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Multipliers in complex Banach spaces and structure of the unit balls

by

KRZYSZTOF JAROSZ (Warszawa)

Abstract. Let X be a Banach space such that $\dim \operatorname{Mult}(X) = \infty$. We construct an into isometry from the space c_0 or c into X; we also prove that the sum of the images of such isometries is dense in X.

1. Introduction. Let X be a complex Banach space. We denote by $B(X)$ the closed unit ball in X and by $E(X)$ the set of all extreme points of $B(X)$. By a *multiplier* on X we mean any continuous linear map $S: X \to X$ such that there is a function $a_S: E(X^*) \to \boldsymbol{C}$ with

$$S^*(x^*) = a_S(x^*)x^* \quad \text{for all } x^* \text{ in } E(X^*).$$

Note that a_S is uniquely determined, bounded and can be extended to a weak* continuous function on $\Delta := \overline{E(X^*)} \setminus \{0\}$, where the closure is taken in the weak* topology. $\operatorname{Mult}(X)$ denotes the algebra of all multipliers on X. It is obvious that the map

$$\operatorname{Mult}(X) \ni S \mapsto a_S \in C(\Delta)$$

is an isometric algebra isomorphism from $\operatorname{Mult}(X)$ onto a closed subalgebra of $C(\Delta)$.

Multipliers have been investigated in different branches of mathematics ([1–3, 5–6]). The fundamental result in this field states that any Banach space can be considered, in a canonical way, as a module over $\operatorname{Mult}(X)$. If $\operatorname{Mult}(X)$ is finite-dimensional we have

(M) $\quad X = X_1 \oplus X_2 \oplus \ldots \oplus X_k \quad$ with $\|(x_1, \ldots, x_k)\| = \sup\{\|x_j\|: 1 \leqslant j \leqslant k\}$

where $k = \dim \operatorname{Mult}(X)$ and $\operatorname{Mult}(X_j) = \boldsymbol{C} \cdot \mathrm{Id}_{X_j}$ for $1 \leqslant j \leqslant k$. In [3] Behrends proved that if $\dim \operatorname{Mult}(X) = \infty$ then for any $\varepsilon > 0$ there is a linear map Φ_ε from c_0, the Banach space of all sequences convergent to zero, into X such that $\|a\| \leqslant \|\Phi_\varepsilon(a)\| \leqslant (1+\varepsilon)\|a\|$ for any $a \in c_0$. In this paper we prove that there is always an isometric embedding. This result gives an affirmative answer to the problem whether $\operatorname{Mult}(X) = \boldsymbol{C} \cdot \mathrm{Id}_X$ for any strictly convex Banach space. To give more information about the structure of the unit sphere in a Banach space X with $\operatorname{Mult}(X)$ infinite-dimensional we prove that for any $x \in B(X)$ there are, in $B(X)$, isometric copies of $B(c_0)$ or of $B(c)$ arbitrary close to x; by c we mean the Banach space of all convergent sequences.

2. The results.

THEOREM. *Let X be a Banach space with $\dim \operatorname{Mult}(X) = \infty$. Then for any $x_0 \in X$ with $\|x_0\| = 1$ and any $\varepsilon_0 > 0$ there is an into isometry Φ from c_0 or c into X such that $\|\Phi(y) - x_0\| < \varepsilon_0$ for some y of norm one from the domain of Φ.*

COROLLARY 1. *Let X be a Banach space and assume that there is an open subset U of $\partial B(X)$, the boundary of the unit ball in X, such that U does not intersect any segment of length two contained in $\partial B(X)$. Then* $\operatorname{Mult}(X) = \boldsymbol{C} \cdot \mathrm{Id}_X$.

Proof. By our theorem we get $\dim \operatorname{Mult}(X) < \infty$, and then from the remark (M) we get $\dim \operatorname{Mult}(X) = 1$.

COROLLARY 2. *Let X be a Banach space and assume that $B(X)$ contains no segment of length two. Then* $\operatorname{Mult}(X) = \boldsymbol{C} \cdot \mathrm{Id}_X$.

COROLLARY 3. *For any strictly convex Banach space X we have* $\operatorname{Mult}(X) = \boldsymbol{C} \cdot \mathrm{Id}_X$.

Remark 1. Note that the theorem cannot be generalized to state that "... there is an into isometry from c_0 into X such that ...". To get a simple example put $X = c$ and $x_0 = \mathbf{1}$.

Remark 2. Neither can the theorem be strengthened to "... $\Phi(y) = x_0$ for some y from the domain of Φ". We give two examples of different nature. The first one is taken from [7].

(a) $X =$ disc algebra, i.e. the algebra of all continuous functions defined on the closed unit disc D on the complex plane which are analytic in $\operatorname{int} D$, and $x_0 = 1$.

(b) Let $f\colon [0, 1] \to \boldsymbol{R}$ be a C^∞ function such that

$$f(0) = 1, \quad f(1) = 0, \quad f^{(k)}(0) = 0 \text{ for } k = 1, 2, \ldots,$$
$$f \text{ is strictly decreasing.}$$

Let X' be the disc algebra with norm given by

$$|||g||| = \sup\{f(|z|)\,|g(z)|\colon z \in D\}$$

and let X be the completion of $(X', |||\cdot|||)$. X can be represented as a subspace of $Y = \{h \in C(D)\colon h|_{\partial D} \equiv 0\}$. It is evident that $E(X^*) \subset D$ and $\operatorname{Mult}(X) = H^\infty(D)$. We prove that there is no into isometry from c_0 nor from c into X such that the image of the unit ball contains $\mathbf{1}$. To this end assume that Φ is such an isometry and let $a \in B(c_0)$ $(a \in B(c))$ be such that $\Phi(a) = \mathbf{1} \in X'$. Let e_n^* be the usual Schauder basis of the space $c^* = l^1$ and put

$$e_\infty^*\colon c \to \boldsymbol{C}, \quad e_\infty^*((a_1, a_2, \ldots)) = \lim a_n.$$

We consider two possibilities:

(i) There is exactly one n in $\boldsymbol{N} \cup \{\infty\}$ such that $|e_n^*(a)| = 1$.

(ii) There are $n \neq m$ in $\boldsymbol{N} \cup \{\infty\}$ such that $|e_n^*(a)| = |e_m^*(a)| = 1$.

