), \mathrm{Bl\,Hol}(D)]_{[\theta]} = \mathrm{Hol}_q^t(D)$, *where*

$$\theta,\ s,\ p \text{ are as above},\ t=(1-\theta)s,\ q=\frac{p}{1-\theta}.$$

4) $[\mathrm{Hol}_p^s(D), L^\infty \mathrm{Hol}(D, |\varrho|^r)]_{[\theta]} = \mathrm{Hol}_q^t(D)$, *where*

$$\theta,\ s,\ p \text{ are as above},\ 0<r<1,\ t=(1-\theta)s-\theta r,\ q=\frac{p}{1-\theta}.$$

5) $[\Lambda_s \mathrm{Hol}(D), L^\infty \mathrm{Hol}(D, |\varrho|^r)]_{[\tilde{\theta}]}$

$$\begin{aligned} &= \Lambda_{(1-\theta)s-\theta r}\mathrm{Hol}(D) && \text{if } (1-\theta)s>\theta r,\\ &= \mathrm{Bl\,Hol}(D) && \text{if } (1-\theta)s=\theta r,\\ &= L^\infty \mathrm{Hol}(D, |\varrho|^t),\ t=\theta r-(1-\theta)s, && \text{if } (1-\theta)s<\theta r\ (s>0,\ 0<r<1). \end{aligned}$$

We have $L^p\mathrm{Hol}(D, |\varrho|^r) = \mathrm{Hol}_p^{-r/p}(D)$ for $1<p<\infty$ and $r>-1$, so we can interpolate between $L^p\mathrm{Hol}(D, |\varrho|^r)$ and Hölder or Bloch spaces of holomorphic functions. Theorem 3 is an extension of Theorem 8 from [14]. If D is as above, then Propositions 2 and 3 together with Corollary 3 remain valid if we replace the projection P by the Bergman projection B and the Sobolev spaces $\mathrm{Harm}_s^p(D)$ by the spaces $\mathrm{Hol}_s^p(D)$. We shall end this paper with remarks concerning some applications of the above results.

II. Proofs.

1) Proof of Proposition 1 and of Theorem 1. Proposition 1 will be proved in the same manner as Proposition 2 in [11]. Let $f\in L^\infty(D, |\varrho|^s)\cap C^\infty(\bar{D})$. In [11] it was proved (in the proof of Proposition 2) that $Pf = \Delta v$, where v is a biharmonic function on D such that

$$v(y) = c\int_D \frac{f(x)\,dV_x}{(|x-y|^2+\varrho(x)\varrho(y))^{n/2-1}} = w_1(y),$$

$$\frac{\partial v}{\partial n}(y) = \frac{\partial}{\partial n}w_1(y) + c\int_D \frac{f(x)|\varrho(x)|(\partial\varrho/\partial n)(y)\,dV_x}{(|x-y|^2+\varrho(x)\varrho(y))^{n/2}} = w_2(y)$$

for $y\in\partial D$. We can assume that $|\nabla\varrho|\equiv 1$ on ∂D. We shall show that $w_1\in\Lambda_{2-s}(D)$ and $\|w_1\|_{\Lambda_{2-s}(D)}\lesssim\|f\|_{L^\infty(D,|\varrho|^s)}$, and that $w_2\in\Lambda_{1-s}(D)$ and $\|w_2\|_{\Lambda_{1-s}(D)}\lesssim\|f\|_{L^\infty(D,|\varrho|^s)}$.

We have for all $i, j \leqslant n$

$$\left|\frac{\partial^2}{\partial y_i\, \partial y_j} w_1(y)\right| = \left|c \frac{\partial^2}{\partial y_i\, \partial y_j} \int_D \frac{f(x)\,|\varrho(x)|^s\, dV_x}{|\varrho(x)|^s \left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2-1}}\right|$$

$$\leqslant c_1 \|f\|_{L^\infty(D,|\varrho|^s)} \int_D \frac{dV_x}{|\varrho(x)|^s\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2}}.$$

The last integral can be estimated in the following manner:

$$\int_D \frac{dV_x}{|\varrho(x)|^s\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2}}$$

$$= \int_D \frac{\left(1-|\nabla\varrho|^2(x)\right)dV_x}{|\varrho(x)|^s\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2}} + \int_D \frac{|\nabla\varrho|^2(x)\, dV_x}{|\varrho(x)|^s\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2}}$$

$$\leqslant c_2\Bigg(\int_D \frac{|\varrho(x)|^{1-s}\, dV_x}{\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2}}$$

$$+\int_D \sum_i \frac{(\partial\varrho/\partial x_i)(x)\,(\partial/\partial x_i)\left(-|\varrho(x)|^{1-s}\right)dV_x}{\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2}}\Bigg).$$

The last integral can be estimated via integration by parts by

$$\int_D \frac{|\varrho(x)|^{1-s}\, dV_x}{\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2+1/2}}.$$

It now follows that the whole expression can be estimated by $c_3/|\varrho(y)|^s$, which can be proved in the following way. There exists $c > 1$ such that $|\varrho(x) - \varrho(y)| \leqslant \sqrt{c}\,|x-y|$. Hence

$$(\varrho(x)-\varrho(y))^2 = \varrho^2(x)+\varrho^2(y)-2\varrho(x)\varrho(y) \leqslant c\,|x-y|^2.$$

Thus

$$\frac{\varrho(x)\varrho(y)}{c} \geqslant \frac{\varrho^2(x)+\varrho^2(y)}{2c} - \tfrac{1}{2}|x-y|^2$$

and

$$\frac{|\varrho(x)|^{1-s}}{\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2+1/2}} \leqslant \frac{|\varrho(x)|^{1-s}}{\left(|x-y|^2+\varrho(x)\varrho(y)/c\right)^{n/2+1/2}}$$

$$\leqslant \frac{c_1}{\left(|x-y|^2+(\varrho^2(x)+\varrho^2(y))/c\right)^{n/2+s/2}} \leqslant \frac{c_1}{\left(|x-y|^2+c_2\varrho^2(y)\right)^{n/2+s/2}}.$$

