Hence $\Phi \in L^p(0, \infty)$. To prove (ii) and (iii) we observe that it is enough to show $\Phi(\xi) \leq C \xi^{-N/p}/|N|$ and $\Phi(\xi) \leq C \xi^{-N/p}/|N|^2$ respectively. For $N > 0$ we use Lemma 4 to derive the estimates. For $N < 0$ we use Lemma 3.

References


Received May 5, 1986

STUDIA MATHEMATICA, T. LXXXVII, (1987)

On $A$-uniform convexity and drop property

by

S. ROLEWICZ (Warszawa)

Abstract. Let $(X, \| \|)$ be a real Banach space. The norm $\| \|$ is called $A$-uniformly convex if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each convex set $E$ contained in the unit ball $B$ with measure of noncompactness greater than $\varepsilon$, $\inf \{ \|x\| : x \in E \} < 1 - \delta$. It is shown that the norm $\| \|$ is $A$-uniformly convex if and only if it satisfies uniformly a certain condition (a) equivalent to the drop property. The paper contains an example of a reflexive space in which there is no $A$-uniformly convex norm equivalent to the given one.

Let $(X, \| \|)$ be a real Banach space. The norm $\| \|$ is called uniformly convex [2] if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ such that $\|x\| = \|y\| = 1$ and

$$\|x - y\| > \varepsilon,$$

we have

$$\|\frac{1}{2}(x + y)\| < 1 - \delta.$$ (2)

Of course, in this definition we can replace condition (2) by

$$\inf \{ \|z\| : z \in \text{conv}(\{x, y\}) \} < 1 - \delta$$

where $\text{conv}(A)$ denotes the convex hull of a set $A$.

Indeed, (2) trivially implies (3). On the other hand, if (3) holds then there is $z \in \text{conv}(\{x, y\})$ such that

$$\|z\| < 1 - \delta.$$ (4)

We have two possibilities: either

$$\frac{1}{2}(x + y) = (1 - t)x + tz$$

for some $t$, $0 < t < 1$, or

$$\frac{1}{2}(x + y) = (1 - t)y + tz$$

for some $t$, $0 < t < 1$.

In both cases $\delta > \frac{1}{2}$ and the norm of $\frac{1}{2}(x + y)$ can be estimated as follows:

$$\|\frac{1}{2}(x + y)\| \leq (1 - t) + t(1 - \delta) = 1 - t\delta < 1 - \frac{1}{2}\delta,$$

(5)

and we obtain (2) with $\delta$ replaced by $\frac{1}{2}\delta$.

Goebel and Sękowski [8] extend the definition of uniform convexity replacing condition (1) by a condition involving the Kuratowski measure of noncompactness.
Let $A$ be a set in a Banach space $X$. The Kuratowski measure of noncompactness of $A$ is the infimum $\alpha(A)$ of those $\varepsilon > 0$ for which there is a covering of $A$ by a finite number of sets $A_i$ such that $\text{diam}(A_i) = \sup \{||x - y|| : x, y \in A_i\} < \varepsilon$. It has the following properties (see for example [1]):

(a) $\alpha(A) = 0$ if and only if the closure $\bar{A}$ of $A$ is compact.

(b) $\alpha(A) = \alpha(\bar{A})$.

(c) $\alpha(\text{conv}(A)) = \alpha(A)$.

(d) $\alpha(A + B) = \alpha(A) + \alpha(B)$.

(e) $\alpha(\lambda A) = |\lambda| \alpha(A)$.

A norm $\| \cdot \|$ in a Banach space $X$ is $\lambda$-uniformly convex [8] if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each convex set $E$ contained in the closed unit ball $B = \{x \in X : ||x|| \leq 1\}$ such that

$$\alpha(E) \geq \varepsilon,$$

we have

$$\inf \{||x|| : x \in E\} < 1 - \delta.$$

Goebel and Sękowski [8] have shown that if $\| \cdot \|$ is $\lambda$-uniformly convex, then each nonexpansive mapping $T$ of a closed convex set $C \subset X$ into itself has a fixed point.

We say that a Banach space $X$ is superreflexive ($\lambda$-uniformly convexifiable) if there is a norm $\| \cdot \|$ which is equivalent to the given one and uniformly convex ($\lambda$-uniformly convex).

