Multilinear singular integrals involving a derivative of fractional order

by

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Abstract. In this paper, we obtain $L^2$ estimates for certain multilinear singular integrals, which are analogues of the Calderón commutators involving a derivative of fractional order. The estimates are obtained by an application of the tent space theory of Coifman, Meyer, and Stein.

1. Introduction. For $\lambda \in [0, 1]$, consider the derivative of fractional order $\lambda$, defined for tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ by

\[ (D^\lambda f)(x) = |x|^\lambda \hat{f}(x). \]

Here, $\hat{f}$ denotes the Fourier transform, defined according to the normalization

\[ \hat{f}(\xi) = \int e^{-ix\xi} f(x) dx. \]

Let $A_1, \ldots, A_n : \mathbb{R} \to C$ be locally integrable functions; let $M_j$, for $1 \leq j \leq n$, denote the operator of pointwise multiplication by $A_j$. If $T$ is an operator, let $T_j(T) = [M_j T] = M_j T - T M_j$; let $\delta_0$ denote the iterated commutator $\delta_0 = \delta_0 \circ \delta_0 \circ \cdots \circ \delta_0$. We consider the multilinear operators

\[ C_{\lambda,n}(A_1, \ldots, A_n) = A_n(D^\lambda)^n, \]

\[ \tilde{C}_{\lambda,n}(A_1, \ldots, A_n) = A_n(H(D)^n) \]

where $H$ denotes the Hilbert transform, defined by

\[ (Hf)(\xi) = -\text{sgn} \xi \hat{f}(\xi). \]

It is easily seen that $C_{\lambda,n} = 0$ if $\lambda n$ is an even integer, and $\tilde{C}_{\lambda,n} = 0$ if $\lambda n$ is an odd integer. For all other positive integers $n$, it is easily seen that

\[ C_{\lambda,n}(f)(\xi) = \gamma_n(\lambda) \text{ p.v. } \int K_n(x, y) f(y) dy, \]

\[ \tilde{C}_{\lambda,n}(f)(\xi) = \tilde{\gamma}_n(\lambda) \text{ p.v. } \int \tilde{K}_n(x, y) f(y) dy \]

where $\gamma_n(\lambda), \tilde{\gamma}_n(\lambda)$ are constants depending on $n$ and $\lambda$, and

\[ K_n(x, y) = |x-y|^{-n-1} \prod_{j=1}^{n} (A_j(x) - A_j(y)), \]

\[ \tilde{K}_n(x, y) = \text{sgn}(x-y) K_n(x, y) \]
(see [8], Chapter 3). These operators are generalizations of the so-called Calderón commutators, which arise when \( A \) is taken equal to 1; these commutators have been extensively studied by Calderón, Coifman, McIntosh, Meyer, and others (see [1], [3], [4]). In particular, it is well known that \( C_{1,1} \) is bounded on \( L^p(R) \) if and only if \( A_j \in \text{Lip}_1(R) \), i.e., \( A_j \in L^{\alpha}(R) \) (see [1] and [6]). Coifman, McIntosh, and Meyer have shown (in [3], Theorem III) that \( C_{1,1} \) (for odd \( n \)) and \( \tilde{C}_{1,1} \) (for even \( n \)) are bounded on \( L^2(R) \) provided that \( A_1, \ldots, A_n \in \text{Lip}_1(R) \). \( L^2 \) estimates for Calderón commutators have also been obtained as a straightforward consequence of the amazing theorem of David and Journé ([5]).

Cohen, Gosselin, and others have asked whether it is possible to obtain \( L^p \) estimates for the operators \( C_{1,1} \) and \( \tilde{C}_{1,1} \) under the assumption that the functions \( A_1, \ldots, A_n \) all have differing degrees of smoothness. They found that the only way to obtain such estimates is to replace each occurrence of the quotient \( (A_j(x) - A_j(y))(x - y)^{-1} \) with an appropriately adjusted Taylor series remainder of \( A_j \). They were then able to obtain the \( L^p \) norms of these modified operators in terms of the BMO norms of the higher derivatives of the \( A_j \) (see [2]).

The case of \( \lambda < 1 \) is fundamentally different. One might well expect that \( A_j \in \text{Lip}_1(R) \) is a necessary and sufficient condition for the \( L^p \)-boundedness of \( C_{1,1} \) and \( \tilde{C}_{1,1} \). But the author has recently shown ([6]) that these operators are bounded on \( L^p \) if and only if \( A_j = [D^j] A_j \in \text{BMO}(R) \); i.e., \( A_j \in \text{I}_j \text{BMO} \), the BMO Sobolev space studied by Strichartz ([9]) which is properly contained in \( \text{Lip}_1(R) \).

We consider the restriction of the multilinear operators \( C_{1,n} \) and \( \tilde{C}_{1,n} \) to the diagonal \( A_1 = A_2 = \cdots = A_n = A \); it is easy to obtain an estimate of the form

\[
\|C_{1,n}(A, \ldots, A)f\|_l^2, \|\tilde{C}_{1,n}(A, \ldots, A)f\|_l^2 \leq C\|A\|^{-\lambda} \|f\|_l^2
\]

(1.10) where \( \|\cdot\|_l \) denotes the norm on \( \text{Lip}_1(R) \), \( \|\cdot\|_p \) denotes the BMO norm, \( \xi = [D^1] A_j \) and \( C \) is a constant independent of \( A_j \). The author has shown (in [7], Chapter 2) that the estimate (1.10) is valid for \( n = 2 \); R. R. Coifman has pointed out that (1.10) for \( n > 2 \) is an immediate consequence, since \( \|K_{x_1, x_2}(y)\| < \|K_{x_1, x_2}(y)\|_{L^2} \) for \( n > 2 \) in the diagonal case.

It is natural to ask whether it is possible to obtain \( L^p \) estimates for \( C_{1,n} \) and \( \tilde{C}_{1,n} \) when the functions \( A_1, \ldots, A_n \) have differing degrees of smoothness. In this paper we answer the question affirmatively and prove the following result for \( n = 2 \) or 3:

**Main Theorem.** Suppose \( n \) is a positive integer, and let \( \lambda_j \in (0, 1) \) for \( 1 \leq j \leq n \). Let \( \lambda = n^{-1}(\lambda_1 + \cdots + \lambda_n) \) and suppose \( A_j \in \text{I}_j \text{BMO} \) with \( \xi_j = [D^j] A_j \) for \( 1 \leq j \leq n \). Then

\[
\|C_{1,n}(A, \ldots, A)f\|_l^2, \|\tilde{C}_{1,n}(A, \ldots, A)f\|_l^2 \leq C\|f\|_l^2 \prod_{j=1}^n \|\xi_j\|.
\]

(1.11) where \( C \) is a constant independent of \( A_1, \ldots, A_n \).

The proof of the Main Theorem may be extended to the case of arbitrary \( n \), but in the interest of relative simplicity we give the proof in the case of \( n = 2, 3 \); the case \( n = 1 \) is the result already cited (see [6]). It should be noted that estimates of the form (1.11) cannot be obtained from the powerful theorem of David and Journé (see [5]).

We begin by showing that \( C_{1,n} \) and \( \tilde{C}_{1,n} \) may be expressed in terms of operators of the form

\[
\int \frac{Y_{x_1}(t) \cdots Y_{x_n}(t) \psi(t) dt}{t^{n-1}}.
\]

Here, \( Y_j \) denotes the symmetric group of degree \( n \); for \( 1 \leq j \leq n, \|Y_j\|_p \) is the operator of pointwise multiplication by \( y_j = A_j \), and, for \( 0 \leq j \leq n, Y_j \in \{P_j, Q_j\} \) where \( P_j = (I + i t D)^{j-1}, Q_j = i D P_j \) and \( D = -i d/ds \). Expressions of the form (1.12) are obtained by means of the symbolic calculus developed in [3]. Then, following [3], we show that the problem of estimating the operator norm of (1.11) may be reduced to certain estimates in the upper half-plane. The necessary quadratic estimates follow from certain remarkable identities involving the operators \( P_j, Q_j, \) together with the Tent Space techniques introduced by Coifman, Meyer, and Stein ([4]). These estimates are computed explicitly in the cases \( n = 2, 3 \); we then indicate how the proofs may be extended to the case of more general \( n \).