If $R \geqslant \operatorname{diam} D$ then for every $y \in \bar{D}$,

$$\int_D \frac{|\varrho(x)|^{1-s}\, dV_x}{\left(|x-y|^2+\varrho(x)\varrho(y)\right)^{n/2+1/2}} \leqslant c_1 \int_D \frac{dV_x}{\left(|x-y|^2+c_2\varrho^2(y)\right)^{n/2+s/2}}$$

$$\leqslant c_1 \int_{B(y,R)} \frac{dV_x}{\left(|x-y|^2+c_2\varrho^2(y)\right)^{n/2+s/2}} \leqslant \frac{c_3}{|\varrho(y)|^s}.$$

Hence by the Hardy–Littlewood lemma $w_1 \in \Lambda_{2-s}(D)$ and $\|w_1\|_{\Lambda_{2-s}(D)} \lesssim \|f\|_{L^\infty(D,|\varrho|^s)}$.

Exactly the same kind of estimates permits us to show that $w_2 \in \Lambda_{1-s}(D)$ and $\|w_2\|_{\Lambda_{1-s}(D)} \lesssim \|f\|_{L^\infty(D,|\varrho|^s)}$. The estimates from [1] yield that the biharmonic function v belongs to $\Lambda_{2-s}(D)$ and

$$\|v\|_{\Lambda_{2-s}(D)} \lesssim \|w_1\|_{\Lambda_{2-s}(\partial D)} + \|w_2\|_{\Lambda_{1-s}(\partial D)}.$$

Thus $\|v\|_{\Lambda_{2-s}(D)} \lesssim \|f\|_{L^\infty(D,|\varrho|^s)}$. The lemma from the proof of Theorem 2 of [11] implies that $\Delta v \in L^\infty(D, |\varrho|^s)$. Since $Pf = \Delta v$ we have

$$\|Pf\|_{L^\infty(D,|\varrho|^s)} \leqslant c\,\|f\|_{L^\infty(D,|\varrho|^s)} \quad \text{for each } f \in L^\infty(D, |\varrho|^s) \cap C^\infty(\bar{D}).$$

But for every $f \in L^\infty(D, |\varrho|^s)$ we can find a sequence of functions $f_n \in C^\infty(\bar{D})$ such that $f_n \to f$ in $L^p(D)$, $1 < p < 1/s$, and

$$\|f_n\|_{L^\infty(D,|\varrho|^s)} \lesssim \|f\|_{L^\infty(D,|\varrho|^s)} \quad \text{for each } n.$$

Hence $\|Pf\|_{L^\infty(D,|\varrho|^s)} \leqslant c\,\|f\|_{L^\infty(D,|\varrho|^s)}$ and Proposition 1 is proved.

In order to prove Theorem 1, it suffices to make the following observations:

(a) $[L^\infty(D, |\varrho|^s), L^\infty(D, |\varrho|^{-s})]^{\sim}_{1/2} = L^\infty(D)$.

(b) By the lemma from the proof of Theorem 2 in [11], for each $1 \leqslant i \leqslant n$ the mapping $f \to (\partial/\partial x_i) Pf$ maps continuously $L^\infty(D, |\varrho|^s)$ into $L^\infty(D, |\varrho|^{1+s})$.

(c) By Proposition 2 of [11], the mapping $f \to (\partial/\partial x_i) Pf$ also maps $L^\infty(D, |\varrho|^{-s})$ into $L^\infty(D, |\varrho|^{1-s})$.

(d) $[L^\infty(D, |\varrho|^{1+s}), L^\infty(D, |\varrho|^{1-s})]^{\sim}_{1/2} = L^\infty(D, |\varrho|)$.

Hence, by interpolation, the mapping $f \to (\partial/\partial x_i) Pf$ maps continuously $L^\infty(D)$ into $L^\infty(D, |\varrho|)$ for every $1 \leqslant i \leqslant n$, and thus P maps continuously $L^\infty(D)$ into $\mathrm{Bl\,Harm}(D)$.

The proof of Corollary 1 is exactly the same as the proof of Theorem 1 in [15] or of Proposition 2 in [13].

2) Proof of Proposition 2 and of Theorem 2. Let us consider the operator

$$Hu = P\left(\Delta\left(\varrho u\frac{\varphi}{|\nabla\varrho|^2}\right)\right),$$

where φ is a function from $C^\infty(\mathbf{R}^n)$ equal to 1 in a neighbourhood of ∂D and equal to zero in a neighbourhood of the set $\{\nabla\varrho = 0\}$. The operator H maps $\mathrm{Harm}_p^s(D)$ into $\mathrm{Harm}_p^{s-1}(D)$ for $s \geqslant 1$. We shall prove the following properties of H:

(a) $\ker H = \{0\}$.

(b) *For each integer $k \geqslant 0$ the operator $u \to P(\varrho^k H^k u)$ is a Fredholm isomorphism of the space* $\mathrm{Harm}_p^{s+k}(D)$, $s \geqslant 0$.

Let us prove (a). If $Hu = 0$ then for every $w \in L^2\,\mathrm{Harm}(D) \cap C^\infty(\bar{D})$

$$\int_D \Delta\left(\varrho u\frac{\varphi}{|\nabla\varrho|^2}\right)\bar{w}\,dV = 0.$$

It follows from the Green formula that

$$\int_D \Delta\left(\varrho u\frac{\varphi}{|\nabla\varrho|^2}\right)\bar{w}\,dV = \int_{\partial D} u\bar{w}\frac{1}{|\nabla\varrho|^2}\,d\sigma = 0.$$

The last equality implies that the trace of u on ∂D is equal to zero, and so $u \equiv 0$ on D.