Let $\langle X, \| \cdot \| \rangle$ be a Banach space. We say that the norm has the drop property [10] if for any closed set $C$ disjoint with the unit ball there is a point $a \in C$ such that

$$D(a, B) \cap C = \{a\}$$

where for brevity we have put

$$D(a, B) = \text{conv}(\{a\} \cup B);$$

we call $D(a, B)$ a drop [3].

It was shown by Rolewicz [10] and Montesinos [9] that a Banach space is reflexive if and only if there is a norm $\| \cdot \|$ equivalent to the given one such that $\| \cdot \|$ has the drop property.

In the present paper we shall discuss the relations between the drop property, the $\lambda$-uniform convexity and uniform convexity of norms, as well as the relations between reflexivity, $\lambda$-uniform convexifiability and superreflexivity.

Let $(X, \| \cdot \|)$ be a Banach space. We say that the norm $\| \cdot \|$ satisfies condition (a) if for each continuous linear functional $f$ of norm one

$$\lim_{\varepsilon \to 0} \alpha(S(f, \varepsilon)) = 0,$$

where $S(f, \varepsilon)$ denotes the "slice"

$$S(f, \varepsilon) = \{x \in X : ||x|| \leq 1, f(x) \geq 1 - \varepsilon\}.$$

Theorem 1 ([9], [10]). The norm $\| \cdot \|$ has the drop property if and only if it satisfies condition (a).

Corollary 1 ([9]). Let $(X, \| \cdot \|)$ be a Banach space. Let $(Y, \| \cdot \|)$ be a subspace of $X$. If the norm $\| \cdot \|$ has the drop property then

(i) The norm $\| \cdot \|$ restricted to $Y$ has the drop property.

(ii) The norm $\| [x] \|_0 = \inf \{||x + y|| : y \in Y\}$ in the quotient space $X/Y$ also has the drop property.

Proof (i) Let $f$ be an arbitrary functional of norm 1 defined on $Y$. Let $f'$ be a norm one extension of $f$ to $X$. Then $S(f, \varepsilon) \subseteq Y \cap S(f', \varepsilon)$ and

$$\alpha(S(f, \varepsilon)) \leq \alpha(S(f', \varepsilon))$$

which tends to zero, because the norm $\| \cdot \|$ on $X$ has the drop property.

(ii) Let $f$ be a functional of norm one defined on the quotient space $X/Y$. It induces a functional $f'$ of norm one on $X$ by the formula $f'(x) = f([x])$. Observe that for each $\varepsilon$

$$S(f, \varepsilon) = [x] : x \in S(f', \varepsilon).$$

Since $\text{diam} ([x] : x \in A) \leq \text{diam} A$, we have

$$\alpha(S(f, \varepsilon)) \leq \alpha(S(f', \varepsilon))$$

and the drop property of the norm $\| \cdot \|$ implies the drop property of the quotient norm.

Theorem 2. Let $(X, \| \cdot \|)$ be a Banach space. Let $x_0$ be a point of norm greater than 1. Let

$$B_0 = \text{conv}([x_0], -x_0) \cup B).$$

The set $B_0$ induces a new norm $\| \cdot \|$ equivalent to the given one. If the norm $\| \cdot \|$ has the drop property, then so does the norm $\| \cdot \|_0$.

The proof is based on some propositions.

Let $f$ be a continuous linear functional on $X$ of norm 1. We write

$$g(f) = \alpha(S(f, \varepsilon)).$$

Proposition 1. For $0 < \lambda < 1$ and $0 < \varepsilon < 1$,

$$g((\lambda)) \geq \lambda g(f).$$

Proof. Let $\delta$ be such that $0 < \delta < \varepsilon$. Let $x_0$ be an element of norm 1 such that

$$f(x_0) = 1 - \delta.$$
By the convexity of the unit ball
\[ x^*_\varepsilon := \left( \frac{1 - \varepsilon - \delta}{\varepsilon} \right) S(f, \varepsilon \cdot x^*_\varepsilon) \subseteq S(f, \varepsilon x^*_\varepsilon). \]
Thus
\[ S(f, \varepsilon x^*_\varepsilon) \subseteq S(f, \varepsilon). \]

Letting \( \varepsilon \) tend to 0, we get (13).