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**2. Integral representation formulas for the commutators.** In this section, we use the techniques of Coifman, McIntosh, and Meyer to obtain integral representation formulas for the commutators. Following [3] and [6], we define, for \( t \neq 0 \), the operators \( P_j = (I + i t D)^{j-1} \) and \( Q_j = i D P_j \). Then \( P_j \) and \( Q_j \) are the operators of convolution with \( p_j \) and \( q_j \), respectively, where \( p_j(x) = \frac{1}{|x|^{n-1}} \) and \( q_j(x) = \frac{1}{|x|^{n-1}} \). We also define \( R_j = I - P_j \), which may be thought of as convolution with \( \delta_j = -p_j \), where \( \delta_j \) is the Dirac measure concentrated at \( 0 \); and we set \( L_j = P_j + i Q_j \). We observe that

\[
L_j = (I + i t D)^{j-1}, \quad P_j = \frac{i}{2} (I + i t D)^{j-1} + \frac{i}{2} (I - i t D)^{j-1}, \quad Q_j = \frac{i}{2} (I + i t D)^{j-1} - \frac{i}{2} (I - i t D)^{j-1}.
\]
We obtain the following result:

**Lemma 2.1.** Let \( r \in (0, 2) \) and set \( \varphi_r = (2/\pi) \sin(r\pi/2) \). Then

\[
|D|^r = \varphi_r \int_0^\infty R_t t^{-r-1} dt,
\]

and

\[
|D|^r = iH|D|^{-r} = \varphi_r \int_0^\infty Q_t t^{-r} dt.
\]

**Proof.** Note that

\[
\varphi_r \int_0^\infty \frac{1}{1 + (r/2) - r} t^{-r-1} dt = \int_0^\infty \int_0^\infty \frac{1}{1 + (r/2)} - r \int_0^\infty \int_0^\infty \frac{1}{1 + (r/2)} t^{-r} dt = \int_0^\infty \int_0^\infty \frac{1}{1 + (r/2)} - r \int_0^\infty Q_t t^{-r} dt.
\]

If we set \( \varphi_r = \int_0^\infty (1 + t)^{-1} dt \), then (2.4) and (2.5) yield

\[
|\xi|^r = \varphi_r \int_0^\infty (1 - \int_0^\infty (2\xi) t^{-r-1} dt,
\]

and

\[
|\xi|^r = \varphi_r \int_0^\infty (1 - \int_0^\infty (2\xi) t^{-r} dt.
\]

A calculation using residues shows that \( \varphi_r = (2/\pi) \sin(r\pi/2) \). The lemma now follows from the definition of \( R_r \) and \( Q_r \).

If \( h \) is any locally integrable function, we denote by \( M_h \) the operator of pointwise multiplication by \( h \). If \( h \in \mathcal{S}(\mathbb{R}) \), then \( M_h \) is an element of the algebra of continuous linear operators on the Schwartz class \( \mathcal{S}'(\mathbb{R}) \).

As in [5], we may assume without loss of generality that the functions \( A_1, A_2, \ldots, A_n \in C_c^\infty(\mathbb{R}) \). For \( T \in \mathcal{A} \) and \( 1 \leq j \leq n \), we define

\[
\delta_j(T) = [M_{A_j}, T] = M_{A_j} T - T M_{A_j}.
\]

It is easy to see that \( \delta_j \) is a derivation of the complex algebra \( \mathcal{A} \). We shall enumerate some of its most important properties (see also [3]). For ease of notation, let \( A_\mu \) denote the iterated commutator \( \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n \); for \( 1 \leq j \leq n \), let \( A_j = A_j \). Then we obtain the following (see [3]):

**Lemma 2.2.** Let \( \alpha, \beta \in \mathbb{C} \) and \( S, T \in \mathcal{A} \). Let \( n \) be a positive integer, let \( 1 \leq j, k \leq n \), and let \( 0 \leq i \leq n - 1 \). Then, with notation as above, we have

\[
\delta_j(\alpha S + \beta T) = \alpha \delta_j(S) + \beta \delta_j(T),
\]

and

\[
\delta_j(ST) = \delta_j(S) T + S \delta_j(T).
\]

(c) \( \delta_j \circ \delta_j = \delta_j \circ \delta_j \).

(d) If \( S \) is invertible, \( \delta_j(S^{-1}) = -S^{-1} \delta_j(S) S^{-1} \).

(e) \( \delta_j(D) = i M_{A_j} \).

(f) \( \delta_j(M_{A_j}) = 0 \).

(g) \( A_j(D) = 0 \).

(h) For \( t \neq 0 \), \( \delta_j(L_t^\infty) = \mp t L_t^\infty M_{A_j} L_t^\infty \).

(i) For \( t \neq 0 \), \( \delta_j(L_t^\infty) = (\mp t)^{\nu} \sum_{j=0}^{\nu} \frac{L_t^\infty M_{A_{j+1}} L_t^\infty M_{A_{j+2}} \cdots L_t^\infty M_{A_{n+1}} L_t^\infty}{j!} \).

(j) For \( t \neq 0 \), \( \delta_j((tD)^k L_t^\infty) = (\mp t)^{\nu} \sum_{j=0}^{\nu} \frac{L_t^\infty M_{A_{j+1}} L_t^\infty M_{A_{j+2}} \cdots L_t^\infty M_{A_{n+1}} L_t^\infty}{j!} \).

**Proof.** Properties (a), (c), (d), (e), (f), and (g) follow from (h), (i), and (j). Property (i) follows from (h) by a simple induction argument. We shall indicate the proof of (i). Notice that

\[
[(tD)^k - i] L_t^\infty = (tD)^k [(D)^k - i] (I + i tD)^{-1} = (tD)^k [(D)^k - i] (I - (itD)^{-1})^{-1} = (tD)^k [I - (itD)^k] (I - (itD)^{-1})^{-1} = -(itD)^{k-1} \sum_{j=0}^{k-1} (itD)^j.
\]

Thus, by (a) and (g),

\[
A_j((tD)^k L_t^\infty) = (tD)^k A_j(L_t^\infty) = (tD)^k \sum_{j=0}^{k-1} (itD)^j A_j(D)^k = (tD)^k A_j(L_t^\infty).
\]

Similarly,

\[
[(tD)^k - (itD)^k] L_t^\infty = (tD)^k [(itD)^k - i] (I - itD)^{-1} = -(itD)^{k-1} \sum_{j=0}^{k-1} (itD)^j (I - itD)^{-1} = -(itD)^{k-1} \sum_{j=0}^{k-1} (itD)^j = \sum_{j=0}^{k-1} (itD)^j,
\]

so that, by (a) and (g),

\[
A_j((tD)^k L_t^\infty) = (itD)^k A_j(L_t^\infty) - (itD)^k \sum_{j=0}^{k-1} (itD)^j A_j(D)^k = (itD)^k A_j(L_t^\infty).
\]

Then (i) follows from (i), (2.1), and (2.12).
let \( q_n = (2/n) \sin(n\pi/2) \). Then

(a) If \( \nu \neq 0 \) and \( K(n, \lambda) = -\frac{1}{t} (1) \mu^{\nu/2} q_n \), then

\[
C_{\lambda, n} = K(n, \lambda) \int_{0}^{\infty} \left[ \sum_{m=0}^{\infty} \left[ L_\nu^* M_{n+1} L_\nu^* M_{n+2} \cdots L_\nu^* M_{n+\mu} L_\nu^* \right] (t^{-1} - \mu) \frac{dt}{t} \right].
\]

(b) If \( \mu \neq 0 \) and \( \tilde{K}(n, \lambda) = \frac{1}{t} (1) \mu^{\nu+1/2} q_n \), then

\[
\tilde{C}_{\lambda, n} = \tilde{K}(n, \lambda) \int_{0}^{\infty} \left[ \sum_{m=0}^{\infty} \left[ L_\nu^* M_{n+1} L_\nu^* M_{n+2} \cdots L_\nu^* M_{n+\mu} L_\nu^* \right] (t^{-1} - \mu) \frac{dt}{t} \right].
\]

Proof. If \( \nu \neq 0 \), then \( \nu \in (0, 2) \); since \( n\lambda - \nu \) is even, we have

\[
|D^{\lambda}| = D^{\lambda - 1} |D| = q_n \int_{0}^{\infty} \left[ \sum_{m=0}^{\infty} \left[ L_\nu^* M_{n+1} L_\nu^* M_{n+2} \cdots L_\nu^* M_{n+\mu} L_\nu^* \right] (t^{-1} - \mu) \frac{dt}{t} \right].
\]

by Lemma 2.1. Consequently

\[
C_{\lambda, n} = \tilde{C}_{\lambda, n} = -q_n \int_{0}^{\infty} \left[ \sum_{m=0}^{\infty} \left[ L_\nu^* M_{n+1} L_\nu^* M_{n+2} \cdots L_\nu^* M_{n+\mu} L_\nu^* \right] (t^{-1} - \mu) \frac{dt}{t} \right].
\]

Combining (2.16) and Lemma 2.2, we obtain (2.13).

If \( \mu \neq 0 \), then \( \nu \in (0, 2) \); since \( n\lambda + 1 - \mu \) is even, we have

\[
H |D^{\lambda}| = H |D^{\lambda - 1} |D| = q_n \int_{0}^{\infty} \left[ \sum_{m=0}^{\infty} \left[ L_\nu^* M_{n+1} L_\nu^* M_{n+2} \cdots L_\nu^* M_{n+\mu} L_\nu^* \right] (t^{-1} - \mu) \frac{dt}{t} \right].
\]

by Lemma 2.1. Thus

\[
\tilde{C}_{\lambda, n} = \frac{1}{t} q_n \int_{0}^{\infty} \left[ \sum_{m=0}^{\infty} \left[ L_\nu^* M_{n+1} L_\nu^* M_{n+2} \cdots L_\nu^* M_{n+\mu} L_\nu^* \right] (t^{-1} - \mu) \frac{dt}{t} \right].
\]

Combining (2.18) and Lemma 2.2 yields (2.14).

3. Reduction to estimates in the upper half-plane. By Lemma 2.3, \( C_{\lambda, n} \) and \( \tilde{C}_{\lambda, n} \) may be written as sums of symmetric multilinear operators of the form

\[
S(a_1, \ldots, a_n) = \int_{0}^{\infty} \left[ (t^{-1} - \mu) \right] M_{a_1}(a_1, \ldots, a_n) \frac{dt}{t}
\]

where, for \( f \in L^2(R) \),

\[
M_{a_1}(a_1, \ldots, a_n) f = \sum_{\sigma \in S_n} \sigma(a_1, a_2, \ldots, a_n) f(x_1, x_2, \ldots, x_n) \in [P, Q],
\]

with \( X_1, X_2, \ldots, X_{n+1} \in [P, Q] \). We aim to show that \( S(a_1, \ldots, a_n) f \) satisfies the estimate

\[
\|S(a_1, \ldots, a_n) f\|_2 \leq K(n) \left( \prod_{j=1}^{n} \|x_j\| \right) \|f\|_2
\]

where \( K(n) \) is a constant depending only upon \( n \). Our Main Theorem is an immediate consequence of this.