Assume first that (i) holds and let $b \in c_0$, $b \neq 0$, be such that

$$\|a + \lambda b\| \leqslant 1 \quad \text{for all } \lambda \text{ in } \boldsymbol{C} \text{ with } |\lambda| = 1.$$

Any element of X can be viewed as an analytic, possibly unbounded function defined on $\operatorname{int} D$. Hence there is $\lambda_0 \in \boldsymbol{C}$, $|\lambda_0| = 1$, such that the first nonzero derivative at the point $0 \in D$ of the function $G_0 = \lambda_0 \Phi(b)$ is positive. We have also

$$g_0(z) = z^k(\alpha_0 + zh(z)) \quad \text{for } z \text{ in } D,$$

where k is a nonnegative integer, $\alpha_0 > 0$ and h is an analytic function on $\operatorname{int} D$. By our assumption we have

$$\|a + \lambda_0 b\| \leqslant 1, \quad \Phi(a + \lambda_0 b) = \mathbf{1} + g_0.$$

To get a contradiction we show that $|||1 + g_0||| > 1$. We have

$$\begin{aligned} |||1 + g_0||| &= \sup\{f(|z|)\,|1 + g_0(z)|\colon z \in D\} \\ &= \sup\{|(1 - [1 - f(|z|)])|\,1 + z^k(\alpha_0 + zh(z))|\colon z \in D\} \\ &\geqslant \sup\{|1 + \alpha_0 z^k| - \varphi(z)\colon z \in D\} \end{aligned}$$

where

$$\varphi(z) = |z^{k+1} h(z)| + (1 - f(|z|))(1 + |\alpha_0 z^k|) \quad \text{for } z \in D.$$

By our assumption about f there is $C > 0$ such that

$$\varphi(z) \leqslant C|z^{k+1}| \quad \text{if } |z| \leqslant \tfrac{1}{2}.$$

Hence

$$|||1 + g_0||| \geqslant \sup\{|1 + \alpha_0 z^k| - C|z^{k+1}|\colon |z| \leqslant \tfrac{1}{2}\} > 1.$$

Assume now that (ii) holds and let $n \neq m \in \boldsymbol{N} \cup \{\infty\}$ be such that $|e_n^*(a)| = |e_m^*(a)| = 1$. Let F_n, F_m be the norm one functionals on X, given by the Hahn–Banach theorem, such that

$$e_m^*(b) = F_m(\Phi(b)), \quad e_n^*(b) = F_n(\Phi(b))$$

for all b from the domain of Φ. We have

$$|F_n(1)| = |e_n^*(a)| = 1 = |e_m^*(a)| = |F_m(1)|,$$

and on the other hand $Y \supset X \ni g \mapsto F(g) = g(0) \in \boldsymbol{C}$ is the unique norm one functional on X such that $F(\mathbf{1}) = \mathbf{1}(0) = 1$; hence F_n and F_m are proportional which is absurd.

Remark 3. As was mentioned in the introduction, any Banach space X is a module over some function algebra, and if X is actually a function algebra, we get a trivial representation, i.e. $\mathrm{Mult}(X) = X$. In this situation it can be deduced from the Theorem of [7] that our theorem can be extended as follows.

For any function algebra X

1) *$X = C(S)$ for some compact set S, or*

2) *For any compact metric space K there is an isometric embedding of $C(K)$ into X.*

The above result does not hold in general. That is, there is a Banach space X such that $\mathrm{Mult}(X)$ is a function algebra not of the form $C(S)$ but X^* is separable, so X contains no $C(K)$ space with K uncountable. An example of such a space X is the space from Remark 2b. In order to prove that X^* is separable it can be shown that for any countable dense subset Λ of $\mathrm{int}\, D$ the set of all linear combinations of evaluations at the points from Λ is norm dense in X^*.

Finally, we note that we only consider the complex case since in the real case all the results presented here are well known (and easy).

3. Proof of the theorem. Before proving the theorem we need some definitions and notation.

For a Hausdorff space S by a *function algebra* on S we mean any algebra of bounded functions defined on S, which contains the unit and which is complete in the usual sup norm topology. For any bounded function f defined on S and any subset S' of S we define

$$\|f\|_{S'} = \sup\{|f(s)|:\ s \in S'\}.$$

For any bounded subset G of the complex plane C we denote by $\hat{G}$ the *polynomially convex hull* of G, i.e.

$$\hat{G} = \{z \in C:\ |p(z)| \leqslant \|p\|_G \text{ for any polynomial } p\}.$$

For any such G we denote by $A(G)$ the closure in the sup norm on G of the algebra of all polynomials. We obviously have $A(G) \cong A(\hat{G})$.

By $\mathrm{Ch}\, A$ and ∂A we denote the Choquet and Shilov boundaries, respectively, of a function algebra A.

For a complex number w and a positive number r we put

$$D(w, r) = \{z \in C:\ |z - w| \leqslant r\}$$

and we write D in place of $D(0, 1)$.

For any $w \neq z \in \mathrm{int}\, D$ we define

$$A_{w,z} = \{f \in A(D):\ f(w) = f'(w) = f(1) = 0 = f'(z) = f''(z) \text{ and } f(z) = 1\}.$$

For any $w \in \mathrm{int}\, D$ we denote by B_w the corresponding Blaschke factor, i.e.

$$B_w(z) = (w - z)/(1 - z\bar{w}) \quad \text{for } z \text{ in } D.$$

Our proof is rather technical so we divide it into a number of steps. We will use the following propositions; the first three are well known.