We do not know whether the function \( g^0 \) is always concave.

**Proposition 2.** Let \( (X, \| \cdot \|) \) be a Banach space. Let \( X_1 = X \times \mathbb{R} \), where the norm \( \| \cdot \|_{1 \times} \) on \( X_1 \) is defined by
\[ \| (x, t) \|_{1 \times} = \| x \| + |t|. \]
If the norm \( \| \cdot \| \) has the drop property, then so does the norm \( \| \cdot \|_{1 \times} \).

**Proof.** Let \( f \) be an arbitrary linear functional of norm one in \( X_1 \), \( \| f \|_{1 \times} = 1 \). Let \( f_0 \) denote the restriction of \( f \) to \( X, f_0 = f|_X \). Of course, \( \| f_0 \| \leq 1 \). We write
\[ S(f_0, \varepsilon) = \{ x \in X : \| x \| \leq 1, f_0(x) \geq 1 - \varepsilon \}. \]
Of course, if \( \| f_0 \| < 1 \) the set \( S(f_0, \varepsilon) \) is void for sufficiently small \( \varepsilon \). Now we have two possibilities:
(i) Neither \((0, 1)\) nor \((0, -1)\) is a point of support of the functional \( f \).
(ii) Either \((0, 1)\) or \((0, -1)\) is a point of support of \( f \).

In case (i) it is easy to observe that for sufficiently small \( \varepsilon \)
\[ S(f, \varepsilon) = \text{conv}((0, 1), (0, -1), \cup S(f_0, \varepsilon)) \]
and by property (b) of the measure of noncompactness
\[ \alpha(S(f_0, \varepsilon)) \leq \alpha(S(f_0, \varepsilon)). \]

Now we consider case (ii). Without loss of generality we may assume that \( f(0, 1) = 1 \). Let \( t \) be an arbitrary number, \(-1 \leq t \leq 1\). Let \( A_t = \{ x \in X : (x, t) \in S(f, \varepsilon) \} \). Of course
\[ S(f, \varepsilon) = \bigcup_{-1 \leq t \leq 1} A_t \times \{ t \}. \]
Using a compactness argument we can easily show that (20) implies
\[ \alpha(S(f, \varepsilon)) = \max_{-1 \leq t \leq 1} \alpha(A_t). \]
Now we shall estimate \( \alpha(A_t) \). We divide the interval \([-1, 1]\) into three sections: \([-1, 0), [0, 1 - \varepsilon), [1 - \varepsilon, 1] \). In \([-1, 0) \]
\[ A_t \times \{ t \} = \text{conv}((0, 1), \cup S(f_0, \varepsilon) \times \{ 0 \}) \]
and so
\[ \alpha(A_t) \leq \alpha(S(f_0, \varepsilon)). \]
In \([1 - \varepsilon, 1] \), \( A_t = (1 - t)B \), where \( B \) denotes the closed unit ball in \( X \). Thus
\[ (23) \alpha(A_t) \leq 2\varepsilon. \]
The most complicated case is the interval \([0, 1 - \varepsilon)\). In this section we obtain \( A_t \) by cutting off a piece of the ball \((1 - t)B\) by a hyperplane with distance from the center not smaller than \( \varepsilon \). In other words,
\[ (25) A_t = (1 - t)S(f_0, \varepsilon, \frac{\varepsilon}{1 - t}). \]
and by (b)
\[ (26) \alpha(A_t) \leq (1 - t)g^{0}(\frac{\varepsilon}{1 - t}). \]

By Proposition 1
\[ (27) \sup_{0 \leq t \leq 1} \alpha(A_t) \leq g^0(\varepsilon). \]
Therefore by (23), (24), (27) and (21)
\[ (28) \alpha(S(f, \varepsilon)) \leq \alpha(S(f_0, \varepsilon)). \]
Thus the norm \( \| \cdot \|_{1 \times} \) satisfies condition (a), which finishes the proof of Proposition 2.

**Proof of Theorem 2.** We embed \( X \) into the space \( X_1 \) described in Proposition 2. Let \( T \) be a projection of \( X_1 \) onto \( X \) such that \( T(0, 1) = x_0 \). It is easy to see that
\[ T(B_1) = \text{conv}(\{ x_0, -x_0 \} \cup B), \quad B_1 = \{ (x, t) : \| (x, t) \|_{1 \times} \leq 1 \}; \]
and by Corollary 1 the norm \( \| \cdot \|_0 \) in \( X \) has the drop property.