We begin the proof of (b) with the following

Lemma. *The mapping $f \to P(\varrho^k f)$ maps continuously $W_p^s(D)$ into $W_p^{s+k}(D)$ and $\Lambda_s(D)$ into $\Lambda_{s+k}(D)$ for $s \geqslant 0$ (k is an integer, $k \geqslant 0$).*

Proof. Let $r \geqslant 0$ be an integer. In [10], Remark 1, we proved the following fact: Each $f \in W_p^r(D)$ can be written in the form

$$f = h_0 + \varrho h_1 + \ldots + \varrho^{r-1} h_{r-1} + w,$$

where $h_i \in \mathrm{Harm}_p^{r-i}(D)$, $w \in \mathring{W}_p^r(D)$ and the correspondences $f \to h_i$, $f \to w$ are uniquely determined and continuous if the defining function ϱ is fixed. In [10] the above fact was stated and proved for $p = 2$, but its proof for $p \neq 2$ is exactly the same. As was mentioned in [14], $w = \varrho^r v$, where $v \in L^p(D)$ and the correspondence $w \to v$ is continuous. In [11], Remark 2, it was observed that there exists a uniquely determined decomposition of $f \in \Lambda_\alpha(D)$,

$$f = h_0 + \varrho h_1 + \ldots + \varrho^s h_s + w,$$

where $s = [\alpha]$ (the integer part of α), $h_k \in \Lambda_{\alpha-k}\,\mathrm{Harm}(D)$ and w belongs to the space $\mathring{\Lambda}_\alpha(D)$ of functions from $\Lambda_\alpha(D)$ which vanish on ∂D up to order $[\alpha]$. This implies that $w = |\varrho|^\alpha v$, where $v \in L^\infty(D)$.

We have

$$P(\varrho^k f) = \sum_{i=0}^{r-1} P(\varrho^{i+k} h_i) + P(\varrho^{r+k} v) \quad \text{if } f \in W_p^r(D).$$

Theorem 1 and Proposition 1 of [14] yield that $\varrho^{i+k} h_i \in W_p^{r+k}(D)$ and $P(\varrho^{r+k} v) \in \mathrm{Harm}_p^{r+k}(D)$. The results of [11] imply our lemma for $f \in \Lambda_\alpha(D)$ in exactly the same way. Interpolation permits us to prove the lemma for $W_p^s(D)$ when s is noninteger.

We shall now prove (b) by induction on k. For $k = 0$, $H^k = \mathrm{Id}$ and (b) is obvious; suppose that (b) holds for $k-1$. We have

$$\begin{aligned} P(\varrho^k H^k u) &= P\left(\varrho^k P\Delta\left(\varrho H^{k-1} u\frac{\varphi}{|\nabla\varrho|^2}\right)\right) \\ &= P\left(\varrho^k \Delta\left(\varrho H^{k-1} u\frac{\varphi}{|\nabla\varrho|^2}\right)\right) - P\left(\varrho^k \Delta G_2 \Delta^2\left(\varrho H^{k-1} u\frac{\varphi}{|\nabla\varrho|^2}\right)\right) \\ &= P\left(\varrho^k \Delta\left(\varrho H^{k-1} u\frac{\varphi}{|\nabla\varrho|^2}\right)\right) - F_k' u. \end{aligned}$$

Our lemma implies that the operator F_k' maps continuously $\mathrm{Harm}_p^{s+k}(D)$ into $\mathrm{Harm}_p^{s+k+1}(D)$ since $\Delta^2(\varrho H^{k-1} u\varphi/|\nabla\varrho|^2) \in W_p^{s-1}(D)$ and thus

$$\Delta G_2 \Delta^2\left(\varrho H^{k-1} u\frac{\varphi}{|\nabla\varrho|^2}\right) \in W_p^{s+1}(D),$$

where G_2 is the operator solving the Dirichlet problem $\Delta^2 u = f$ on D, $u = \partial u/\partial n = 0$ on ∂D. Hence F_k' is a compact operator from $\mathrm{Harm}_p^{s+k}(D)$ into itself.

Since $P(\Delta\varrho^{k+1} H^{k-1} u\varphi/|\nabla\varrho|^2) = 0$, we obtain

$$P\left(\varrho^k \Delta\left(\varrho H^{k-1} u\frac{\varphi}{|\nabla\varrho|^2}\right)\right) = -kP(\varrho^{k-1} H^{k-1} u) + F_k'' u,$$

where F_k'' maps continuously $\mathrm{Harm}_p^{s+k}(D)$ into $\mathrm{Harm}_p^{s+k+1}(D)$. This follows by an elementary calculation from our lemma and from the fact that $\varphi \equiv 1$ on a neighbourhood of ∂D. Hence

$$P(\varrho^k H^k u) = -kP(\varrho^{k-1} H^{k-1} u) + F_k u,$$

where F_k is a compact operator from $\mathrm{Harm}_p^{s+k}(D)$ into itself. By the inductive hypothesis $u \to P(\varrho^k H^k u)$ is a Fredholm operator. We have $\ker P(\varrho^k H^k u) = 0$ because $\ker H = \{0\}$. Hence $u \to P(\varrho^k H^k u)$ is an isomorphism and (b) is proved.

Observe that we have already proved the first part of Proposition 2. The above construction applied to the Hölder spaces $\Lambda_s\,\mathrm{Harm}(D)$ gives the proof of Proposition 2 for these spaces. Proposition 1 together with the above construction and interpolation gives the proof of the last part of Proposition 2. The details are exactly the same as above and thus are left to the reader.