Proposition 1. *Let A be a function algebra on a Hausdorff space S. If $\dim A = \infty$ then there is an f in A such that the set $f(S)$ is infinite.*

Proposition 2. *Let G be an open, bounded, connected subset of the complex plane and assume that ∂G, the boundary of G, is homeomorphic to a circle. Then there is a homeomorphism g from $\bar{G}$ onto D such that $g|_G$ is analytic.*

Corollary. *Let G be a bounded infinite subset of the complex plane. Then there is a homeomorphism f in $A(\bar{G})$ which maps G onto a set G' such that $\bar{G}' \subset \mathrm{int}\, D \cup \{1\}$ and 1 is a cluster point of G'.*

Proof. By an appropriate translation of the complex plane we can assume, without loss of generality, that 0 is a cluster point of G and that there are no cluster points of G in the set $C_+ = \{z \in C:\ \mathrm{Re}\, z > 0\}$. The set K of all isolated points of $\bar{G}$ is at most countable so there is a half-line L such that $L \cap K = \{0\}$. By moving G again we can assume that $L = \{z \in C:\ \mathrm{Re}\, z \geqslant 0,\ \mathrm{Im}\, z = 0\}$. We define $\chi:\ R \to R$ by $\chi(t) = \mathrm{dist}((t, 0), G)$. Then χ is continuous and

$$\bar{G} \cap \{z \in C:\ |\mathrm{Im}\, z| \leqslant \chi(\mathrm{Re}\, z),\ \mathrm{Re}\, z \geqslant 0\} = \{0\}.$$

Hence using χ, $-\chi$ and some arc we can define a Jordan curve J such that $0 \in J$ and $\bar{G} \setminus \{0\}$ is contained in S, the bounded component of $C \setminus J$. By Proposition 2 there is a homeomorphism g, in $A(S)$, from S onto D and we can assume that $g(0) = 1$. To end the proof we put $f = g|_{\bar{G}}$.

Proposition 3. *Let A be a function algebra on a Hausdorff space S, let $f \in A$ and let $g \in A(f(S))$. Then $g \circ f \in A$.*

Proposition 4. *For any $w_0 \in \mathrm{int}\, D$ and any sequence $(w_n)_{n=1}^{\infty}$ in $\mathrm{int}\, D$ with $\lim w_n = 1$ there are a sequence $f_n \in A_{w_n, w_0}$ and a sequence $g_n \in A_{w_0, w_n}$ such that $\lim \|f_n\| = \lim \|g_n\| = 1$ and $f_n \to 1$ and $g_n \to 0$ uniformly on compact subsets of $D \setminus \{1\}$.*

Proof. We need the following statement, which is an immediate consequence of Lemma 1 of [4]:

For any $\varepsilon > 0$ and any open neighbourhood U of 1 in D there is a p in $A(D)$ such that

$$\|p\| = 1+\varepsilon, \quad p(1) = 1, \quad |p(w)| \leqslant \varepsilon \quad \text{for } w \in D \setminus U,$$

$$\|p - \mathrm{Re}^+ p\| \leqslant \varepsilon$$

where for a complex number w we put $\mathrm{Re}^+ w = \max(0, \mathrm{Re}\, w)$.

Let w_0 be as in our proposition and fix any open neighbourhood U of 1. Without loss of generality we can assume, in the above statement, that $w_0 \notin U$ and then, by putting $(p - p(w_0))^3$ in place of p, we can also assume that

$$p(w_0) = p'(w_0) = p''(w_0) = 0.$$

Fix now $n \in N$ such that $\mathrm{Re}\, p(w_n) \geqslant 1 - \varepsilon$. By the same argument as above there is a q in $A(D)$ such that

$$\|q\| \leqslant 1+\varepsilon, \quad q(1) = 1, \quad \|q - \mathrm{Re}^+ q\| \leqslant \varepsilon,$$

$$|q(w)| \leqslant \varepsilon \quad \text{for all } w \in D \text{ such that } \mathrm{Re}\, p(w) \leqslant \mathrm{Re}\, p(w_n),$$

$$q(w_0) = q'(w_0) = q''(w_0) = 0.$$

Put

$$\tilde{f}_n = (p-q)/(p(w_n) - q(w_n)), \quad f_n = 1 - (1 - \tilde{f}_n)^3.$$

By a direct computation it is easy to verify that $f_n \in A_{w_n, w_0}$ and $\|f_n\| \leqslant 1 + 100\varepsilon$.

The construction of a sequence $(g_n)_{n=1}^{\infty}$ is analogous.

PROPOSITION 5. *Let A be a function algebra on a compact Hausdorff space S, let $S' \subset S$ be a peak set for A and let p be a lower semicontinuous and strictly positive function defined on S with $p|_{S'} \equiv 1$. Then there is an f in A such that $f(s) = 1$ for $s \in S'$ and $|f(s)| \leqslant p(s)$ for $s \in S \setminus S'$.*

Proof. The above proposition is very well known in the case when p is continuous [9, p. 61].

Let p be as in our proposition and let $q\colon S \to \mathbf{R}$ be defined by

$$q(s) = \begin{cases} 1 & \text{for } s \in S', \\ \inf\{p(s)\colon s \in S\} & \text{for } s \in S \setminus S'. \end{cases}$$

We have $q \leqslant p$ and q is upper semicontinuous, so by the theorem of Tong [10] there is a continuous function p' defined on S such that

$$0 < q \leqslant p' \leqslant p.$$

The function p' is continuous, strictly positive and $p'|_{S'} = 1$, hence there is an f in A such that $f|_{S'} = 1$ and $|f(s)| \leqslant p'(s) \leqslant p(s)$ for $s \in S \setminus S'$.

For the proof of our theorem we also need the following lemma.

LEMMA 1. *Let f be a real, nonnegative function defined on a set G contained in the complex plane. Assume that $1 \in \bar{G} \subset \operatorname{int} D \cup \{1\}$, $\|f\|_G = 1$ and $f(w) \to 0$ as $w \to 1$. Then for any $\varepsilon > 0$ there are $\tilde{f} \in A(D)$, $z_0 \in \bar{G}$ and $\delta > 0$ such that*

(i) $$\|f - f\tilde{f}\|_G < \varepsilon, \quad \|f\tilde{f}\| = 1,$$

(ii) $$|f\tilde{f}(w)| + \delta |B_{z_0}^2(w)| \leqslant 1 \quad \text{for } w \in G.$$

Proof. Assume without loss of generality that $\varepsilon < 0.1$. Put

$$t_0 = \inf\{t > 0\colon t(1 - \varepsilon \mathrm{Re}\, w) > f(w) \text{ for all } w \in G\},$$

and let $z_0 \in \bar{G}$ be such that

$$t_0(1 - \varepsilon \mathrm{Re}\, z_0) = \limsup_{w \to z_0} f(w).$$

Note that $1 - 2\varepsilon < t_0 < 1 + 2\varepsilon$ and that by our assumptions $z_0 \neq 1$. Let

$$h(w) = k(w - a_0)^2 \quad \text{for } w \in \mathbf{C}$$

where $k > 0$ and $a_0 \in \mathbf{C}$ are such that the plane in $\mathbf{C} \times \mathbf{R}$ given by $w \mapsto t_0(1 - \varepsilon \mathrm{Re}\, w)$ is tangent to the surface $w \mapsto |h(w)|$ at the point $(z_0, 1 - \mathrm{Re}\, z_0)$. By a direct computation we get

$$k = \varepsilon^2 t_0 (1 - \mathrm{Re}\, z_0)^{-1}/4,$$

$$a_0 = \mathrm{Re}\, z_0 + 2(1 - \mathrm{Re}\, z_0)/\varepsilon + i \operatorname{Im} z_0.$$