Let \( R_n^2 = R \times (0, \infty) \), and let \( \|f\|_2 \) denote the norm on \( L^2(R_n^2, dx dt/t) \). In this section, we contend that (3.3), and hence the Main Theorem, are consequences of the following:

**Main Lemma.** With notation as above, and under the hypotheses of the Main Theorem, there exists a constant \( K \) independent of \( A_1, \ldots, A_n, f \), such that, for \( X \in [P, Q] \),

\[
\|f(t^{-1} - \mu) X M_{a_1} X_{a_2} M_{a_3} \cdots X_{a_n} M_{a_{n+1}} f\|_2 \leq K \left( \prod_{j=1}^{n} \|x_j\| \right) \|f\|_2.
\]

We (convene that, for \( n = 1 \), \( M_{a_1} \) is simply \( P \) or \( Q \)).

In the interest of simplicity we restrict our attention to the case \( n = 2 \); the general case is similar. For notational ease we write \( a_1 = a, a_2 = b, X_1 = X, X_2 = Y, X_3 = Z \). Abusing notation in the usual way, we do not distinguish between \( a \) and \( M_{a_1} \), \( b \) and \( M_{a_2} \), \( P \) and \( M_{a_1} \). We are interested in estimating the \( L^2 \) norm of

\[
S(a, b) f = \int_{0}^{\infty} \left[ \sum_{m=0}^{\infty} \left[ L_\nu^* M_{n+1} L_\nu^* M_{n+2} \cdots L_\nu^* M_{n+\mu} L_\nu^* \right] (t^{-1} - \mu) \frac{dt}{t} \right].
\]

We claim, first of all, that an expression of the form (3.5) can be written as a sum of expressions of the following types:

\[
L(a, b) f = \int_{0}^{\infty} \left[ (Q, X, aY, bZ) + Q, X, bY, aZ \right] (t^{-1} - \mu) \frac{dt}{t},
\]

\[
R(a, b) f = \int_{0}^{\infty} \left[ (X, aY, bZ) + X, bY, aZ \right] (t^{-1} - \mu) \frac{dt}{t},
\]

\[
I(a, b) f = \int_{0}^{\infty} \left[ (X, aW, bZ) + X, bW, aZ \right] (t^{-1} - \mu) \frac{dt}{t},
\]

where \( W, X, Y, Z \in [P, Q] \), and the \( X, Y, Z \) occurring in (3.6)–(3.8) need not be the same as those occurring in (3.5).
We can easily compute the $L^2$ norms of (3.6)–(3.8) by duality. If $f, g$ are complex-valued functions in $L^2(\mathbb{R})$, let us define the (real) inner product of $f$ and $g$ by setting

$$\langle f | g \rangle = \int_{\mathbb{R}} f(x)g(x)\,dx.$$  

With respect to this inner product, $P_0^* = P$, $Q_0^* = -Q$, and multiplication operators are selfadjoint. Let us compute the norm of the operator $L(a, b)$ by duality; it is equal to

$$\sup_{1 < t < \infty} |\langle L(a, b) f | g \rangle|.$$  

Now note that

$$|\langle L(a, b) f | g \rangle| = \int_0^\infty \langle Q_1(X, aY, bZ, f + X, bY, aZ, f) | g \rangle \,dt_{t^{-\frac{1}{2}}}|\langle Q_1(g) | g \rangle| \,dt = \int_0^\infty \langle Q_1(X, aY, bZ, f + X, bY, aZ, f) | g \rangle \,dt_{t^{-\frac{1}{2}}}.$$  

$$\leq \int_0^\infty \langle Q_1(Y, aZ, f) | g \rangle \,dt_{t^{-\frac{1}{2}}}.$$  

$$\leq \sup_{1 < t < \infty} |\langle L(a, b) f | g \rangle|.$$  

where we have used the fact that $Q_0^* = -Q$, together with the Schwarz inequality and the triangle inequality. An application of the Plancherel theorem shows that

$$\|Q_1(g)\|_2 \leq \frac{1}{\sqrt{2}} \|g\|_2.$$  

(see [3], Proposition 4). Thus estimating the operator norm of $L(a, b)$ is reduced to estimating

$$|\langle Q_1(Y, aZ, f) | g \rangle| \quad \text{and} \quad |\langle Q_1(Y, aZ, f) | g \rangle|.$$  

The problem of estimating the operator norm of $R(a, b)$ is completely analogous, in view of the fact that $R(a, b)$ and $L(a, b)$ are "essentially" adjoint to one another.

It remains to estimate the operator norm of $I(a, b)$. We have

$$|\langle I(a, b) f | g \rangle| = \int_0^\infty \langle X, aW, Y, bZ, f + X, bW, aZ, f | g \rangle \,dt_{t^{-\frac{1}{2}}}.$$  

$$\leq \int_0^\infty \langle X, aW, Y, bZ, f + X, bW, aZ, f | g \rangle \,dt_{t^{-\frac{1}{2}}}.$$  

$$\leq \int_0^\infty \langle X, aW, Y, bZ, f | g \rangle + \langle X, bW, aZ, f | g \rangle \,dt_{t^{-\frac{1}{2}}}.$$  

If we let $\lambda_1 = \delta$, $\lambda_2 = \varepsilon$, then $(t^{-\frac{1}{2}})^2 = t^{1-\delta}, t^{-\frac{1}{2}}$, and, by the Schwarz inequality, we obtain

$$\langle f | g \rangle \leq \left( \int_0^\infty \langle X, aW, Y, bZ, f | g \rangle \,dt_{t^{-\frac{1}{2}}} \right)^{\frac{1}{2}} \left( \int_0^\infty \langle X, aW, Y, bZ, f | g \rangle \,dt_{t^{-\frac{1}{2}}} \right)^{\frac{1}{2}}.$$  

Hence the problem of estimating the operator norm of $I(a, b)$ is reduced to that of estimating expressions such as

$$\left( \int_0^\infty \|Y, aZ, f\|_2^2 \right)^{\frac{1}{2}} \left( \int_0^\infty \|Y, bZ, f\|_2^2 \right)^{\frac{1}{2}}.$$  

where $Y, Z \in \{P, Q\}$.

Thus it remains for us to establish our claim that any expression of the form (3.5) may be written as a sum of expressions of the form (3.6)–(3.8). We shall make use of the following identities:

**Lemma 3.1.**

(a) $P_0 = P_0^2 + Q_0^2$.

(b) $\frac{d}{\partial t} P = -2Q^2$.

(c) $\frac{d}{\partial t} Q = 2P, Q = -Q$.

**Proof.** Identities (b) and (c) are given in Proposition 2 of [3]. To prove (a), note that the symbol of $P_0^2 + Q_0^2$ is given by

$$\left( 1 + \frac{1}{t^2} \right)^{\frac{1}{2}}.$$  

which is the symbol of $P_0$. \[\Box\]

To prove our claim, we consider various cases, corresponding to the various possible values of $X, Y, Z$ and $Z$ in (3.5):

**Case 1:** $Y = P$. In this case,

$$I(a, b, f) = \int_0^\infty \langle X, aP, bZ, f + X, bP, aZ, f | g \rangle \,dt_{t^{-\frac{1}{2}}}.$$  

(3.18)
We may use Lemma 3.1(a) to write \( S(a, b) f \) as the sum of \( I_0(a, b) f \) + \( I_1(a, b) f \), where, for \( j = 0 \) or \( 1 \),
\[
(3.19) \quad I_j(a, b) f = \int_0^\infty \left( X(a, b) Y_j(a, b) Z_j(a, b) f(t^{-1}) \right) d\tau
\]
with \( Y_0 = P, \ Y_1 = Q \). \( I_0(a, b) \) and \( I_1(a, b) \) have the same structure as \( I(a, b) \).

Case 2: \( X = Y = Q, \ Z = P. \) In this case,
\[
(3.20) \quad S(a, b) f = \int_0^\infty (Q, a, b, P, b, a) f(t^{-1}) d\tau
\]
we integrate by parts, using Lemma 3.1(b),(c). Let \( du = t^{-2 \lambda} dt \) and \( v = (Q, a, b, P, b, a) f \); we obtain
\[
(3.21) \quad S(a, b) f = \frac{1}{1 - \lambda} L_2(a, b) f + \frac{1}{1 - \lambda} I_2(a, b) f + \frac{1}{1 - \lambda} R_2(a, b) f
\]
where
\[
(3.22) \quad L_2(a, b) f = \int_0^\infty (Q, P, a, b, P, b, a) f(t^{-1}) t^{-2 \lambda} d\tau
\]
\[
(3.23) \quad I_2(a, b) f = \int_0^\infty (Q, a, b, P, b, a, a) f(t^{-1}) t^{-2 \lambda} d\tau
\]
\[
(3.24) \quad R_2(a, b) f = \int_0^\infty (Q, a, b, a, P, b, a) f(t^{-1}) t^{-2 \lambda} d\tau
\]
Consequently, we have
\[
(3.25) \quad S(a, b) f = \frac{1}{1 - \lambda} L_2(a, b) f + \frac{1}{1 - \lambda} I_2(a, b) f + \frac{1}{1 - \lambda} R_2(a, b) f
\]
where \( L_2(a, b), I_2(a, b), \) and \( R_2(a, b) \) have the same structure as \( L(a, b), \ I(a, b), \) and \( R(a, b) \) respectively.