Hence it only remains to prove that $R_k u = P(\varrho^k u)$ extends to an isomorphism between $\mathrm{Harm}_p^s(D)$ and $\mathrm{Harm}_p^{s+k}(D)$ for $s<0$.

Assume first that $s+k\leqslant 0$. Theorem 2 of [14] implies that for every $h\in \mathrm{Harm}_p^s(D)\cap C^\infty(\bar{D})$

$$\|P(\varrho^k h)\|_{s+k} = \sup_{\substack{v\in \mathrm{Harm}_q^{-s-k}(D)\\ \|v\|\leqslant 1}} |\langle P(\varrho^k h), v\rangle| = \sup_{\substack{v\in \mathrm{Harm}_q^{-s-k}(D)\\ \|v\|\leqslant 1}} |\langle h, P(\varrho^k v)\rangle|$$

$$= |||h|||, \quad q=\frac{p}{p-1}.$$

Since $v\to P(\varrho^k v)$ is an isomorphism between $\mathrm{Harm}_q^{-s-k}(D)$ and $\mathrm{Harm}_q^{-s}(D)$, the norm $|||h|||$ is equivalent to $\|h\|_p^s$. The functions from $\mathrm{Harm}_p^s(D)\cap C^\infty(\bar{D})$ are dense in $\mathrm{Harm}_p^s(D)$. Hence R_k extends to an isomorphism between $\mathrm{Harm}_p^s(D)$ and $\mathrm{Harm}_p^{s+k}(D)$ if $s+k\leqslant 0$.

We now use Remark 4 from [14] to interpolate between $\mathrm{Harm}_p^{-k}(D)$ and $L^p\,\mathrm{Harm}(D)$, and Theorem 3 from [14] to interpolate between $L^p\,\mathrm{Harm}(D)$ and $\mathrm{Harm}_p^k(D)$. This interpolation gives the rest of the proof of Proposition 2.

We can now prove Theorem 2. For $s, s_1, s_2\geqslant 0$ Theorem 2 was already proved in [14] (see Theorem 3 and Proposition 3). If, in 1), s_1 or s_2 is negative, then we have by Proposition 2

$$[\mathrm{Harm}_{p_1}^{s_1}(D), \mathrm{Harm}_{p_2}^{s_2}(D)]_{[\theta]} = R_k^{-1}\big([R_k(\mathrm{Harm}_{p_1}^{s_1}(D)), R_k(\mathrm{Harm}_{p_2}^{s_2}(D))]_{[\theta]}\big)$$
$$= R_k^{-1}\big([\mathrm{Harm}_{p_1}^{s_1+k}(D), \mathrm{Harm}_{p_2}^{s_2+k}(D)]_{[\theta]}\big) = R_k^{-1}\big(\mathrm{Harm}_q^{t+k}(D)\big) = \mathrm{Harm}_q^t(D).$$

In exactly the same manner we prove the other four items of Theorem 2 always using Proposition 2.

Remark. It follows immediately from Proposition 1.5 of [18] that the operator H defined above does not depend on the choice of a defining function ϱ or of a function φ. The operator H seems to be an important one. It not only defines an isomorphism between $\mathrm{Harm}_p^s(D)$ and $\mathrm{Harm}_p^{s-1}(D)$, but also has the following property: If $u, v\in \mathrm{Harm}_2^{1/2}(D)$ then

$$\int_{\partial D} u\bar{v}\,d\sigma = \langle Hu, v\rangle_0 = \langle u, Hv\rangle_0.$$

Thus it is also useful in the study of Hardy spaces (see Remark 2 at the end of this paper).

3) Proof of Proposition 3 and of Corollary 3. In the case of Sobolev spaces we prove Proposition 3 and Corollary 3 for integer s by induction with respect to s.

If $s=0$ then Proposition 2 of [14] is exactly the needed result. Suppose now that Proposition 3 and Corollary 3 hold for each $s_1\leqslant s$ and $t>0$. Just as in the proof of Proposition 2, we have

$$P\left(\Delta\left(|\varrho|^{t+2}h\frac{\varphi}{|\nabla\varrho|^2}\right)\right)=0 \quad \text{for } h\in \mathrm{Harm}_p^{s+1}(D).$$

Hence

$$(t+1)(t+2)P(|\varrho|^t h\varphi) = (t+2)\left[2P\left(\sum_i \frac{\partial\varrho}{\partial x_i}\frac{\partial h}{\partial x_i}\frac{\varphi|\varrho|^{t+1}}{|\nabla\varrho|^2}\right)\right.$$
$$+P\left(\Delta\varrho\,|\varrho|^{t+1}h\frac{\varphi}{|\nabla\varrho|^2}\right)+2P\left.\left(|\varrho|^{t+1}h\sum_i\frac{\partial\varrho}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\varphi}{|\nabla\varrho|^2}\right)\right)\right]$$
$$+2P\left(|\varrho|^{t+2}\sum_i\frac{\partial h}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\varphi}{|\nabla\varrho|^2}\right)\right)+P\left(|\varrho|^{t+2}h\Delta\left(\frac{\varphi}{|\nabla\varrho|^2}\right)\right).$$

By the inductive assumption the operators on the right map $W_p^{s+1}(D)$ continuously into $\mathrm{Harm}_p^{s+1+t}(D)$. Since $P(|\varrho|^t h) = P(|\varrho|^t\varphi h)+P(|\varrho|^t(1-\varphi)h)$ and $\varphi\equiv 1$ in a neighbourhood of ∂D, Proposition 3 is proved for $s+1$.