Hence we have

$$|1 - h(w)| < 4\varepsilon \quad \text{for any } w \text{ in } D.$$

Put

$$\varphi(w) = k|w - a_0|^2 - t_0(1 - \varepsilon \mathrm{Re}\, w) \quad \text{for } w \in \mathbf{C}.$$

The map φ defines a rank two surface in $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ which is tangent to the plane $\mathbf{C} \times \{0\}$ at the point z_0, so for any sufficiently small δ' we have

$$\varphi(w) \leqslant 2\delta' |w - z_0|^2 \quad \text{for } w \in D.$$

Hence

$$\frac{t_0(1 - \varepsilon \mathrm{Re}\, w)}{k|w - a_0|^2} + \delta' |w - z_0|^2 \leqslant 1 \quad \text{for } w \in D.$$

So to end the proof of the lemma it is sufficient to put $\tilde{f} = 1/h$ and to take $\delta > 0$ such that

$$\delta |B_{z_0}(w)|^2 \leqslant \delta' |w - z_0|^2 \quad \text{for any } w \text{ in } D.$$

Now to prove our theorem, fix $\varepsilon_0 > 0$, $x_0 \in \partial B(X)$ and assume $\dim \mathrm{Mult}(X) = \infty$. By Proposition 1 there is a T in $\mathrm{Mult}(X)$ such that the set $G = a_T(\Delta)$ is infinite. For any x in X and any w in G we define

$$\hat{x}(w) = \sup\{|x^*(x)|:\ x^* \in \Delta,\ a_T(x^*) = w\}$$

and we extend $\hat{x}$ to $\bar{G}$ by

$$\hat{x}(w_0) = \limsup_{w \in G, w \to w_0} \hat{x}(w) \quad \text{for } w_0 \in \bar{G} \setminus G.$$

Note that $\hat{x}$ is an upper semicontinuous function on $\bar{G}$ and that $\|x\| = \|\hat{x}\|_G$. $\mathrm{Mult}(X)$ is isomorphic to a function algebra on Δ so, by Proposition 3, we have $f(T) \in \mathrm{Mult}(X)$ whenever $f \in A(G)$. Moreover, for any such f and for any x^* in $E(X^*)$ we have

$$x^*\big(f(T)(x)\big) = f(a_T)(x^*) \cdot x^*(x).$$

Hence, for any $f \in A(G)$, we have

$$(*) \qquad \big(f(T)(x)\big)\hat{}\,(w) = |f|\,\hat{x}(w) \quad \text{for all } w \text{ in } G.$$

The above observation $(*)$ will play a fundamental role in the whole proof.

The idea of the proof is the following:

Using "peaking" functions of the algebra $A(G)$ we construct a sequence $x_1, x_2, \ldots$ of norm one elements of X and a sequence $w_1, w_2, \ldots$ of elements of $\bar{G}$ such that $\hat{x}_n(w_n) = 1$ and the supports of $\hat{x}_n$ are "almost disjoint", i.e. the sets $\{z \in G:\ \hat{x}_n(z) > \varepsilon\}$ are pairwise disjoint. Then, using $(*)$, by the same method as in Lemma 1 we perturb x_n slightly to obtain a sequence $x'_1, x'_2, \ldots$ of norm one elements of X also with "almost disjoint" supports and such that we can estimate their behaviour near their peak points $w'_n \approx w_n$:

$$(!) \qquad \hat{x}'_n(w) \leqslant 1 - \delta\,|B^2_{w'_n}(w)| \quad \text{for all } w \text{ in } G.$$

Next, by Proposition 4, we find, for each $n \in N$, a function g_n from $A(G)$ such that $\|g_n\|$ is very close to 1 and

$$g_n(w'_n) = 1,$$

$$g_n(w'_m) = 0 \quad \text{for } n \neq m,$$

$$g'_n(w'_m) = g''_n(w'_m) = 0 \quad \text{for all } n, m \text{ in } N.$$

We put $y_n = \big(g_n(T)\big)(x'_n)$. By the Schwarz Lemma, for all w in G we have

$$|g_n(w)| \leqslant (1+\varepsilon)|B^2_{w'_m}(w)| \quad \text{for } n \neq m,$$

$$|g_n(w) - 1| \leqslant (1+\varepsilon)|B^2_{w'_n}(w)|.$$

Hence for any w in G we get

$$(!!) \qquad \hat{y}_n(w) \leqslant 1 - \delta'\,|B^2_{w'_n}(w)|, \quad \hat{y}_n(w) \leqslant \delta'\,|B^2_{w'_m}(w)| \quad \text{for } n \neq m.$$

Finally, using the "almost disjointness" of the supports of $\hat{y}_n$ and (!) and (!!) we prove that

$$\sum_{n=1}^{\infty} \hat{y}_n(w) \leqslant 1 \quad \text{for any } w \in G.$$

The above inequality, together with $\|y_n\| = 1$, is equivalent to the statement that

$$c_0 \ni (a_1, a_2, \ldots) \mapsto \sum_{n=1}^{\infty} a_n y_n$$

is an isometric embedding of c_0 into X and will end the proof.

We divide the proof into two parts according to the following conditions:

A. There is a cluster point a_0 of $\partial\hat{G}$ such that

$$\lim_{w \to a_0} \hat{x}_0(w) = 0.$$

B. For any cluster point a of $\partial\hat{G}$ we have

$$\limsup_{w \to a} \hat{x}_0(w) > 0.$$

Part A. By the corollary from Proposition 2 and by Proposition 3, composing T, at the very beginning, with an appropriate analytic map we can assume that

$\bar{G} \subset \mathrm{int}\, D \cup \{1\}$,

1 is a cluster point of $\partial\hat{G}$,

$\hat{x}(w) \to 0$ as $w \to 1$.