If the convergence in formula (10) is uniform with respect to all \( f, \| f \|_1 \)
= 1, then we say the norm \( \| \cdot \|_0 \) satisfies the uniform condition (a).

More precisely, we say that a norm \( \| \cdot \| \) in a Banach space \((X, \| \cdot \|)\)
satisfies the uniform condition (a) if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for each continuous linear functional \( f \) of norm one
\[ (29) \alpha(S(f, \delta)) \leq \varepsilon. \]

**Theorem 3.** Let \((X, \| \cdot \|)\) be a Banach space. The norm \( \| \cdot \| \) is \( \Lambda \)-uniformly convex if and only if it satisfies the uniform condition (a).

**Proof.** Observe that the norm is \( \Lambda \)-uniformly convex if and only if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for each convex subset \( E \) of the unit
ball $B$

$$\inf \|x\| : x \in E \geq 1 - \delta$$

implies

$$\alpha(E) < \varepsilon.$$  

(31)

Let $E$ be an arbitrary convex subset of the unit ball satisfying (30). Then by the separation theorem there is a continuous linear functional $f$ of norm one such that

$$E = S(f, \delta).$$

(32)

Thus the uniform condition (a) implies (31). On the other hand, $S(f, \delta)$ is a convex subset of the unit ball satisfying (30). Thus the $\Delta$-uniform convexity implies the uniform condition (a).

In a similar way as in Corollary 1 we obtain

**Proposition 3.** Let $(X, \| \cdot \|)$ be a Banach space. Let $(Y, \| \cdot \|)$ be a subspace of $X$. If the norm $\| \cdot \|$ is $\Delta$-uniformly convex then:

(i) The norm $\| \cdot \|$ restricted to $Y$ is also $\Delta$-uniformly convex.

(ii) The quotient norm $\|\cdot\|_0 = \inf \|x+y\| : y \in Y$ is $\Delta$-uniformly convex in the quotient space $X/Y$.

**Proposition 4.** Let $(X, \| \cdot \|)$ be a Banach space. Let $X_1 = X \times \mathbb{R}$ with the norm $\| (x, t) \|_1 = \| x \| + |t|$. If the norm $\| \cdot \|$ is $\Delta$-uniformly convex, then so is the norm $\| \cdot \|_1$.

**Proof.** The proof is a slight modification of the proof of Proposition 2. Let $\varepsilon$ be a small positive number. Let $f$ be an arbitrary continuous linear functional of norm one defined on $X_1$.

Without loss of generality we may assume that $f(0, 1) \geq 0$. We shall consider two cases:

(i) $f(0, 1) \leq -\varepsilon$.

(ii) $f(0, 1) > -\varepsilon$.

As previously, we denote by $f_0$ the restriction of $f$ to $X$.

In case (i)

$$S(f, \delta) = \text{conv}(\{(0, 1), (0, -1)\} \cup S(f_0, \delta))$$

for $\delta < \varepsilon$ and by property (b) of the Kuratowski measure of noncompactness

$$g^1(\delta) \leq g^0(\delta)$$

for $\delta < \varepsilon$.

(33)

In the second case we also introduce the sets

$$A_1 = \{x \in X : (x, t) \in S(f, \delta)\}.$$

We divide the interval $[-\varepsilon, 1]$ into three sections $[-1, 0], [0, 1-\varepsilon], [1-\varepsilon, 1]$.
The functional $f$ has norm one. Thus (38) implies

$$\|y\| \geq 1 - 2\varepsilon.$$  

By (37) and (39),

$$\|z\|^p = \|x\|^p - \|y\|^p \leq 1 - (1 - 2\varepsilon)^p < 2\varepsilon,$$

and so

$$\|z\| < \sqrt[2p]{\varepsilon}.$$

Thus by (37), (38), (41),

$$S(f, e) = S(f, 2e) \cap Y + \{z \in Z : \|z\| \leq \sqrt[2p]{\varepsilon}\}.$$

The set $S(f, 2e) \cap Y$ is compact since $Y$ is finite-dimensional. Thus by properties (a) and (d) of the Kuratowski measure of noncompactness

$$\alpha(S(f, e)) = \alpha\left(\{z \in Z : \|z\| \leq \sqrt[2p]{\varepsilon}\}\right) = 2 \sqrt[p]{2}\varepsilon.$$

**Corollary 2.** There is a $\Delta$-uniformly convex space which is not superreflexive.

**Proof.** Taking for $X_n$ $n$-dimensional spaces either with the $c_0$ norm, i.e.

$$\|x\| = \sup_{1 \leq k \leq n} |x_k|, \quad x = (x_1, \ldots, x_n),$$

or with the $l^p$ norm, i.e.