In order to prove Corollary 3 in this case we take the same decomposition of $f\in W_p^{s+1}(D)$ as in the proof of Proposition 2:

$$f = h_0+\sum_{i=1}^{s}\varrho^i h_i+\varrho^{s+1}v, \qquad h_i\in \mathrm{Harm}_p^{s-i+1}(D),\ v\in L^p(D).$$

Thus

$$|\varrho|^t f = |\varrho|^t h_0+\sum_{i=1}^{s}(-1)^i|\varrho|^{t+i}h_i+(-1)^{s+1}|\varrho|^{t+s+1}v$$

and Proposition 3, together with Proposition 2 of [14], yields the required result.

The same procedure permits us to prove Proposition 3 and Corollary 3 for Hölder spaces. The only difference is that in order to start our induction we must observe that Proposition 2 from [11] implies that $h\to P(|\varrho|^t h)$ maps $L^\infty(D, |\varrho|^s)$ into $\Lambda_{t-s}(D)$ if $t>s$ and into $L^\infty(D, |\varrho|^{s-t})$ if $t\leqslant s$.

4) Proof of Proposition 4 and of Theorem 3. The proof of Proposition 4 is based on the same methods as the proof of Hölder estimates for the Bergman projection in [9] and Bloch norm estimates in [13]. In order not to repeat ourselves we outline it very briefly.

In [9] we used the Kerzman–Stein integral formula [6] to construct another projection G from $L^2(D)$ onto $L^2\,\mathrm{Hol}(D)$ and we got the representation

$$Bf = \big(I-(G-G^*)\big)^{-1}G^*f = G\big(I+(G-G^*)^{-1}\big)f.$$

We proved that G is an integral operator with kernel $G(w, z)$ holomorphic in w and such that $|G(w, z)| \lesssim 1/|F(w, z)-\varrho(z)|^{n+1}$ if w is sufficiently close to z and to ∂D, where

$$F(w, z) = \sum_{i=1}^{n} \frac{\partial\varrho}{\partial z_i}(z_i-w_i)+\tfrac{1}{2}\sum_{i,j}\frac{\partial^2\varrho}{\partial z_i\,\partial z_j}(w_i-z_i)(z_j-w_j).$$

Hence the operator $G-G^*$ is also an integral operator and its kernel $A(w, z)$ satisfies the estimates

$$|A(w, z)| \lesssim \frac{1}{|F(w, z)-\varrho(z)|^{n+1/2}}, \qquad |\mathrm{grad}_w A(w, z)| \lesssim \frac{1}{|F(w, z)-\varrho(z)|^{n+3/2}}$$

if w is near z and near ∂D.

We shall need the smooth change of coordinates $v(z)$ used by S. Krantz in [7]:

$$v_1 = \varrho(z)-\varrho(w)+i\,\mathrm{Im}\,F(w, z) = t_1+it_2,$$
$$v_i = t_{2i-1}+it_{2i},$$
$$v(w) = 0,$$

and next the spherical coordinates in the variables

$$t_2, \ldots, t_{2n}, \qquad r = t_2^2+\ldots+t_{2n}^2, \qquad t_2 = r\cos\theta.$$

We then have

$$|F(w, z)-\varrho(z)| \geqslant c\left([-t_1+2|\varrho(w)|+(t_1^2+r^2)]^2+r^2\cos^2\theta\right)^{1/2}.$$

Now we have for $f\in L^\infty(D, |\varrho|^s)$, $0<s<1$,

$$|Gf(w)| = \left|\int_D \frac{G(w, z)}{|\varrho(z)|^s}|\varrho(z)|^s f(z)\,dV_z\right| \leqslant \|f\|_{L^\infty(D,|\varrho|^s)}\int_D \frac{|G(w, z)|}{|\varrho(z)|^s}\,dV_z.$$

From Krantz's estimates [7, 9] it follows that there exist c and R independent of w such for all $f\in L^\infty(D, |\varrho|^s)$ and $w\in D$

$$|Gf(w)| \leqslant c\|f\|_{L^\infty(D,|\varrho|^s)}$$
$$\times\int_{-R}^{|\varrho(w)|} dt_1 \int_0^{(R^2-t_1^2)^{1/2}} dr\int_0^{\pi}\frac{r^{2n-2}\sin^{2n-3}\theta\,d\theta}{(-t_1+|\varrho(w)|)^s([2|\varrho(w)|-t_1+t_1^2+r^2]^2+r^2\cos^2\theta)^{(n+1)/2}}$$

By putting $s=\cos\theta$, we can estimate the last integral by

$$\int_{-R}^{|\varrho(w)|} dt_1 \int_0^{(R^2-t_1^2)^{1/2}} dr\int_{-1}^{1}\frac{r^{2n-2}\,ds}{(-t_1+|\varrho(w)|)^s([2|\varrho(w)|-t_1+t_1^2+r^2]^2+r^2s^2)^{(n+1)/2}}.$$

After the same elementary estimates and integration with respect to s and r as in Krantz [7], we get the estimate

$$|Gf(w)| \leqslant c\int_{-R}^{|\varrho(w)|}\frac{\|f\|_{L^\infty(D,|\varrho|^s)}\,dt_1}{(-t_1+|\varrho(w)|)^s(2|\varrho(w)|-t_1)} \leqslant c_1\frac{\|f\|_{L^\infty(D,|\varrho|^s)}}{|\varrho(w)|^s}.$$

Thus G maps continuously $L^\infty(D, |\varrho|^s)$ into itself. In exactly the same manner we can prove that

$$|(G-G^*)f(w)| \leqslant \frac{c\|f\|_{L^\infty(D,|\varrho|^s)}}{|\varrho(w)|^s}, \qquad |\mathrm{grad}(G-G^*)f(w)| \leqslant \frac{c\|f\|_{L^\infty(D,|\varrho|^s)}}{|\varrho(w)|^{1/2+s}}.$$

This last estimate implies that $G-G^*$ is a compact operator from $L^\infty(D, |\varrho|^s)$ into itself. Hence $I-(G-G^*)$ and $I+(G-G^*)$ are Fredholm isomorphisms of $L^\infty(D, |\varrho|^s)$ and $B = G(I+(G-G^*))^{-1}$ maps continuously $L^\infty(D, |\varrho|^s)$ onto $L^\infty\,\mathrm{Hol}(D, |\varrho|^s)$, $0<s<1$. Proposition 4 is proved.