Let $(w_n)_{n=1}^{\infty}$ be a sequence of complex numbers and let $(r_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that

(1) $\quad w_n \in \partial\hat{G} \cap \mathrm{int}\, D \quad$ for all n in N,

(2) $\quad \lim w_n = 1$,

(3) $\quad D(w_n, r_n) \cap D(w_m, r_m) = \emptyset \quad$ for all $n \neq m$.

To simplify the notation we will write D_n in place of $D(w_n, r_n)$.

Put $f = \hat{x}_0$, $\varepsilon = \varepsilon_0/2$ and let $\tilde{f}$, δ and $z_0 \in \bar{G}$ be as in Lemma 1.

By Proposition 4, taking an appropriate subsequence of $(w_n)_{n=1}^{\infty}$ we can assume without loss of generality that there are f_n in A_{w_n, z_0} and $f_{n,m}$ in A_{w_n, w_m} such that

$$(4) \qquad \Big\|\prod_{n=1}^{\infty} f_n\Big\|_G \leqslant 1 + \delta/4, \quad \Big|\prod_{n=1}^{\infty} f_n(w) - 1\Big| < \varepsilon \quad \text{if } \hat{x}_0(w) \geqslant \varepsilon,$$

(5) $\|\prod_{n<m}^{\infty} f_{n,m}\| < 2,$

(6) $\inf\{|B^2_{z_0}(w)|:\ w \in \bigcup_{n=1}^{\infty} D_n\} > 1/2,$

(7) $|\tilde{f}|\,\hat{x}_0(w) < 0.01\delta \quad \text{for } w \in \bigcup_{n=1}^{\infty} D_n.$

Taking r_n smaller if necessary, we can also assume that

(8) $$\sup\{|B^2_{w_n}(w)|:\ z \in D_n\} = \beta\delta \cdot 0.01/2^n \quad \text{for all } n \text{ in } \boldsymbol{N},$$

where β is an absolute constant which we will define later on.

We will now define an isometric embedding Ψ of c_0 into X in two steps. In the first one we define a sequence $(x_n)_{n=0}^{\infty}$ of elements of X such that

$$c_0 \ni (a_0, a_1, a_2, \ldots) \overset{\Psi}{\mapsto} \sum_{j=0}^{\infty} a_j x_j \in X$$

is an isomorphic embedding of c_0 into X with $\|\Psi\|\,\|\Psi^{-1}\|$ close to one, and in the second step, using the functions f_n, $f_{n,m}$ and $\tilde{f}$ we slightly modify our sequence $(x_n)_{n=1}^{\infty}$ to get a sequence $(y_n)_{n=1}^{\infty}$ in X which defines an isometric embedding of c_0 into X.

We now define by induction a sequence $(x_n)_{n=1}^{\infty}$ of elements of X and a sequence $(z_n)_{n=1}^{\infty}$ of elements of G such that

(9) $\|x_n\| = 1 = \hat{x}_n(z_n) \quad \text{for } n \in \boldsymbol{N},$

(10) $\hat{x}_n(w) + \frac{1}{10}|B^2_{z_n}(w)| \leqslant 1 \quad \text{for } n \in \boldsymbol{N},\ w \in G,$

(11) $\hat{x}_n(w) \leqslant 0.01\delta/2^{n+2} \quad \text{for } w \in G \setminus D_n.$

Assume we have defined $x_1, \ldots, x_{n-1}$ and $z_1, \ldots, z_{n-1}$ (if $n = 1$ there is no assumption). Put $G_n = B_{w_n}(G)$ and let $p \in A(G_n) = A(\hat{G}_n)$ be such that

(12) $p(0) = 1 = \|p\|_{G_n}, \quad |p(w)| \leqslant 0.01\delta/2^{n+3} \quad \text{for } w \in G_n \setminus B_{w_n}(D_n).$

Such a p exists since $0 = B_{w_n}(w_n)$, $w_n \in \partial\hat{G}$ and $\partial\hat{G}_n = B_{w_n}(\partial\hat{G})$, and moreover the Choquet boundary of $A(G_n)$ is equal to the topological boundary of $\hat{G}_n$. By Proposition 5 we can also assume that

$$|p(w)| \leqslant |w-1|^{-3} \quad \text{for } w \text{ in } B_{w_n}(D_n).$$

Hence we can put the function $w \mapsto p(w)(1-w)^3$ in place of p to get a function in $A(G_n)$ such that (12) is still satisfied and moreover we have

(13) $$|p(w)| \leqslant 1 - \operatorname{Re} w \quad \text{for } w \in G_n \setminus D(0, 1/2).$$

Let $x_0^* \in E(X^*)$, $x' \in B(X)$ be such that

$$|p \circ B_{w_n}(a_T(x_0^*))| \geqslant 0.99, \qquad |x_0^*(x')| \geqslant 0.99.$$

Put

$$y = (p \circ B_{w_n}(T))(x').$$

We have

$$\hat{y} = |p \circ B_{w_n}|\,\hat{x}', \qquad 1 \geqslant \|y\| \geqslant \hat{y}(a_T(x_0^*)) \geqslant 0.98.$$

We have $B_{w_n} \circ B_{w_n} = \mathrm{Id}_D$ and we can define $g\colon G_n \to \boldsymbol{R}$ by

$$g = \hat{y} \circ B_{w_n} = |p|\,\hat{x}' \circ B_{w_n}.$$

Put

$$t_0 = \sup\{t \geqslant 0:\ 1 - \operatorname{Re} w \geqslant t g(w) \text{ for all } w \text{ in } G_n\}.$$

By (13) we have $t_0 < \infty$. Put $g_0 = t_0 g$ and let $w_0 \in G_n$ be such that

$$1 - \operatorname{Re} w_0 = g_0(w_0);$$

such a w_0 exists since g_0 is upper semicontinuous. From (12) and (13) we have

(14) $$w_0 \in B_{w_n}(D_n), \qquad 0.9 \leqslant t_0 \leqslant 1.1.$$

Put

$$h(w) = \frac{(w + \bar{w}_0 - 2)^2}{4(1 - \operatorname{Re} w_0)} \quad \text{for } w \in \boldsymbol{C}.$$

Note that the plane in $\boldsymbol{C} \times \boldsymbol{R}$ given by $w \mapsto 1 - \operatorname{Re} w$ is tangent to the surface $w \mapsto |h(w)|$ at the point $(w_0, 1 - \operatorname{Re} w_0)$. We put

$$x_n = \left(\frac{1}{h} \circ B_{w_n}(T)\right)(t_0 y);$$

we have

(15) $$\hat{x}_n = \left|\frac{1}{h} \circ B_{w_n}\right| t_0 \hat{y} = \left|\frac{p}{h}\right| \circ B_{w_n} \cdot t_0 \hat{x}',$$