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p},$$

with $p_n \to \infty$, we obtain the classical examples of Day [5] of nonsuperreflexive spaces. By Theorem 5, those spaces are $\Delta$-uniformly convex.

**Theorem 6.** Let $X_\alpha = l^p$ with the standard norm. Assume that $p_n \to \infty$.

Then the space

$$X = (X_1 \times X_2 \times \ldots)_{\ell_p}, \quad 1 < p < +\infty,$$

is not $\Delta$-uniformly convex.

**Proof.** We shall denote the standard norm by $\| \|$. Suppose that there is a $\Delta$-uniformly convex norm $1$ in $X$ equivalent to $\| \|$. This means that there are two positive numbers $m, M$ such that $\|x\| \leq m\|x\| \leq M\|x\|$. Replacing $\| \|$ by $m\| \|$ we may assume without loss of generality that

$$\|x\| \leq M\|x\|.$$

This means that the unit ball in the standard norm contains the unit ball in the new norm $1$ and that the unit ball in the new norm contains the ball of radius $\alpha = 1/M$ in the standard norm.

We shall denote by the same symbols $\| \|$ and $1$ the restrictions of the norms $\| \|$ and $1$ to each component $X_n = l^p$. The calculation will be done in one space $X_n = l^p$ with $p_n$ sufficiently large. The choice of $p_n$ will follow from the construction. For brevity we put

$$p_n = \tilde{p}, \quad X_n = \tilde{X}.$$

We decompose $\tilde{X}$ into two infinite-dimensional subspaces by decomposing the set of natural numbers into two disjoint infinite sets $N_1, N_2$ and putting

$$Y = \{x \in \tilde{X} : x_i = 0, \quad i \in N_2\}, \quad Z = \{x \in \tilde{X} : x_i = 0, \quad i \in N_1\}.$$

Of course, $\tilde{X} = Y + Z$ and for $y \in Y, z \in Z$

$$\|y + z\|^p = \|y\|^p + \|z\|^p.$$

Now let $\varepsilon = 1/\alpha$. Since we have assumed that the norm $1$ is $\Delta$-uniformly convex there is $\delta > 0$ such that for each convex set $E \subset \{x : \|x\| \leq 1\}$ such that $\alpha(E) < \varepsilon$ we have

$$\inf \{\|x\| : x \in E\} < 1 - \delta.$$

Let $y$ be an arbitrary element of $Y$ such that

$$\|y\| < \alpha(1 - 1/2)^{1/p}.$$

Then, of course, for an arbitrary $z \in Z$ such that $\|z\| < 1/\alpha$

$$\|y + z\| < \alpha.$$

Thus

$$y + z \in \{z \in Z : \|z\| \leq 1\} \subset \{z \in \tilde{X} : \|z\| \leq \alpha\} \subset \{x \in \tilde{X} : \|x\| \leq 1\}.$$

The set $y + z$ has the Kuratowski measure of noncompactness in the standard norm not larger than $\varepsilon = 1/\alpha$. By (43) the same is true in the new norm. Thus there is $z \in Z$, \(\|z\| < \sqrt[2p]{\varepsilon}\), such that $y + z = 1 - \delta$. Of course $1 - \delta < 1$ and finally

$$1 - \delta < \frac{1}{\sqrt[p]{2}} \|y + z\| < 1 - \frac{1}{\sqrt[p]{2}}.$$

Thus $\|y\| < \alpha(1 - 1/\delta)^{1/p}$ and we have shown that for each $y \in Y$ such that

$$\|y\| < \alpha \frac{(1 - 1/\delta)^{1/p}}{1 - \frac{1}{\sqrt[p]{2}}},$$

we have $\|y\| < 1$.