Theorem 3 is now a direct consequence of the regularity of the Bergman projection in Sobolev and Hölder norms, of Proposition 4 and of the fact proved in [14] that for all $s<0$ and $1<p<\infty$, the projection B extends to a continuous projection from $\mathrm{Harm}_p^s(D)$ onto $\mathrm{Hol}_p^s(D)$. Hence Theorem 2 implies Theorem 3.

5) Proof of Propositions 2 and 3 and of Corollary 3 for spaces of holomorphic functions. The fact that $f\to B(|\varrho|^s f)$ maps $W_p^r(D)$ into $\mathrm{Hol}_p^{r+s}(D)$ and $\Lambda_r(D)$ into $\Lambda_{r+s}\mathrm{Hol}(D)$ for $r, s>0$ if D is a smooth bounded strictly pseudoconvex domain follows immediately from Proposition 3 and Corollary 3 since $Bf = BPf$. We must only prove that if k is an integer then $h\to B(\varrho^k h)$ is an isomorphism between $\mathrm{Hol}_p^s(D)$ and $\mathrm{Hol}_p^{s+k}(D)$ and between $\Lambda_s\mathrm{Hol}(D)$ and $\Lambda_{s+k}\mathrm{Hol}(D)$.

The proof is the same as that of Proposition 2 for harmonic functions with one significant difference.

We define

$$H(u) = B\left(\Delta\left(\varrho u\frac{\varphi}{|\nabla\varrho|^2}\right)\right)$$

and in the proof of (b) we show that for $u\in\mathrm{Hol}_p^{s+k}(D)$

$$B(\varrho^k H^k u) = B\left(\varrho^k\Delta\left(\varrho H^{k-1}u\frac{\varphi}{|\nabla\varrho|^2}\right)\right)-B\left(\varrho^k T\bar\partial\Delta\left(\varrho H^{k-1}u\frac{\varphi}{|\nabla\varrho|^2}\right)\right)$$
$$= -kB(\varrho^{k-1}H^{k-1}u)-Fu,$$

where Tw is the canonical solution of the $\bar\partial$-problem $\bar\partial Tw = w$, $Tw\perp L^2\mathrm{Hol}(D)$, and F maps continuously $\mathrm{Hol}_2^{s+k}(D)$ into $\mathrm{Hol}_2^{s+k+1/2}(D)$ and $\Lambda_{\alpha+k}\mathrm{Hol}(D)$ into $\Lambda_{\alpha+k+1/2}\mathrm{Hol}(D)$. Then the same procedure as that used for harmonic functions shows that $u\to B(\varrho^k u)$ is an isomorphism between

$\mathrm{Hol}_2^s(D)$ and $\mathrm{Hol}_2^{s+k}(D)$ and between $\Lambda_\alpha \mathrm{Hol}(D)$ and $\Lambda_{\alpha+k}\mathrm{Hol}(D)$. Theorem 3 and duality arguments now yield that $u \to B(\varrho^k u)$ is an isomorphism between $\mathrm{Hol}_p^s(D)$ and $\mathrm{Hol}_p^{s+k}(D)$ for all s and p, $1 < p < \infty$. It can also be proved by use of Proposition 4 that $u \to B(\varrho^k u)$ is an isomorphism between $L^\infty \mathrm{Hol}(D, |\varrho|^s)$ and $\Lambda_{k-s}\mathrm{Hol}(D)$.

The fact that $Fu = B(\varrho^k T\bar{\partial}\Delta(\varrho u))$ maps continuously $\mathrm{Hol}_2^{s+1}(D)$ into $\mathrm{Hol}_2^{s+k+1/2}(D)$ follows from Kohn's estimates of the canonical solution of the $\bar{\partial}$-problem (see [5a]) and Proposition 3. It was proved by Henkin, Grauert and Lieb that there exists an operator T_1 solving the $\bar{\partial}$-problem which maps $\Lambda_{\alpha,(0,1)}(D)$ into $\Lambda_{\alpha+1/2}(D)$ (see for example [7] or [2]). We have $T = (I - B)T_1$. Since B maps $\Lambda_\alpha(D)$ into $\Lambda_\alpha(D)$ (see [2], [16] or [9]), Proposition 3 implies that F maps $\Lambda_{\alpha+1}\mathrm{Hol}(D)$ into $\Lambda_{\alpha+k+1/2}\mathrm{Hol}(D)$.

III. Remarks.

1. Proposition 4 and Theorem 3 remain valid if we replace $\mathrm{Hol}_p^s(D)$, $\Lambda_s\mathrm{Hol}(D)$, $L^\infty\mathrm{Hol}(D, |\varrho|^s)$, $\mathrm{Bl\,Hol}(D)$ by the spaces $PH_p^s(D)$, $\Lambda_s PH(D)$, $L^\infty PH(D, |\varrho|^s)$, $\mathrm{Bl}\,PH(D)$ of functions pluriharmonic on D (i.e. of functions f on D with $\partial\bar{\partial}f \equiv 0$) or by the spaces $\mathrm{Re\,Hol}_p^s(D)$, $\mathrm{Re}\,\Lambda_s\mathrm{Hol}(D)$, $\mathrm{Re}\,L^\infty\mathrm{Hol}(D, |\varrho|^s)$, $\mathrm{Re\,Bl\,Hol}(D)$ of the real parts of holomorphic functions, and the Bergman projection B by the orthogonal projection Q from $L^2(D)$ onto $L^2 PH(D)$ or by the real projection S_r from $L_r^2(D)$ onto $\mathrm{Re}\,L^2\mathrm{Hol}(D)$ (cf. [12] and [13]).