(16) $$\hat{x}_n \circ B_{w_n} = g_0/|h|.$$

Hence we get

$$\|x_n\| = \|\hat{x}_n\|_G = \|\hat{x}_n \circ B_{w_n}\|_{G_n} = \sup \frac{g_0}{|h|}(w)$$

$$\leqslant \sup \frac{1 - \operatorname{Re} w}{|h(w)|} = \frac{1 - \operatorname{Re} w_0}{|h(w_0)|} = 1$$

and

$$\hat{x}_n(B_{w_n}(w_0)) = 1$$

so we can put $z_n = B_{w_n}(w_0)$ and (9) is fulfilled. Inequality (11) is a consequence of (15), (12) and (14). To check (10) it is sufficient, by (15), to show that

$$\frac{g_0(w)}{|h(w)|} + \tfrac{1}{10}|B_{w_0}^2(w)| \leqslant 1 \quad \text{for } w \in G_n, \tag{17}$$

and since by the definition of t_0 and g_0 we have $g_0(w) \leqslant 1 - \operatorname{Re} w$ for any w in G_n, it is sufficient to show that

$$\frac{4(1-\operatorname{Re} w)(1-\operatorname{Re} w_0)}{|w+\bar{w}_0-2|^2} + \tfrac{1}{10}|B_{w_0}^2(w)| \leqslant 1 \quad \text{for } w \in D. \tag{18}$$

Note that for $w_0 = 0$ we have

$$\frac{4(1-\operatorname{Re} w)}{|w-2|^2} + \tfrac{1}{9}|w|^2 \leqslant 1 \quad \text{for } w \in D.$$

By a direct computation it is easy to deduce from the above inequality that there is a constant β' such that (18) is satisfied whenever $|w_0| \leqslant \beta'$; on the other hand, from (8) and (14) we have $|w_0| < \beta^2$ so to get (18) it is sufficient to define β to be equal to β'.

Now we slightly modify the sequence $(x_n)_{n=0}^{\infty}$, which satisfies (9)–(11), and we get a sequence $(y_n)_{n=0}^{\infty}$ in X which defines an isometric embedding of c_0 into X. To this end we put

$$f_0 = \prod_{n=1}^{\infty} f_n, \quad g_n = \prod_{j=0, j\neq n}^{\infty} f_{j,n},$$

$$y_0 = f_0 \tilde{f}(T)(x_0), \quad y_n = g_n(T)(x_n), \quad n = 1, 2, \ldots$$

We have

$$\hat{y}_0 = |f_0 \tilde{f}|\,\hat{x}_0, \quad \hat{y}_n = |g_n|\,\hat{x}_n.$$

By (4) and Lemma 1 we have

$$\|y_0 - x_0\| = \|(y_0-x_0)\hat{\ }\|_G = \|(f_0 \tilde{f} - 1)\,\hat{x}_0\|_G \leqslant \varepsilon_0.$$

So to end this part of the proof we have to show that the map $\Phi\colon c_0 \to X$ defined by

$$\Phi((a_0, a_1, \ldots)) = \sum_{j=0}^{\infty} a_j y_j \quad \text{for } (a_0, a_1, \ldots) \in c_0$$

is a well defined into isometry. We have

$$\|y_n\| \geqslant \hat{y}_n(z_n) = 1 \quad \text{for } n = 0, 1, 2, \ldots,$$

so we only have to show that

$$\sum_{n=0}^{\infty} \hat{y}_n(w) \leqslant 1 \quad \text{for any } w \text{ in } G. \tag{19}$$

From the Schwarz Lemma and by (5), for any $n \neq m$, we have

$$|g_n(w)| \leqslant 2|B_{z_m}^2(w)| \quad \text{for } w \text{ in } D \tag{20}$$

and by (8) we get

$$|g_n(w) - 1| \leqslant 3|B_{z_n}^3(w)| \leqslant \frac{0.03}{2^n}\,\delta\,|B_{z_n}^2(w)| \quad \text{for } w \in D_n. \tag{21}$$

From (4) and (6) we also get

$$|f_0(w)| \leqslant 1 + \tfrac{1}{2}\delta|B_{z_0}^2(w)| \quad \text{for } w \in G \setminus \bigcup_{n=1}^{\infty} D_n, \tag{22}$$

$$|f_0(w)| \leqslant 2|B_{z_n}^2(w)| \quad \text{for } w \in D. \tag{23}$$

Let w be any point of $G \setminus \bigcup_{n=1}^{\infty} D_n$. By (11), (20), (22) and Lemma 1 we get

$$\begin{aligned}\sum_{n=0}^{\infty} \hat{y}_n(w) &= \hat{y}_0(w) + \sum_{n=1}^{\infty} |g_n(w)|\,\hat{x}_n(w) \leqslant |f_0|\,|\tilde{f}|\,\hat{x}_0 + 2\sum_{n=1}^{\infty} |B_{z_0}^2(w)|\frac{0.01\delta}{2^{n+2}} \\ &\leqslant |f_0|\,(1-\delta|B_{z_0}^2(w)|) + 0.01\delta|B_{z_0}^2(w)| \\ &\leqslant (1+\tfrac{1}{2}\delta|B_{z_0}^2(w)|)(1-\delta|B_{z_0}^2(w)|) + 0.01\delta|B_{z_0}^2(w)| \leqslant 1.\end{aligned}$$

Assume now that $w \in D_k$; successively by (23), (7), (20), (11) and (21) we get

$$\begin{aligned}\sum_{n=0}^{\infty} \hat{y}_n(w) &= |f_0|\,|\tilde{f}|\,\hat{x}_0 + \sum_{n=1}^{\infty} |g_n(w)|\,\hat{x}_n(w) \\ &\leqslant 2|B_{z_k}^2(w)|\cdot 0.01\delta + 2\sum_{n=1, n\neq k}^{\infty} |B_{z_k}^2(w)|\frac{0.01\delta}{2^{n+2}} + |g_k(w)|\,\hat{x}_k(w) \\ &\leqslant 0.03\delta|B_{z_k}^2(w)| + (1+0.03\delta|B_{z_k}^2(w)|)(1-0.1|B_{z_k}^2(w)|) \leqslant 1.\end{aligned}$$

Part B. For this part of the proof we need the following two auxiliary results. The first one is an immediate consequence of the Michael–Pełczyński theorem [7, § 4].