Case 3: \( X = P, \ Y = Z = Q. \) This case is essentially adjoint to Case 2. An analogous integration by parts shows that \( S(a, b) \) is again expressible as a sum of operators of the form \( L(a, b), \ I(a, b), \) and \( R(a, b) \).

Case 4: \( X = Y = Z = Q. \) In this case
\[
(3.26) \quad S(a, b) f = \int_0^\infty (Q, a, b, Q, b, a) f(t^{-1}) t^{-2 \lambda} d\tau
\]
Once again, we integrate by parts, using Lemma 3.1(c), letting \( du = t^{-1 - 2 \lambda} dt \), \( v = (Q, a, b, Q, b, a) f \). In this manner we obtain
\[
(3.27) \quad S(a, b) f = \frac{3}{2 - 2 \lambda} L_2(a, b) f - \frac{1}{1 - \lambda} L_2(a, b) f + \frac{1}{1 - \lambda} R_2(a, b) f
\]
where
\[
(3.28) \quad L_2(a, b) f = \int_0^\infty (Q, P, a, b, Q, P, P, b, a) f(t^{-1}) t^{-1 - 2 \lambda} d\tau
\]
\[
(3.29) \quad I_2(a, b) f = \int_0^\infty (Q, a, b, P, b, P, b, a) f(t^{-1}) t^{-1 - 2 \lambda} d\tau
\]
\[
(3.30) \quad R_2(a, b) f = \int_0^\infty (Q, a, b, P, P, b, a) f(t^{-1}) t^{-1 - 2 \lambda} d\tau
\]
Consequently, provided \( \lambda \neq \frac{1}{2} \), we have
\[
(3.31) \quad S(a, b) f = \frac{2}{2 \lambda - 1} L_2(a, b) f + \frac{2}{2 \lambda - 1} I_2(a, b) f + \frac{2}{2 \lambda - 1} R_2(a, b) f
\]
Note that, if \( \lambda = \frac{1}{2} \), the operator \( \tilde{C}_{1,2} = 0 \); moreover, consideration of the formula (2.13) shows that, regardless of the value of \( \lambda \), the operator (3.26) does not arise in the expansion of \( C_{1,2} \). Thus, whenever the operator (3.26) arises, it can be expressed as a sum of operators having the same form as \( L(a, b), \ I(a, b), \) and \( R(a, b) \).

This establishes our claim, and thereby shows that, for \( n = 2 \), the proof of the Main Theorem can be reduced to proving the Main Lemma.

We make a few remarks concerning the case of more general \( n \). Analogous arguments, making use of Lemma 3.1, can be used to show that, in general, any operator of the form (3.1) arising in the expansion of \( C_{1,4} \), or \( \tilde{C}_{1,4} \), may be expressed as the sum of operators of the form
\[
(3.32) \quad I(a_1, \ldots, a_n) = \int_0^\infty \prod_{j=1}^n Y_{j,a_j} M_{a_j} \ldots Y_{n,a_n} M_{a_n} f(t^{-1}) t^{-2 \lambda} d\tau
\]
in which, for some \( j \in \{2, 3, \ldots, n\}, Y_{j,a_j} \in \{P_j, Q_j, P, Q\}, \) and for all other values of \( j, \ Y_j \in \{P, Q\}; \) and operators of the form
\[
(3.33) \quad \int_0^\infty (t^{-1 - 2 \lambda} P, M_{a_1} \ldots, a_n) d\tau \quad \text{or} \quad \int_0^\infty (t^{-1 - 2 \lambda} P, a_1, \ldots, a_n) d\tau
\]
with \( M_{a_j} \) defined as in (3.2). Duality arguments may then be used to show that the Main Theorem follows from the Main Lemma in the case of \( n \geq 3 \).

4. The tent space. In order to prove the Main Lemma, we will make use of certain ideas from the theory of tent spaces of Coifman, Meyer, and Stein [43] together with facts from Hardy space theory. We begin with some definitions.

**Definition 4.1.** Let \( f : \mathbb{R}^n \rightarrow C \) be a measurable function with respect to the measure \( dx dt \).
(a) The square area function of \( f \), \( S(f) \), is given by

\[
S(f)(x) = \left[ \int_{|x-y|<1} |f(y)|^2 2^{s-1} dy \right]^{1/2}.
\]

(b) We say that \( f \) is an element of the tent space \( T_{2,1} \) if and only if \( S(f) \in L^1() \). We define

\[
\|f\|_{T_{2,1}} = \|S(f)\|_1.
\]

(c) We say that \( f \) is an atom for \( T_{2,1} \) if and only if there is a finite interval \( I \subseteq \mathbb{R} \) such that \( f \) is supported in \( I = [x, y] \in \mathbb{R} \): \( [x-r, x+r] \subseteq I \) and

\[
\left[ \int_I |f(y)|^2 2^{s-1} dy \right]^{1/2} \leq |I|^{-1/2}.
\]

The set \( I \) is called the tent based on \( I \).

Coifman, Meyer, and Stein have obtained the following useful characterization of \( T_{2,1} \) (see [4], Lemmas 1 and 2 and Theorem 2):

**Proposition 4.1.** Let \( f : \mathbb{R}^2 \rightarrow C \) be a measurable function with respect to \( dx dt \).

(a) \( \|S(f)\|_1 = \sqrt{2} \|f\|_1 \) : moreover, if \( f \) is a \( T_{2,1} \)-atom, then \( f \in T_{2,1} \) and \( \|f\|_{T_{2,1}} \leq \sqrt{2} \).

(b) \( f \) is an element of \( T_{2,1} \) if and only if there is a sequence \( \{a_n\} \) of \( T_{2,1} \)-atoms and a sequence \( \{\lambda_n\} \) of complex coefficients such that

\[
f = \sum_{n=1}^{\infty} \lambda_n a_n,
\]

\[
\sum_{n=1}^{\infty} |\lambda_n| < +\infty.
\]

Moreover, the \( T_{2,1} \) norm of \( f \) is equivalent to the infimum over all representations (4.4) of the sums (4.5).

There is an intimate relation between the space \( T_{2,1} \) and the Hardy space \( H^1 \), defined in terms of atoms. We shall recall a few theorems and definitions from Hardy space theory (see [10], section 2).

**Definition 4.2.** Suppose \( q \geq 1 \), \( s \) is a nonnegative integer, \( \sigma > \max \{s, 0\} \), and \( x_0 \in \mathbb{R} \). Let \( f \) be a locally integrable function on \( \mathbb{R} \), and let \( f^w \) be a \( w \) function on \( \mathbb{R} \), where \( f^w(x) = f(x)|x-x_0|^w \).

(a) \( f \) is called a \( (1, q, s) \)-atom centered at \( x_0 \) if and only if \( f \) is supported in a finite interval \( I \) centered at \( x_0 \), and

\[
\|f\|_q \leq |I|^{1/q-1},
\]

\[
\int_I f(x) x^s dx = 0 \quad \text{for all nonnegative integers} \quad j \leq s.
\]
Suppose, moreover, that $0 < \epsilon < \beta$.

(a) If $f_j$ is a $T_{2,1}$-atom, then $g$ is a $(\epsilon, 2, 0, \epsilon)$-molecule, and

$$\Omega(g) \leq C(\epsilon, \beta) B$$

where $C(\epsilon, \beta)$ is a constant depending only upon $\epsilon, \beta$.

(b) If $f_j$ is any $T_{2,1}$ function, then $g \in H^1$, and

$$\|g\|_{H^1} \leq C(\epsilon, \beta) B \|f_j\|_{T_{2,1}}$$

where $C(\epsilon, \beta)$ depends only upon $\epsilon, \beta$.

Proof: Note first that (b) is immediate from (a) by Proposition 4.1 and Proposition 4.2. Thus it suffices to prove (a).

We begin by observing that, since $g$ is integrable, we must have

$$\int g(x) \, dx = 0 \tag{4.19}$$

because $\psi(0) = 0$. The integrability of $g$ will follow from our estimate of $\Omega(g)$.