2. Theorem 1 implies that all results of [15] remain valid if we replace the unit ball in $\boldsymbol{R}^n$ by an arbitrary smooth bounded domain in $\boldsymbol{R}^n$. In particular, Theorem 3 of [15] which is an extension of Theorem 5.12 from [3] remains valid for such general domains.

The proof is the same as in the case of the unit ball except the lemma in the proof of part (b) of Theorem 3 in [15]. We now prove this lemma in the general case.

LEMMA. *Let* $\mathrm{Harm}^p(\partial D)$ *denote the Hardy space of harmonic functions on* D *with trace on* ∂D *belonging to* $L^p(\partial D)$, $1 < p \leqslant \infty$. *Then*:

(a) $\mathrm{Harm}^p(\partial D) \subset \mathrm{Harm}_p^{1/p}(D)$ *if* $\infty > p \geqslant 2$.

(b) $\mathrm{Harm}^p(\partial D) \supset \mathrm{Harm}_p^{1/p}(D)$ *if* $1 < p \leqslant 2$.

Proof. (a) It is well known that $\mathrm{Harm}^2(\partial D) = \mathrm{Harm}_2^{1/2}(D)$. We also have $\mathrm{Harm}^\infty(\partial D) \subset \mathrm{Bl\,Harm}(D)$. Theorem 2 implies that

$$\mathrm{Harm}^{2/\theta}(\partial D) \subset \mathrm{Harm}_{2/\theta}^{\theta/2}(D), \qquad 0 < \theta < 1.$$

(b) We can assume that the defining function ϱ of D is such that $|\nabla\varrho| \equiv 1$ on ∂D. In the proof of Proposition 2 (part (a)) we have already proved that

$$\int_{\partial D} u\bar{w}\,d\sigma = \int_D \Delta\left(\varrho u \frac{\varphi}{|\nabla\varrho|^2}\right)\bar{w}\,dV = \int_D P\left(\Delta\left(\varrho u \frac{\varphi}{|\nabla\varrho|^2}\right)\right)\bar{w}\,dV = \langle Hu, w\rangle_0.$$

We have

$$\|w\|_{\mathrm{Harm}\,p(\partial D)} = \sup_{\substack{u\in \mathrm{Harm}^q(\partial D)\\ \|u\|\leqslant 1}} \left|\int_{\partial D} u\bar{w}\,d\sigma\right| = \sup_{\substack{u\in \mathrm{Harm}^q(\partial D)\\ \|u\|\leqslant 1}} |\langle Hu, w\rangle_0| \leqslant c\,\|w\|_p^{1/p},$$

since $q = p/(p-1) \geqslant 2$, $\mathrm{Harm}^q(\partial D) \subset W_q^{1/q}(D)$, the mapping H maps continuously $\mathrm{Harm}_q^{1/q}(D)$ into $\mathrm{Harm}_q^{1/q-1}(D) = \mathrm{Harm}_q^{-1/p}(D)$ and

$$\|w\|_p^{1/p} = \sup_{\substack{v\in \mathrm{Harm}_q^{-1/p}(D)\\ \|v\|\leqslant 1}} |\langle v, w\rangle_1|$$

($\langle\ ,\ \rangle_1 = \langle\ ,\ \rangle_0$ on $L^2\mathrm{Harm}(D)$).

3. Proposition 1 yields immediately that $L^\infty\mathrm{Harm}(D, |\varrho|^s)$, $0 < s < 1$, represents the dual to $\mathring{L}^1\mathrm{Harm}(D, |\varrho|^{-s})$ via the pairing $\langle\ ,\ \rangle_0$. The space $\mathring{L}^1\mathrm{Harm}(D, |\varrho|^{-s})$ is the closure of $L^2\mathrm{Harm}(D)$ in $L^1(D, |\varrho|^{-s})$. Hence we get the following interpolation theorem "dual" to Theorem 2:

$$[L^1\mathrm{Harm}(D, |\varrho|^{-r}), \mathrm{Harm}_p^s(D)]_{[\theta]} = \mathrm{Harm}_{p/(\theta p+1-\theta)}^{\theta r+(1-\theta)s}(D)$$

for $-\infty < r < 1$, $-\infty < s < +\infty$, $1 < p < \infty$.

The same fact remains true if the spaces of harmonic functions are replaced by spaces of holomorphic or pluriharmonic functions on a smooth bounded strictly pseudoconvex domain D (see Theorem 3 and Remark 1).

Addendum. In our next paper *On duality and interpolation for spaces of polyharmonic functions* we shall prove that $L^1\mathrm{Harm}(D)$ represents the dual to $\mathrm{Bl}^0\mathrm{Harm}(D)$ which is the closure of $C^\infty(\bar{D}) \cap \mathrm{Harm}(D)$ in $\mathrm{Bl\,Harm}(D)$ via the pairing $\overline{\langle v, u\rangle_1}$, $v \in L^1\mathrm{Harm}(D)$, $u \in \mathrm{Bl}^0\mathrm{Harm}(D)$. If D is a strictly pseudoconvex domain in $\boldsymbol{C}^n$, then $L^1\mathrm{Hol}(D)$ represents the dual to $\mathrm{Bl}^0\mathrm{Hol}(D)$ which is the closure of $C^\infty(\bar{D}) \cap \mathrm{Hol}(D)$ in $\mathrm{Bl\,Hol}(D)$, via the same pairing. The space $\mathrm{Bl}^0\mathrm{Harm}(D)$ can be characterized as the subspace of $\mathrm{Bl\,Harm}(D)$ consisting of functions u for which $\varrho\,\mathrm{grad}\,u \to 0$ as $\varrho \to 0$. The same fact holds for $\mathrm{Bl}^0\mathrm{Hol}(D)$. In particular, if D is the unit disc in $\boldsymbol{C}$ then $\mathrm{Bl}^0\mathrm{Hol}(D)$ is equal to the classical Bloch class B_0. Straube's observation (see Remark after the statement of Corollary 1) yields that

$$\overline{\langle v, u\rangle_1} = \lim_{\varepsilon\to 0^+} \int_{D_\varepsilon} u\bar{v}, \qquad D_\varepsilon = \{x \in D\colon |\varrho(x)| > \varepsilon\}.$$

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. 12 (1959), 623–727.