THEOREM (Michael–Pełczyński). *Let G be a compact subset of the unit disc D and let $(z_j)_{j=0}^{\infty}$ be a sequence of distinct points from $\partial\hat{G}$ such that $z_j \to z_0$ as $j \to \infty$. Then, for any $\varepsilon > 0$, there is a sequence $(f_j)_{j=1}^{\infty}$ in $A(G)$ such that*

(24)
$$\|f_j\| = 1 = f_j(z_j) \quad \textit{for all } j \textit{ in } N,$$
$$\Big|\sum_{j=1}^{\infty} f_j(w)-1\Big| < \varepsilon \quad \textit{for all } w \textit{ in } G,$$

and

$$c \ni (a_1, a_2, \ldots) \mapsto \sum_{j=1}^{\infty} a_j f_j \in A(G)$$

is a well-defined into isometry.

LEMMA 2. *Let G be a subset of D, let p be an upper semicontinuous nonnegative function on G and let $S \subset \partial\hat{G} \cap \partial D$ be a peak set for $A(G)$ such that $p|_S \equiv K > 0$. Assume that there are an $\varepsilon > 0$ and a $w_0 \in G \cap \operatorname{int} D$ such that*

(25) $$\|p\|_G = 1 = p(w_0), \quad p(w) \leqslant 1-2\varepsilon|B^2_{w_0}(w)| \quad \textit{for } w \textit{ in } G.$$

Then there are f_0, g_0 in $A(G)$ such that

(26) $$f_0|_S \equiv 1, \quad \|f_0+g_0-1\|_G \leqslant 4\varepsilon,$$

(27) $$\|pg_0\|_G = 1 = pg_0(w_0),$$

(28) $$|pg_0(w)|+\frac{p(w)}{K}|f_0(w)| \leqslant 1 \quad \textit{for all } w \textit{ in } G.$$

Proof of Lemma 2. Let $k \in A(G)$ be such that

$$\|k\| = 1, \quad k|_S \equiv 1, \quad |k(w)| < 1 \quad \text{for all } w \text{ in } G\setminus S.$$

Fix a positive integer n and define

$$U_n = \{z \in C\colon |1-z^n| < (1+\varepsilon)(1-|z^n|)\}.$$

U_n is an open set which contains the segment $[0, 1)$ on the real axis, so by the same argument as in the proof of the corollary of Proposition 2, there is an l in $A(D) \subset A(k(G))$ such that

$$\|l\| = 1, \quad l(1) = 1, \quad l(k(G)\setminus\{1\}) \subset U_n.$$

Composing l with an appropriate Blaschke factor we can also assume that $l(k(w_0)) = 0$. Put $f = (l \circ k)^n$. We have

$$\|f\| = 1, \quad f|_S \equiv 1, \quad f(G) \subset U_1,$$
$$f(w_0) = f'(w_0) = \ldots = f^{(n)}(w_0) = 0.$$

Hence, by the Schwarz Lemma, we get

$$|f(w)| \leqslant |B_{w_0}(w)|^{n+1} \quad \text{for all } w \text{ in } G.$$

Note that the sequence $(B^n_{w_0})_{n=1}^{\infty}$ tends uniformly to zero on any compact subset of $\operatorname{int} D$ so, taking n sufficiently large and since p is upper semicontinuous, we can assume that

(29) $$|f(w)| \leqslant K\varepsilon|B^2_{w_0}(w)| \quad \text{for any } w \in G \text{ with } p(w) \geqslant (1+\varepsilon)K.$$

Put

$$q(w) = K/p(w)-|f(w)|\big(1-2\varepsilon-K(1+\varepsilon)\big)-K(1+\varepsilon) \quad \text{for } w \in G.$$

Then q is a lower semicontinuous function such that

$$q|_S \equiv 2\varepsilon, \quad q(w) > \varepsilon \quad \text{for any } w \in G \text{ such that } p(w) < (1+\varepsilon)K.$$

By Proposition 5 there is an h in $A(G)$ such that

(30)
$$h|_S \equiv 2\varepsilon, \quad \|h\|_G = 2\varepsilon,$$
$$|h(w)| \leqslant q(w) \quad \text{for } w \in G \text{ with } p(w) < (1+\varepsilon)K.$$

We define $g_0 = 1-f$ and $f_0 = (1-2\varepsilon)f+hf$. Now (26) is evident. From (29) we have $f(w_0) = 0$ hence $pg_0(w_0) = g_0(w_0) = 1$ and $\|pg_0\|_G = 1$ will follow from (28). We have to check (28). To this end let $w \in G$ and assume first that $p(w) \geqslant (1+\varepsilon)K$. By (25) and (29) we have

$$|pg_0(w)|+\frac{p(w)}{K}|f_0(w)| \leqslant \big(1-2\varepsilon|B^2_{w_0}(w)|\big)\big(1+K\varepsilon|B^2_{w_0}(w)|+\varepsilon|B^2_{w_0}(w)|\big) \leqslant 1.$$

Assume now that $p(w) < (1+\varepsilon)K$. Since $f(G) \subset U_1$ we have $|g_0(w)| < (1+\varepsilon)(1-|f(w)|)$, hence by (30) and the definition of q we get

$$\begin{aligned}|pg_0(w)|&+p(w)|f_0(w)|/K\\ &\leqslant p(w)\big[K(1-|f(w)|)(1+\varepsilon)+(1-2\varepsilon)|f(w)|+|fh(w)|\big]/K\\ &\leqslant p(w)\big[|f(w)|(1-2\varepsilon-K(1+\varepsilon))+K(1+\varepsilon)+q(w)\big]/K \leqslant 1\end{aligned}$$

and this ends the proof of Lemma 2.