Suppose that $f_j$ is a $T_{2,1}$-atom supported in a tent $I$, and let $x_0$ be the center of $I$. We shall show that $g$ is a $(1, 2, 0, \epsilon)$-molecule centered at $x_0$. If $g = 2$ in Definition 4.2, then $\omega = \epsilon + 1/2$, $\epsilon/\omega = 2(2\epsilon + 1)^{-1}$, \(1 - \epsilon/\omega = (2\epsilon + 1)^{-1} \). Thus

$$\Omega(g) = \|g\|_{L^2(2\epsilon + 1)^{-1}} \|g\|_{L^2(2\epsilon + 1)^{-1}}^{-1} \tag{4.20}$$

where

$$\|g\|_{L^2(2\epsilon + 1)^{-1}}^{-1} = \left[ \int_{x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \right]^{1/(2\epsilon + 2)} \tag{4.21}$$

We compute $\|g\|_{L^2(2\epsilon + 1)^{-1}}$ by duality. For $h \in L^2(\mathbb{R})$, we have

$$\left( \langle h, g \rangle \right) = \left[ \left( \hat{h} * \hat{f}_j \right) dt \right] \left[ \left( \hat{f}_j * \hat{h} \right) dt \right] \tag{4.22}$$

$$= \left[ \left( \hat{h} * \hat{f}_j \right) dt \right] \left[ \left( \hat{h} * \hat{f}_j \right) dt \right] \tag{4.23}$$

$$\leq \left( 2\epsilon \right)^{-1} \left( \int \left( \hat{h}(-t) \hat{f}_j(x) \right) \, dx \right) \left( \int \left( \hat{f}_j(-t) \hat{h}(x) \right) \, dx \right) \tag{4.24}$$

$$\leq \left( 2\epsilon \right)^{-1} \left( \int \left( \hat{h}(-t) \hat{f}_j(x) \right) \, dx \right) \left( \int \left( \hat{f}_j(-t) \hat{h}(x) \right) \, dx \right) \tag{4.25}$$

$$\leq B \|h\|_2 B \|f_j\|_2 \leq B \|h\|_2 \|f_j\|_2^{1/2} \leq B \|h\|_2 \|f_j\|_2^{1/2} \tag{4.26}$$

where we have used Plancherel's Theorem and (4.3). Thus

$$\|g\|_{L^2(2\epsilon + 1)^{-1}} \leq B \|g\|_{L^2(2\epsilon + 1)^{-1}} \leq B \left[ \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \right]^{1/(2\epsilon + 2)} \tag{4.27}$$

Furthermore,

$$\left[ \int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \right]^{1/(2\epsilon + 2)} \tag{4.28}$$

where, letting $|I|$ denote the Lebesgue measure of $I$,

$$I_1 = \int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \tag{4.29}$$

$$\leq 10^{2\epsilon + 1} |I|^{2\epsilon + 1} \|g\|_2 \leq 10^{2\epsilon + 1} B |I|^{2\epsilon} \tag{4.30}$$

$$I_2 = \int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \tag{4.31}$$

For $|x-x_0| \geq 10|I|$, we have

$$\|g\|_{L^2(2\epsilon + 1)^{-1}} \leq B \left[ \int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \right]^{1/(2\epsilon + 2)} \tag{4.32}$$

where $C(\beta)$ is a constant depending on $\beta$. Now

$$\int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \tag{4.33}$$

$$\leq \left[ \int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \right]^{1/2} \left[ \int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \right]^{1/2} \tag{4.34}$$

Thus

$$\|g\|_{L^2(2\epsilon + 1)^{-1}} \leq B \left[ \int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx \right]^{1/(2\epsilon + 2)} \tag{4.35}$$

where $C(\beta)$ is a constant depending on $\beta$. Moreover,

$$\int_{x-x_0} \int \left( \frac{g(x)^2}{|x-x_0|^2} \right) dx = (2\beta - 2) \|g\|_{L^2(2\epsilon + 1)^{-1}} \tag{4.36}$$

whence, by (4.26), (4.29), and (4.30),

$$\|g\|_{L^2(2\epsilon + 1)^{-1}} \leq C(\epsilon, \beta) B^2 \|f_j\|_{L^2(2\epsilon + 1)^{-1}} \tag{4.37}$$

Thus, by (4.21), (4.24), (4.25), and (4.31),

$$\|g\|_{L^2(2\epsilon + 1)^{-1}} \leq C(\epsilon, \beta) B^2 \|f_j\|_{L^2(2\epsilon + 1)^{-1}} \tag{4.38}$$
so that, by (4.20), (4.23), and (4.32),
\[(4.33)\] 
\[\Omega(g) \leq C\sigma B.\]
This completes the proof of (a).  

We are now in a position to prove the following useful generalization of Lemma 9 of [4]:

**Proposition 4.4.** There is a constant \(K\) such that, if
(a) \(\varphi_1, \varphi_2\) are two functions such that, setting \(\varphi_j' = t^{-1} \varphi_j(t^{-1})\), we have
\[(4.34)\] 
\[|\varphi_j(x)| \leq C_j(1 + |x|^2)^{-1},\]
where \(C_j\) is independent of \(x \in \mathbb{R}\) for \(j = 1, 2\);
(b) \(g(x, t) = \tilde{g}(x)\) satisfies \(\gamma = \sup_{t \geq 0} |g(t)| \leq L^2(\mathbb{R})\);
(c) \(f(x, t) = f_t(x)\) satisfies \(f_t \in L^2(\mathbb{R}^2, dx dt)\);
(d) \(M(x, t) = M_t(x)\) satisfies \(M_t = \sup_{t \leq 0} |M_t(x)| < \infty\),
then \(F(x, t) = (\varphi_1 \ast f_t)(x)(\varphi_2 \ast g_t)(x)M_t(x)\) defines a function in \(T_{2, 1}\) with norm dominated by \(\mathcal{K}_C C_2 M_1 \|f\|_2 \|g\|_2\).

**Proof.** Let \(\varphi\) denote the Poisson kernel on \(\mathbb{R}\); i.e.,
\[(4.35)\] 
\[\varphi(x) = \frac{1}{\pi} \frac{t}{t^2 + x^2}.\]
Then, for \(j = 1, 2\) and \(x \in \mathbb{R}\),
\[(4.36)\] 
\[|\varphi_j(x)| \leq \pi C_j|\varphi_j(x)|.\]
Thus
\[(4.37)\] 
\[|F(y, t)| \leq \pi^2 C_1 C_2 M_1 (\varphi_1 * f_t)(\varphi_2 * g_t)(y).\]
If \(|x - y| \leq t\), it is easily seen that
\[(4.38)\] 
\[(\varphi_1 * f_t)(y) \leq S_{\varphi_1}(x), \quad (\varphi_2 * g_t)(y) \leq S_{\varphi_2}(x)\]
where \(\ast\) denotes the Hardy–Littlewood maximal function. Thus the square area function satisfies
\[(4.39)\] 
\[S(F)(x) \leq 25\pi^2 C_1 C_2 M_1 \left[ \frac{1}{|x - t|} \left( \int_{|x - t|}^{x} |f_t(y)|^2 dy dt \right)^{1/2} \right]^{1/2} \leq 25 \sqrt{2\pi} C_1 C_2 M_1 \gamma(x) \left( \int_{0}^{\infty} \left( f_t(y) \right)^{2} \frac{dy dt}{t} \right)^{1/2}
\]
whence
\[(4.40)\] 
\[\|S(F)\|_1 \leq 25 \sqrt{2\pi} C_1 C_2 M_1 \gamma \|f_t\|_2 \|g_t\|_2\]
by the Schwarz inequality. Since the Hardy–Littlewood maximal operator is bounded on \(L^2(\mathbb{R})\), we obtain, by Fubini’s theorem,
\[(4.41)\] 
\[\|S(F)\|_1 \leq KC_1 C_2 M_1 \|f\|_2 \|g\|_2.\]

The following generalization of Theorem 4 of [4] will be crucial to our proof of the Main Lemma:

**Proposition 4.5.** Let \(\beta \in (0, 1)\), and suppose that
(a) \(\beta_1, \beta_2\) is an \(L^1\) function and \(C_n T_j\) are constants such that \(\sup \beta_j \leq [-T_j, T_j]\), \(\beta_j \in C^2(\mathbb{R})\), and, for all \(t \in \mathbb{R}\),
\[(4.42)\] 
\[|\beta_j''(t)| \leq C_j T_j^\beta\]
where the superscript denotes the derivative of order \(s\), and \(s \in [0, 1] \cup \{0\}\);
(b) \(M(x, t) = M_t(x)\) is a function and \(T_j \geq 1\) a constant such that \(\sup M_t \leq [-T_j, T_j]\) and
\[(4.43)\] 
\[M_t = \sup_{t \geq 0} |M_t(x)| < \infty\]
(c) \(g(x, t) = g_t(x)\) satisfies \(\gamma = \sup_{t \geq 0} |g_t(x)| \leq L^2(\mathbb{R})\);
(d) \(f(x, t) = f_t(x)\) satisfies \(f_t \in L^2(\mathbb{R}^2, dx dt)\).
Furthermore, set \(\beta_j(t) = t^{-1} \beta_j(t^{-1})\) for \(j = 1, 2\) and define
\[(4.44)\] 
\[G = H |D|^{\beta_1} \int_{0}^{\infty} M_t(\varphi_1 * f_t)(\varphi_2 * g_t)^{\beta_2 - 1} dt.\]
Then \(G \in H^1(\mathbb{R})\), and
\[(4.45)\] 
\[\|G\|_1 \leq KC_1 C_2 (T_0 + T_1 + T_2)^\beta M_1 \|f\|_2 \|g\|_2\]
where \(K_\beta\) is a constant depending only upon \(\beta\).