Now, $\beta$ ought to be chosen so that

$$\beta = \frac{(1 - 1/\delta)^{1/p}}{1 - \frac{1}{\sqrt[p]{2}}} > 1.$$

This is possible since $p_n \to \infty$. Now repeating the decomposition procedure $n$ times we deduce that there is an infinite-dimensional space $Y_\alpha$ such that for
all $y \in Y_s$ such that
\begin{equation}
\|y\| \leq \alpha \left[ \frac{(1-1/2^p)\|\|}{1-\delta} \right]^{1/p}
\end{equation}
we have
\begin{equation}
\|y\| \leq 1
\end{equation}
By (47) this contradicts (43).

**Corollary 3.** There are reflexive spaces which are not $\Delta$-uniformly convexifiable.

**Proof.** The space $X$ described in Theorem 6 is reflexive [5].

Now we shall distinguish a property lying between uniform convexity and $\Delta$-uniform convexity. The starting point is the following.

**Proposition 5** [10]. Let $(X, \|\|)$ be a Banach space. Let $x$ not to belong to the unit ball. Let
\begin{equation}
R(x) = D(x, B) \setminus B.
\end{equation}
The norm $\|\|$ is uniformly convex if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that
\begin{equation}
\|x\| < 1 + \delta
\end{equation}
implies
\begin{equation}
\text{diam}(R(x)) < \varepsilon.
\end{equation}

Proposition 5 suggests the investigation of the following condition on the norm:
\begin{equation}
(\beta) \quad \text{For each } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that } \|x\| < 1 + \delta \text{ implies }
\alpha(R(x)) < \varepsilon.
\end{equation}

**Proposition 6.** If a norm $\|\|$ satisfies condition $(\beta)$, then it is $\Delta$-uniformly convex.

**Proof.** Suppose that the norm is not $\Delta$-uniformly convex. Then by Theorem 3 it does not satisfy condition (a). This means that there is an $\varepsilon_0 > 0$ and sequences of continuous linear functionals of norm one $\{f_n\}$ and positive numbers $\delta_n > 0$ such that
\begin{equation}
\alpha(S(x_n, \delta_n)) > \varepsilon_0.
\end{equation}

Let $x_n$ be an element such that $1 + 2\delta_n \leq \|x\| \leq 1 + 3\delta_n$ and
\begin{equation}
f_n(x_n) = 1 + 2\delta_n.
\end{equation}

By (55) for each element of the form $\frac{1}{2}(x_n + y), y \in S(x_n, \delta_n)$, we have $f\left(\frac{1}{2}(x_n + y)\right) > 1$ and $\frac{1}{2}(x_n + y) \not\in B$. On the other hand, $\frac{1}{2}(x_n + y) \in D(x_n, B)$. Thus $\frac{1}{2}(x_n + y) \in R(x_n)$. Observe that the set $\{\frac{1}{2}(x_n + y), y \in S(x_n, \delta_n)\}$ is isometric to the set $S(f_n, \delta_n)$ with coefficient $1/2$. Thus, by (54) and property (e) of the Kuratowski measure of noncompactness
\begin{equation}
\alpha(R(x_n)) \geq \varepsilon_0 < 2,
\end{equation}
which completes the proof.

Observe that $\Delta$-uniform convexity does not imply condition $(\beta)$. Indeed, in Proposition 4 we have constructed a $\Delta$-uniformly convex space which is the $l^p$-product of a $\Delta$-uniformly convex space $(X, \|\|)$ by $R$. It is easy to see that for $x = (0, 1 + \delta)$ the closure of $R(x) = D(x, B) \setminus B$ contains the unit sphere in $X$, thus $\alpha(R(x)) \geq 1$ independently of $\delta > 0$.

We shall say that a Banach space $(X, \|\|)$ is a $(\beta)$-space if there is a norm $\|\|_1$ equivalent to $\|\|$ such that $\|\|_1$ satisfies condition $(\beta)$.

We have shown that every superreflexive space is a $(\beta)$-space and every $(\beta)$-space is $\Delta$-uniformly convexifiable. We do not know anything about the converse implications.

**References**


*Institute of Mathematics, Polish Academy of Sciences*

Received June 9, 1986

Revised version July 9, 1986