Now to end the proof of part **B** let x_0 and $\varepsilon_0 > 0$ be as in the Theorem and assume that T and G are such that the assumption of part **B** is fulfilled. Note that

(31) $$\text{If } \limsup_{w\to w_0} \hat{x}_0(w) > 0 \text{ then } w_0 \in G$$

so by our assumption we get $\partial\hat{G} \subset G$ and

$$\inf\{\hat{x}_0(w)\colon w \text{ is a cluster point of } \partial\hat{G}\} > 0.$$

Since $\hat{x}_0$ is upper semicontinuous there is a cluster point z_0 from $\partial\hat{G}$ such that $\hat{x}_0|_{\partial\hat{G}}$ is continuous at this point. By the same argument as in the proof of the corollary of Propositiom 2, and by Proposition 3, composing T

with an appropriate analytic map we can assume that

(32) $G \subset D$, 1 is a cluster point of $G \cap \partial D$, $\|\hat{x}_0\|_{G \cap \operatorname{int} D} = 1$.

Let $(z_j)_{j=1}^{\infty}$ be any sequence of distinct points of $G \cap \partial D$ such that $z_j \to z_0 = 1$ as $j \to \infty$. Put $S = \{z_j: j = 0, 1, 2, \ldots\}$. Any countable closed subset of ∂D is a peak set for $A(D)$; what is more, S is a peak set for $A(G)$.

Put $G' = \{z \in C: 2z \in G\} \cup \{1\}$, define $k: G' \to R$ by $k(1) = 0$ and $k(z) = \hat{x}_0(2z)$ for $2z \in G$, put $\varepsilon_0 = \varepsilon$ and let $\tilde{f}$ be as in Lemma 1. Define $k_1 \in A(D)$ by $k_1(2z) = \tilde{f}(z)$. We have

$$\|k_1(T)(x_0)\| = 1 = |k_1(w_0)|\,\hat{x}_0(w_0), \quad \|k_1(T)(x_0) - x_0\| \leqslant \varepsilon_0,$$

$$|k_1|\,\hat{x}_0(w) + \delta\,|B_{w_0}^2(w)| \leqslant 1 \quad \text{for all } w \text{ in } G.$$

So, by taking $k_1(T)(x_0)$ in place of x_0, we can assume without loss of generality that there are a $\delta > 0$ and a $w_0 \in \bar{G} \cap \operatorname{int} D$ such that $\hat{x}_0(w_0) = 1$ and

(33) $$\hat{x}_0(w) + \delta\,|B_{w_0}^2(w)| \leqslant 1 \quad \text{for all } w \text{ in } G;$$

by (31) we have $w_0 \in G$.

Let $(f_j)_{j=1}^{\infty}$ be as in the Michael–Pełczyński theorem and put

$$f = \sum_{j=1}^{\infty} f_j \cdot (K - \hat{x}_0(z_j))/B_{w_0}^2(z_j).$$

We have $f \in A(G)$ and

$$f(z_j)\,B_{w_0}^2(z_j) = K - \hat{x}_0(z_j) \quad \text{for } j = 1, 2, \ldots,$$

$$\|f\| = \sup\{|K - \hat{x}_0(z_j)|: j \in N\}.$$

Put $x' = ((1 + fB_{w_0}^2)(T))(x_0)$. We have $\|x' - x\| \leqslant \|f\|$ and $\hat{x}'(z_j) = K$ for all j in N. Since $K - \hat{x}_0(z_j) \to 0$ as $j \to \infty$, taking an appropriate subsequence of $(z_j)_{j=1}^{\infty}$ we can assume that

$$\|x' - x_0\| \leqslant \varepsilon_0/2, \quad \|f\|_G = \delta/2,$$

hence by (33) and the definition of x' we have

$$\hat{x}'(w) + \tfrac{1}{2}\delta\,|B_{w_0}^2(w)| \leqslant 1 \quad \text{for all } w \text{ in } G.$$

So, to simplify the notation, we can assume without loss of generality (by putting x' in place of x_0) that

(34) $$\hat{x}_0|_S \equiv K.$$

Put now $p = \hat{x}_0$, $\varepsilon = \min(\delta, \varepsilon_0)/4$ and let f_0, g_0 be as in Lemma 2. Put

$$y_0 = g_0(T)(x_0), \quad y_j = f_0 f_j(T)(x_0)/K$$

and define $\Phi: c \to X$ by

$$\Phi((a_0, a_1, \ldots)) = \sum_{j=0}^{\infty} a_j y_j.$$

By (27) we have $\|y_0\| = |g_0|\,\hat{x}_0(w_0) = 1$ and by (26) and (34) we have $\|y_j\| \geqslant |f_0 f_j|(z_j)\,\hat{x}_0(z_j)/K = 1$. On the other hand, from Lemma 2 we get $\sum_{j=1}^{\infty} |f_j(w)| \leqslant 1$ for $w \in G$ and so, by (28), for any w in G we have

$$\sum_{j=0}^{\infty} \hat{y}_j(w) = |g_0(w)|\,\hat{x}_0(w) + \frac{\hat{x}_0(w)}{K}|f_0(w)| \sum_{j=1}^{\infty} |f_j(w)| \leqslant 1.$$

We have shown that Φ is a well-defined into isometry; to end the proof note that by (26) and (24) we get

$$\|x_0 - \Phi((1, K, K, K, \ldots))\| \leqslant \|g_0 + f_0(\sum_{j=1}^{\infty} f_j) - 1\| \leqslant \varepsilon_0.$$

References

[1] E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces I, II*, Ann. of Math. 96 (1972), 98–173.

[2] E. Behrends, *M-Structure and the Banach–Stone Theorem*, Lecture Notes in Math. 736, Springer, 1979.

[3] —, *Multiplier representations and an application to the problem whether $A \otimes_\varepsilon X$ determines A and/or X*, Math. Scand. 52 (1983), 117–144.

[4] K. Jarosz, *Into isomorphisms of spaces of continuous functions*, Proc. Amer. Math. Soc. 90 (1984), 373–377.

[5] —, *Perturbations of Banach Algebras*, Lecture Notes in Math. 1120, Springer, 1985.

[6] —, *Isometries between injective tensor products of Banach spaces*, Pacific J. Math. 121 (1986), 383–396.

[7] E. A. Michael and A. Pełczyński, *A linear extension theorem*, Illinois J. Math. 11 (1967), 563–579.

[8] A. Pełczyński, *Some linear topological properties of separable function algebras*, Proc. Amer. Math. Soc. 18 (1967), 652–660.

[9] I. Suciu, *Function Algebras*, Noordhoff, 1975.

[10] H. Tong, *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. 19 (1952), 289–292.

INSTYTUT MATEMATYKI UNIWERSYTETU WARSZAWSKIEGO
INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY
PKiN IX p., 00-901 Warszawa, Poland

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