**Proof.** Let \(S = T_0 + T_1 + T_2\). For \(j = 0, 1, 2\), let \(\tilde{\beta}_j = \varphi_{2, \beta_j}\); i.e.,
\[(4.46)\] 
\[\tilde{\beta}_j(x) = S^{-1} \varphi_j(s^\beta), \quad \tilde{\beta}_j(\xi) = \tilde{\beta}_j(s^\beta).\]
Note that \(\sup \tilde{\beta}_j \leq [-1, 1]\). For \(t > 0\), we let \(\beta_j(t) = t^{-1} \tilde{\beta}_j(t^{-1})\).
Let \(\eta \in C_0^\infty(\mathbb{R})\) be an even nonnegative function, supported in \([-2, 2]\) and identically one on \([-1, 1]\). Define \(\Psi\) to be the function for which
\[(4.47)\] 
\[\Psi(\xi) = -\text{sign}(\xi) |\xi|^\beta \eta(\xi).\]
It is not difficult to show that there is a constant \(B\) depending upon \(\beta\) for which \(\Psi\) is a \((B, \beta, \beta)\)-psi function. It is easily seen that
\[(4.48)\] 
\[G(x) = S^\beta \int_{0}^{\infty} \Psi_{012} M_t(\varphi_1 * f_t)(\varphi_2 * g_t) (x) \frac{dt}{t},\]
where
\[\Psi_{012}(\eta_1, \tilde{\beta}_1, \tilde{\beta}_2) = \eta_1 \tilde{\beta}_1 \tilde{\beta}_2 (x), \quad \Psi_{12} = \eta_1 \tilde{\beta}_1 \tilde{\beta}_2 (x) \frac{dt}{t}.\]
Now note that
\begin{equation}
(4.49) \quad \| \Phi_{1,2} \ast f_{0}\|_{2}^{2} = (2\pi)^{1/2} \left[ \int_{\mathbb{R}^{2}} \| \Phi_{1}(t, \lambda) \Phi_{2}(\lambda) \|^{2} \frac{d\lambda dt}{t} \right]^{1/2} \leq C_{1} \| f \|_{2}^{2}.
\end{equation}
Moreover,
\begin{equation}
(4.50) \quad \| \sup \| \Phi_{1,2} \ast g \|_{2} \|_{1} \leq \| \sup \| \Phi_{1} \ast g \|_{2} \|_{1} \leq 4C_{2} \| g \|_{2}
\end{equation}
by an application of Theorem 2, Chapter 3 of [8].

It is easily seen that there is a constant $C_{3} > 0$ for which
\begin{equation}
(4.51) \quad \| f(x) \|_{2} \leq C_{3} (1 + |x|)^{-1}
\end{equation}
Thus we may apply Proposition 4.4 to obtain
\begin{equation}
(4.52) \quad \| M_{f_{0}} \ast (\eta_{1} \ast \Phi_{1,2} \ast f_{0}) \ast (\eta_{2} \ast \Phi_{2,2} \ast g_{0}) \|_{2,1} \leq KC_{1} C_{3} C_{2} M_{\ast} \| f \|_{2} \| g \|_{2}
\end{equation}
where $K$ is a purely geometric constant. The estimate (4.45) then follows from (4.48), (4.52), and Proposition 4.3.

5. $H^{1}$ estimates. We now turn to the proof of the Main Lemma, which involves an estimate in $L^{2}(\mathbb{R}^{2}, dx dt)$ which we can obtain by duality. Note that
\begin{equation}
(5.1) \quad \| (t^{-1+s} f)(x, t) \|_{L^{2}} = \sup_{\|h\|_{2} = 1} \left\| \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \langle M_{\eta_{1}} M_{\eta_{2}} \ast f \rangle \right) \frac{dt}{t} \right)^{1/2} \right\|
\end{equation}
where
\begin{equation}
(5.2) \quad (M_{\eta_{1}} f)(x) = \mathcal{H}(1-\lambda) f(\lambda)\|_{L^{2}}
\end{equation}
Thus the Main Lemma is proved once we have estimates of the form
\begin{equation}
(5.3) \quad \| M_{\ast} \ast (X, x) \|_{L^{2}} \leq K \left\| \langle \mathcal{F}(X) \rangle \|_{L^{2}} \right\| f \|_{L^{2}}
\end{equation}
\begin{equation}
(5.4) \quad \| M_{\ast} \|_{L^{1}} \leq K \left\| \langle \mathcal{F}(X) \rangle \|_{L^{1}} \right\| f \|_{L^{1}}, \quad n \geq 2,
\end{equation}
where $C$ is a constant depending only upon $\lambda$, $K$ is independent of $A_{1}, \ldots, A_{n}$, $h$, and $f$, and $Y$ is equal to $P$ or $Q$. In this section, we will use results from the previous section, together with identities involving $P$ and $Q$, to establish (5.3), and (5.4) for $n = 2, 3$. Finally, we will indicate how (5.4) can be obtained for $n > 4$.

We proceed via a series of lemmas.

**Lemma 5.1.** Suppose that $s \in (0, 1)$, $A \in \mathcal{S}(R) \cap \text{Lip}_{2}(\mathbb{R})$, $a = A^{\ast}$, and $X \in \mathcal{P}(\mathbb{R})$. Then, for all $t > 0$,
\begin{equation}
(5.5) \quad \| t^{1-s} X \|_{\infty} \leq C(1 + |x|)^{-1}
\end{equation}
where $\| \ast \|_{2}$ denotes the norm in $\text{Lip}_{2}$.

**Proof.** Since $tDP = Q_{r}$ and $tDQ_{r} = R_{r} - P_{r}$, we have
\begin{equation}
(5.6) \quad t^{1-s} \langle P_{r} \ast f \rangle = t^{1-s} f_{0} \ast (\eta_{1} \ast \Phi_{1,2} \ast f_{0}) \leq KC_{1} C_{3} C_{2} \| f \|_{2} \| f \|_{2}
\end{equation}
where $K$ is a purely geometric constant. The estimate (4.45) then follows from (4.48), (4.52), and Proposition 4.3.

Now, $\delta_{r}(0) = 0$, $\delta_{r}(0) = 1$, and $\delta_{r}(x) = p_{r}(x) = \frac{1}{2} e^{-|x|}$, so
\begin{equation}
(5.7) \quad \| t^{1-s} P_{r} \ast f \|_{L^{2}} \leq C(1 + |x|)^{-1}
\end{equation}
where $C$ is a constant depending only upon $\lambda$, $K$ is independent of $A_{1}, \ldots, A_{n}$, $h$, and $f$, and $Y$ is equal to $P$ or $Q$. In this section, we will use results from the previous section, together with identities involving $P$ and $Q$, to establish (5.3), and (5.4) for $n = 2, 3$. Finally, we will indicate how (5.4) can be obtained for $n > 4$.

We proceed via a series of lemmas.

**Lemma 5.2.** Let $s \in (0, 1)$. Then $I_{s}(x)$ is properly contained in $\text{Lip}_{2}(\mathbb{R})$. Moreover, there is a constant $C_{3}$ such that if $A \in \mathcal{I}_{s}(\text{BMO})$ and $x = |D|^{1/2}$, then
\begin{equation}
(5.8) \quad \| A \|_{L^{2}} \leq C_{3} \| A \|_{L^{2}}
\end{equation}

**Proof.** See the proof of Theorem 3.4 of [9].

**Lemma 5.3.** Let $p(x) = \frac{1}{2} e^{-|x|}$ and $q(x) = \text{sign} p(x)$, as above. Then there are sequences $(\eta_{x})_{x \in \mathbb{R}}$, $(\eta_{x})_{x \in \mathbb{R}} \in \mathcal{S}(\mathbb{R})$ having the following properties:
(a) $\eta_{x}$ and $\overline{\eta_{x}}$ are supported in $A_{k}$, where $A_{0} = [-1, 1]$ and $A_{k} = \{ x : 2^{k+1} < |x| < 2^{k} \}$ for $k \geq 1$.
(b) $\overline{\eta_{x}} = \sum_{k=0}^{\infty} \eta_{x}$.
(c) There is a constant $C$ such that for all nonnegative integers $k$ and for $j = 0, 1, 2$, we have
\begin{align}
|\mathbf{p}^j_k (\xi)| &\leq C 2^{-\frac{j}{2} + \alpha_j}, \\
|\mathbf{q}^j_k (\xi)| &\leq C 2^{-\frac{j}{1 + \alpha_j}}, \\
|\mathbf{p}_0 (x)| &\leq 2C \inf \{2^{-k}, 8^{-k} |x|^{-2}\}, \\
|\mathbf{q}_0 (x)| &\leq 2C \inf \{14^{-k} |x|^{-2}\}, \\
\|\mathbf{p}_k\|_1 &\leq 4C \cdot 4^{-k}, \\
\|\mathbf{q}_k\|_1 &\leq 4C \cdot 2^{-k},
\end{align}
where, in (5.11) and (5.12), the superscript denotes the derivative of order $j$.

Proof. The functions $\mathbf{p}_k$ and $\mathbf{q}_k$ are defined and discussed in [6], Section 3; properties (a) and (b) and inequalities (5.11) and (5.12) follow from that discussion. Inequalities (5.11) through (5.15) can be shown via direct computation using the inverse Fourier transform.

In what follows, let $\mathbf{p}_k = t^{-\gamma} \mathbf{p}_k (t^{-\gamma})$ and $\mathbf{q}_k = t^{-\gamma} \mathbf{q}_k (t^{-\gamma})$ for each nonnegative integer $k$.