[2] P. Ahern and R. Schneider, *Holomorphic Lipschitz functions in pseudoconvex domains*, Amer. J. Math. 101 (1979), 543–562.
[3] F. Beatrous and J. Burbea, *Sobolev spaces of holomorphic functions in the ball*, preprint 1985, Dissertationes Math., to appear.
[4] S. Bell, *A duality theorem for harmonic functions*, Michigan Math. J. 29 (1982), 123–128.
[5] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, 1976.
[5a] G. Folland and J. J. Kohn, *The Neumann Problem for the Cauchy–Riemann Complex*, Princeton Univ. Press, Princeton 1972.
[6] N. Kerzman and E. Stein, *The Szegö kernel in terms of Cauchy–Fantappié kernels*, Duke Math. J. 45 (1978), 197–223.
[7] S. Krantz, *Optimal Lipschitz and L^p regularity for the equation $\bar{\partial}u = f$ on strongly pseudoconvex domains*, Math. Ann. 219 (1976), 233–260.
[8] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow 1978 (in Russian).
[9] E. Ligocka, *The Hölder continuity of the Bergman projection and proper holomorphic mappings*, Studia Math. 80 (1984), 89–107.
[10] —, *The Sobolev spaces of harmonic functions*, ibid. 84 (1986), 79–87.
[11] —, *The Hölder duality for harmonic functions*, ibid. 84 (1986), 269–277.
[12] —, *On the orthogonal projections onto spaces of pluriharmonic functions and duality*, ibid. 84 (1986), 279–295.
[13] —, *The Bergman projection on harmonic functions*, ibid. 85 (1987), 229–246.
[14] —, *Estimates in Sobolev norms $\| \|_p^s$ for harmonic and holomorphic functions and interpolation between Sobolev and Hölder spaces of harmonic functions*, ibid. 86 (1987), 255–271
[15] —, *On the reproducing kernel for harmonic functions and the space of Bloch harmonic functions on the unit ball in R^n*, this volume, 23–32.
[16] D. Phong and E. Stein, *Estimates for the Bergman and Szegö projections on strongly pseudo-convex domains*, Duke Math. J. 44 (1977), 695–704.
[17] E. Straube, *Harmonic and analytic functions admitting a distribution boundary value*, Ann. Scuola Norm. Sup. Pisa (4) 11 (1984), 559–591.
[18] —, *Orthogonal projections onto subspaces of the harmonic Bergman space*, Pacific J. Math. 123 (1986), 465–476.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Śniadeckich 8, 00-950 Warszawa, Poland

Received March 3, 1986
Revised version October 1, 1986 (2146)

STUDIA MATHEMATICA, T. LXXXVII. (1987)

Nilpotent Lie groups and eigenfunction expansions of Schrödinger operators II *

by

ANDRZEJ HULANICKI (Wrocław) and JOE W. JENKINS (Albany, N.Y.)

Abstract. Let $\mathscr{L} = -d^2/dx^2 + |P(x)|$, where P is a polynomial of degree $d+1$. Following the general pattern of [9] and using new estimates proved in [3] the following theorem is proved.

THEOREM. *Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ be the eigenvalues corresponding to the orthonormal basis $\varphi_1, \varphi_2, \ldots$ of eigenfunctions of $\mathscr{L}$ in $L^2(\boldsymbol{R})$. Let $K \in C^\infty(\boldsymbol{R})$, with $K(0) = 1$, be such that for some $\gamma > 1$ and $R > 0$*

$$\sup_{\lambda > 0}(1+\lambda)^{n(s+1)}|K^{(j)}(\lambda)| \leqslant R^n (n!)^\gamma, \quad j \leqslant n,\ n = 1, 2, \ldots,$$

where $s = [(2+d)(5+d)/4]+1$. Then for every $f \in L^p(\boldsymbol{R})$, $1 \leqslant p < \infty$, we have

$$\lim_{t\to 0}\Big\|\sum_{n=1}^{\infty} K(t\lambda_n)(f, \varphi_n)\varphi_n - f\Big\|_{L^p} = 0.$$

In our previous paper [9] we used nilpotent Lie groups to obtain results on the summability of eigenfunction expansions of Schrödinger operators on $\boldsymbol{R}^n$ whose potentials were sums of squares of polynomials. In an attempt to prove similar results for operators with more general potentials we investigate here the operator

$$\mathscr{L} = -\frac{d^2}{dx^2} + |P(x)|,$$

where P is a polynomial of degree $d+1$, say.

We believe that most of our present results are valid also in higher dimensions but the technique used here is restricted to dimension one. Also our summability results are weaker than those for operators considered in [9]. An application of the methods of the present paper gives the following theorem.

THEOREM. *Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ be the eigenvalues corresponding to the orthonormal basis $\varphi_1, \varphi_2, \ldots$ of eigenfunctions of $\mathscr{L}$ in $L^2(\boldsymbol{R})$. Let $K \in C^\infty(\boldsymbol{R})$,*

* This research was founded in part by National Science Foundation grant DMS 8501518.