Lemma 5.4. For $\delta \in (0, 1)$ there is a constant $C_\delta$ such that for all $A \in \text{Lip}_1 (R)$, and for all $t > 0$,
\begin{align}
t^{-\delta}\|A - p_{\alpha_0} \ast A\|_w &\leq C_\delta \|A\|_w, \\
t^{-\delta}\|p_{\alpha_0} \ast A\|_w &\leq C_\delta 2^{-1 + \delta} \|A\|_w \quad \text{for } k > 0, \\
t^{-\delta}\|q_{\alpha_0} \ast A\|_w &\leq C_\delta 2^{-1 + \delta} \|A\|_w \quad \text{for } k \geq 0.
\end{align}

Proof. Note that, since $\mathbf{p}_0 (0) = 1$, we have
\begin{align}
|A(x) - p_{\alpha_0} \ast A(x)| &= \left| t^{-\gamma} \int p_0 \left[ \frac{x-y}{t} \right] |A(x) - A(y)| \, dy \right| \\
&\leq t^{\gamma} \|A\|_w \int |p_0 (t|z|)| |z|^\gamma \, dz \\
&\leq t^{\gamma} \|A\|_w \cdot 4C \int_0^\infty z^\gamma \, dz + \int_1^{\infty} z^\gamma \, dz \\
&= t^{\gamma} \|A\|_w \cdot 4C \frac{1}{1 - \delta^2},
\end{align}
where the second inequality follows from (5.13). This establishes (5.17). If $k \geq 1$, we have $\mathbf{p}_0 (0) = 0$, so that
\begin{align}
|A(x) - p_{\alpha_k} \ast A(x)| &= \left| t^{-\gamma} \int p_0 \left[ \frac{x-y}{t} \right] |A(x) - A(y)| \, dy \right| \\
&\leq t^{\gamma} \|A\|_w \int |p_0 (t|z|)| |z|^\gamma \, dz \\
&\leq t^{\gamma} \|A\|_w \cdot 4C \int_0^\infty z^\gamma \, dz + \int_1^{\infty} z^\gamma \, dz \\
&= t^{\gamma} \|A\|_w \cdot 4C \frac{1}{1 - \delta^2},
\end{align}
where $\gamma$ denotes the Hardy–Littlewood maximal function. The result follows from the $L^2$ boundedness of the Hardy–Littlewood maximal operator.

Lemma 5.6. Let $n$ be an integer greater than 1, let $Y \in \{P, Q\}$, and let $h_\gamma \in L^2 (R^n, dx \, dt)$. Under the hypotheses of Lemma 5.5, the functions $G$ and $A$...
defined by
\begin{align}
G &= H |D|^{1 - \lambda} \left\{ \int_0^1 [X, h_1] \cdot [Y, g_1] (t^{1 - \lambda_1}) dt \right\}, \\
A &= H |D|^{1 - \lambda} \left\{ \int_0^1 [X, h_1] \cdot [Y, g_1] (t^{1 - \lambda_1}) dt \right\}
\end{align}
are in $H^s$, and satisfy the estimates
\begin{align}
\|G\|_{H^s} &\leq C \left( \prod_{j=1}^{s} \|B\|_{L^2} \right) \|h\|_{H^s} \|g\|_{H^s}, \\
\|A\|_{H^s} &\leq K \|h\|_{H^s} \|g\|_{H^s}
\end{align}
where $K$ is a constant depending on $n$ and $\lambda_1, \ldots, \lambda_s$.

Proof. The proof is an application of Proposition 4.5. Notice that, for each $j \in \{1, 2, \ldots, n-1\}$,
\begin{align}
(t^{1 - \lambda_1}) [X, h_j] b_j &= \sum_{\lambda_0=0}^{n-1} (t^{1 - \lambda_1}) x_{j,\lambda_0} b_j
\end{align}
where $x_{j,\lambda_0} = p_{j,\lambda_0}$ or $q_{j,\lambda_0}$ according as $X_j = P$ or $Q$. In turn, we have
\begin{align}
(t^{1 - \lambda_1}) x_{j,\lambda_0} \star b_j &= \begin{cases} 
(t^{1 - \lambda_1}) (A - p_{j,\lambda_0} \star A) & \text{if } X_j = Q \text{ and } l_j = 0, \\
(t^{1 - \lambda_1}) (q_{j,\lambda_0} \star A) & \text{if } X_j = Q \text{ and } l_j \geq 1, \\
(t^{1 - \lambda_1}) (p_{j,\lambda_0} \star A) & \text{if } X_j = P \text{ and } l_j \geq 0.
\end{cases}
\end{align}

By Lemma 5.4, we have
\begin{align}
\sup_{t \geq 0} \| (t^{1 - \lambda_1}) x_{j,\lambda_0} \star b_j \| &\leq C_{j,\lambda_0} 2^{-l_j} \|B\|_{L^2}.
\end{align}
If we set
\begin{align}
M_1(l_1, \ldots, l_{n-1}) &= \sum_{j=1}^{n-1} (t^{1 - \lambda_1}) x_{j,\lambda_0} \star b_j, \\
M_2(l_1, \ldots, l_{n-1}) &= \sup_{t \geq 0} \| M_1(l_1, \ldots, l_{n-1}) \|_{L^2}
\end{align}
then we have
\begin{align}
\text{supp} M_1(l_1, \ldots, l_{n-1}) \subseteq [-r^{-1} 4^{2^{l_1} + \ldots + 2^{l_{n-1}}}, r^{-1} 4^{2^{l_1} + \ldots + 2^{l_{n-1}}}], \\
M_2(l_1, \ldots, l_{n-1}) &\leq C_{\lambda_1} r^{-1} 4^{(1 + 2^{l_1} + \ldots + 2^{l_{n-1}})} \|B\|_{L^2}.
\end{align}
Moreover, we have
\begin{align}
G &= \sum_{l_0=0}^{\infty} \sum_{l_1=0}^{\infty} \ldots \sum_{l_n=0}^{\infty} G(l_0, l_1, \ldots, l_n),
\end{align}
where
\begin{align}
G(l_0, l_1, \ldots, l_n) &= H |D|^{1 - \lambda} \left\{ \int_0^1 [x_{l_0, \lambda_0} \star h_0 \cdot M_1(l_0, l_1, \ldots, l_{n-1})] (t^{1 - \lambda_1}) dt \right\}.
\end{align}

Using Lemma 5.3(a), we may apply Proposition 4.5 with $\beta = 1 - \lambda_0$, to see that $G(l_0, l_1, \ldots, l_n) \in H^1(R)$, and
\begin{align}
\|G(l_0, l_1, \ldots, l_n)\|_{H^1} &\leq K_{\lambda_0} \left( \sum_{j=0}^{n-1} 2^{-l_j} \right) \|f\|_{H^1} \|g_0\|_{H^1} \\
&\leq K_{\lambda_1, \ldots, \lambda_n} \left( \sum_{j=0}^{n-1} 2^{-l_j} \right) \|f\|_{H^1} \|g_0\|_{H^1} \|h_0\|_{H^1}.
\end{align}
The estimate (5.28) now follows on combining (5.37) and (5.38).

The estimate (5.29) is still easier. Note that we may write
\begin{align}
A(l, m) &= \sum_{\lambda_0=0}^{\infty} \sum_{\lambda_1=\lambda_0}^{\infty} A(l, m),
\end{align}
where
\begin{align}
A(l, m) &= H |D|^{1 - \lambda} \left\{ \int_0^1 [(x_{l_0, \lambda_0} \star h_0) \cdot (x_{l_1, \lambda_1} \star g_1)] (t^{1 - \lambda_1}) dt \right\}.
\end{align}
By a simplification of the argument in Proposition 4.5, we obtain the estimate
\begin{align}
\|A(l, m)\|_{H^1} &\leq K_{\lambda_0} (2^{-l_0} \|h_0\|_{H^1} \|g_1\|_{H^1} \|f\|_{H^1})
\end{align}
combining (5.40) and (5.42) yields (5.29).

To complete the proof of the Main Lemma, we show how the estimate (5.4) can be obtained from Lemma 5.6. To do this, we need the following identities involving $P$ and $Q$:

**Lemma 5.7.** Let $f, g$ be functions in $\mathcal{S}'(R)$, possibly depending upon $t$. Let $D_t = \partial / \partial t$. We have
\begin{align}
(P_t(fg)) &= (P_t f) \cdot g - Q_t([P_t f] \cdot [D_t g]) - P_t([Q_t f] \cdot [D_t g]), \\
(Q_t(fg)) &= (Q_t f) \cdot g + P_t([P_t f] \cdot [D_t g]) - Q_t([Q_t f] \cdot [D_t g]).
\end{align}
\[ P_1Q_2g = \{P_1, f\} \cdot \{Q_1, g\} - Q_1(\{P_1, f\} \cdot \{R_1, g\}) - P_1(\{Q_1, f\} \cdot \{R_1, g\}). \]

(c) \[ P_1Q_2g = \{P_1, f\} \cdot \{Q_1, g\} - Q_1(\{P_1, f\} \cdot \{R_1, g\}) - P_1(\{Q_1, f\} \cdot \{R_1, g\}). \]

\[ P_1Q_2g = \{Q_1, f\} \cdot \{P_1, g\} + P_1(\{Q_1, f\} \cdot \{R_1, g\}) - Q_1(\{E_1, f\} \cdot \{R_1, g\}). \]

\[ Q_1P_2g = \{Q_1, f\} \cdot \{P_1, g\} + P_1(\{Q_1, f\} \cdot \{R_1, g\}) - Q_1(\{E_1, f\} \cdot \{R_1, g\}). \]

Proof. It is not difficult to show that

\[ L^2_f(g) = [L^2_f \cdot f] \cdot g \pm \iota L^2_f([L^2_f \cdot f] \cdot [D_1, g]) \]

where, as before, \( L^2 = P_1 \pm Q_1 = (I \mp iD_1)^{-1} \) (see Section 6 of [3]). From this it is easy to establish (a) and (b). Identities (c) and (e) follow from (a) and (b) by letting \([Q_1, g]\) play the role of \(g\); likewise, (d) and (f) follow from (a) and (b) by letting \([P_1, g]\) play the role of \(g\).

**Lemma 5.8.** Let \(a, b\) be functions in \( \mathcal{F}(R) \), possibly depending on \(u\). We have

(a) \[ Q_1aQ_2b + Q_1bQ_2a = R_1(\{Q_1, a\} \cdot \{Q_2, b\}) + R_1(\{P_1, a\} \cdot \{P_2, b\}). \]

(b) \[ P_1aQ_2b + P_1bQ_2a = R_1(\{Q_1, a\} \cdot \{Q_2, b\}) + R_1(\{Q_1, a\} \cdot \{P_2, b\}). \]

Proof. Since \( Q_1 = D_1P_1 \) and \( R_1 = D_1Q_1 \), (a) will follow from (b). By Lemma 5.7(b), letting \( f = a \) and \( g = b \), we have

\[ P_1aQ_2b = [P_1, a] \cdot \{Q_2, b\} - Q_1(\{P_1, a\} \cdot \{R_2, b\}) - P_1(\{Q_1, a\} \cdot \{R_1, b\}). \]

Moreover, since \( R_1 = I - P_1 \), we have

\[ -Q_1(\{P_1, a\} \cdot \{R_1, b\}) = Q_1(bP_1a + Q_1(\{P_1, a\} \cdot \{R_2, b\})) \]

\[ -P_1(\{Q_1, a\} \cdot \{R_1, b\}) = -P_1(bQ_1a + P_1(\{Q_1, a\} \cdot \{R_2, b\})) \]

whence, substituting into (5.44), we obtain

\[ P_1aQ_2b + P_1bQ_2a = [P_1, a] \cdot \{Q_2, b\} - Q_1(bP_1a + Q_1(\{P_1, a\} \cdot \{R_2, b\}) + P_1(\{Q_1, a\} \cdot \{P_2, b\}). \]

By Lemma 5.7(a), we have

\[ [P_1, a] \cdot \{Q_2, b\} - Q_1(bP_1a + Q_1(\{P_1, a\} \cdot \{R_2, b\}) = Q_1(\{Q_1, a\} \cdot \{Q_2, b\}). \]

Combining (5.47) and (5.48), we obtain (b).

We now use Lemmas 5.7 and 5.8 to show how the estimate (5.4) may be obtained in the cases \(n = 2\) and \(n = 3\). We begin with the case \(n = 2\), in which we are concerned with the function

\[ G_2(a_1, a_2) = H |D_1|^{1-\frac{N}{2}} \int_0^1 [X, h_1] \cdot \{X_1, a_1, X_2, f\} (t^{1-\frac{N}{2}})^2 dt \]

where \(X, X_1, X_2 \in [P, Q]\). By Lemma 5.7(c)-(o), \(G_1(a_1)\) may be rewritten as the sum of

\[ G = H |D_1|^{1-\frac{N}{2}} \int_0^1 [X, h_1] \cdot \{X_1, a_1, X_2, f\} (t^{1-\frac{N}{2}})^2 dt \]

and two functions of the form

\[ A = \pm H |D_1|^{1-\frac{N}{2}} \int_0^1 [X, h_1] \cdot W_1(\{Y_1, a_1\} \cdot [Z, f]) (t^{1-\frac{N}{2}})^2 dt \]

where \(W, W_1 \in [P, Q]\) and \(Z \in [P, Q, R]\). We claim that \(G\) and \(A\) are in \(H^1\), with norm bounded by \(C \|A_1\|_1 \|\|f\|_2\|_2\), where \(C\) depends only on \(\lambda_1\) and \(\lambda_2\). In the case of the function \(G\), this is immediate from Lemma 5.6. For \(A\), we use Lemma 5.5 to see that

\[ w = (t^{1-\frac{N}{2}}) [Y_1, a_1] \cdot [Z, f] \]

satisfies \( |w| = \sup_{x \in \mathcal{F}(R)} \|w\|_2 \leq C_1 \|A_1\|_1 \|\|f\|_2\|_2\). The desired estimate for \(A\) is then a consequence of Lemma 5.6. We obtain (5.4) in the case of \(n = 2\) by applying Lemma 5.2.

In the case \(n = 3\), we consider the function

\[ G_3(a_1, a_2, a_3) = H |D_1|^{1-\frac{N}{2}} \int_0^1 [X, h_1] \cdot M_{xy}(a_1, a_2) f (t^{1-\frac{N}{2}})^2 dt \]

where

\[ M_{xy}(a_1, a_2) = X_1, a_1, X_2, a_2, X_3, f + X_1, a_2, X_2, a_1, X_3, f \]

We make repeated use of Lemma 5.7(c)-(o), beginning with the expression

\[ X_2, a_1, X_2, f \]

In consequence we see that

\[ M_{xy}(a_1, a_2) f = S_{xy}(a_1, a_2) f + E_{xy}(a_1, a_2) f \]

where

\[ S_{xy}(a_1, a_2) f = X_1, a_1, [X_2, a_2] \cdot [X_3, f] + X_1, a_2, [X_2, a_1] \cdot [X_3, f] \]

and \(E_{xy}(a_1, a_2) f\) is a sum of functions of the form

\[ W_1([Y_2, a_2] \cdot Z_2, [Y_3, a_3] \cdot [Z_2, f]) \]

where \(W_1 \in \{P, P, Q\}, Y_2, Y_3 \in \{P, Q\}, Z_2, Z_3 \in \{P, Q, R\}\). Defining

\[ E_2(a_1, a_2) f = H |D_1|^{1-\frac{N}{2}} \int_0^1 [X, h_1] \cdot E_{xy}(a_1, a_2) f (t^{1-\frac{N}{2}})^2 dt \]

we easily obtain an estimate of the form

\[ \|E_2(a_1, a_2) f\|_2 \leq C_2 \|A_1\|_1 \|A_2\|_2 \|\|f\|_2\|_2\]. \]
by Lemmas 5.5 and 5.6. It therefore remains to estimate

\[(5.59) \quad S_2(a_1, a_2) f = H^{1 - \lambda_3} \int_0^\infty \left[ X_t h_t \right] S_2(a_1, a_2) f(t^{1 - \lambda_3} \lambda_t) dt.
\]

It is in estimating this operator that we make crucial use of the fact that \(S_2(a_1, a_2)\) is symmetric in \(a_1\) and \(a_2\), in an application of Lemma 5.8.

By Lemma 5.7(d)(f), \(S_2(a_1, a_2) f\) may be written as a sum of expressions of the form

\[(5.60) \quad W_1([Y_{11} a_1, X_2 a_2 + Y_1 a_2 X_2 a_1], [Z_t, f])
\]

where \(W_1 \in [P, Q_1], Y_1 \in [P_1, Q_1], \) and \(Z_t \in [P_1, Q_1, R].\) If \(X_2 = P_1\), Lemma 5.7(d)(f) shows that \([Y_1 a_1, X_2 a_2 + Y_1 a_2 X_2 a_1]\) may be expressed as a sum of functions of the form

\[(5.61) \quad W_1([Y_{11} a_1, a_1], [Y_{11} a_1, a_1]),
\]

with \(W_1 \in [P, Q_1], Y_1 \in [P_1, Q_1].\) If \(X_2 = Q_1\), then \([Y_1 a_1, X_2 a_2 + Y_1 a_2 X_2 a_1]\) has a similar expression; but in this case the symmetry is crucial and Lemma 5.8 must be used. In any case, Lemmas 5.7 and 5.8 enable us to write \(S_2(a_1, a_2) f\) as the sum of \(H^4\) functions which can be estimated by Lemma 5.6.

In the case of \(n = 2, 3\), we have actually proved the following:

**Lemma 5.9.** Under the hypotheses of the Main Theorem, we have, for \(n \geq 2\),

\[(5.62) \quad \|G_{n-1} (a_1, \ldots, a_{n-1}) f\|_{H^1} \leq K \left( \prod_{j=1}^{n-1} \|A_j\|_{L^2} \|h_j\|_{L^2} \right) \|f\|_{L^2}
\]

where \(K\) is independent of \(A_1, \ldots, A_{n-1}, h_j, \) and \(f.\)

To prove Lemma 5.9 for \(n \geq 4\), we must make repeated use of Lemmas 5.7 and 5.8 to express \(G_{n-1} (a_1, \ldots, a_{n-1}) f\) in terms of \(H^4\) functions which can be estimated by Lemma 5.6. At every stage there will be at least one term for which symmetry in \(a_1, \ldots, a_{n-1}\) is crucial. To treat these terms, analogues to Lemma 5.8 may be developed which show, for example, that the expression

\[(5.63) \quad \sum_{a_1 \ldots a_{n-1}} [Q_1 a^{(1)}_1 Q_2 a^{(2)}_2 \ldots Q_{n-1} a^{(n-1)}_{n-1}] - [Q_1 a_1, Q_2 a_2, \ldots, Q_{n-1} a_{n-1}]
\]

can be written as a sum of functions built up from \(P_1, Q_1, P_1 a_1, \) and \([Q_1 a_1]\), where \(1 \leq j \leq n-1.\) We omit the details.

The Main Lemma, and hence the Main Theorem, now follow from Lemmas 5.9 and 5.2.