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Stieltjes vectors and cosine functions generators

by

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Abstract. Properties of some classes of C^∞-vectors for generators of regular operator cosine functions are presented.

1. Introduction. Let A be a linear operator in a Banach space X, with domain $D(A)$ and let

$$C^\infty(A) = \bigcap_{n=1}^{\infty} D(A^n).$$

We shall distinguish the following subsets of $C^\infty(A)$:

– *analytic vectors*:

$$D_{\mathrm{a}}(A) = \Big\{x;\ \sum_{n=0}^{\infty} \|A^n x\|\, t^n/n! < +\infty \text{ for some } t > 0\Big\};$$

– *semianalytic vectors*:

$$D_{\mathrm{sa}}(A) = \Big\{x;\ \sum_{n=0}^{\infty} \|A^n x\|\, t^n/(2n)! < +\infty \text{ for some } t > 0\Big\};$$

– *Stieltjes vectors*:

$$D_{\mathrm{s}}(A) = \Big\{x;\ \sum_{n=1}^{\infty} \|A^n x\|^{-1/(2n)} = +\infty\Big\}.$$

The analytic vectors were introduced by E. Nelson [7], the semianalytic vectors by B. Simon [11] and the Stieltjes vectors by A. E. Nussbaum [8] and independently by D. Masson and W. K. McClary [6]. It is clear that $D_{\mathrm{a}}(A) \subset D_{\mathrm{sa}}(A) \subset D_{\mathrm{s}}(A)$; moreover, $D_{\mathrm{a}}(A)$ and $D_{\mathrm{sa}}(A)$ are linear subspaces of $C^\infty(A)$, but this is not necessarily true for $D_{\mathrm{s}}(A)$.

Two other subspaces of $D_{\mathrm{a}}(A)$, respectively $D_{\mathrm{sa}}(A)$, are important in our considerations, namely:

$$D_{\mathrm{a}}^0(A) = \Big\{x;\ \sum_{n=0}^{\infty} \|A^n x\|\, t^n/n! < +\infty \text{ for all } t > 0\Big\},$$

$$D_{\mathrm{sa}}^0(A) = \Big\{x;\ \sum_{n=0}^{\infty} \|A^n x\|\, t^n/(2n)! < +\infty \text{ for all } t > 0\Big\}.$$

It is not difficult to see that

$$D_a^0(A) = \{x \in C^\infty(A);\ \|A^n x\|^{1/n} = o(n)\},$$
$$D_{sa}^0(A) = \{x \in C^\infty(A);\ \|A^n x\|^{1/n} = o(n^2)\}.$$

The subspace $D_a^0(A)$ was considered in [4] by G. Lumer and R. S. Phillips; they proved that for generators of groups the set $D_a^0(A)$ is always dense in X; moreover, *if for a dissipative operator* A (i.e. such that for each $x \in D(A)$, there is a linear bounded functional $x^* \in X^*$ with $\langle x^*, x\rangle = \|x\|$, $\|x^*\| = \|x\|$ and $\operatorname{Re}\langle x^*, Ax\rangle \leqslant 0$) *the subspace* $D_a^0(A)$ *is dense in* X, *then* A *generates a strongly continuous semigroup of contractions.*

We shall prove in what follows that for a large class of operators $D_{sa}^0(A)$ is dense, namely for generators of regular operator cosine functions.

This fact and the above result of Lumer and Phillips will permit us to show that a certain class of operators satisfies Assumption 6.2 of H. O. Fattorini ([2]). Finally, we obtain a new proof of a criterion of Masson, McClary and Nussbaum ([6], [8]) for the selfadjointness of symmetric and semibounded operators.

2. Semianalytic vectors and regular cosine functions. Let X be a Banach space and $\mathscr{L}(X)$ the algebra of bounded linear operators on X. A function $C\colon \boldsymbol{R} \to \mathscr{L}(X)$ is called a *regular* (or *strongly continuous*) *operator cosine function* on X if

$C(t+s)+C(t-s) = 2C(t)C(s)$, $\quad t, s \in \boldsymbol{R}$;
$C(0) = I$;
$t \to C(t)x$ is continuous on $\boldsymbol{R}$ for each $x \in X$.

If C is a regular cosine function, then there are constants $M \geqslant 1$ and $\omega \geqslant 0$ such that

(1) $$\|C(t)\| \leqslant Me^{\omega|t|}, \qquad t \in \boldsymbol{R}.$$

The (infinitesimal) *generator* of a regular cosine function is the closed and densely defined operator A given by

$$D(A) = \{x \in X;\ t \to C(t)x \text{ is a twice differentiable function of } t\},$$
$$Ax = C''(0)x.$$

For elementary properties of regular cosine function we refer the reader to [2], [5], [10].

PROPOSITION 1. *Let A be the generator of a regular cosine function. Then the set of $C^\infty(A)$-vectors x with $\|A^n x\|^{1/n} = o(n)$ is dense in X.*

Proof. Consider for $x \in X$ and $\varepsilon > 0$

$$x_\varepsilon = (\varepsilon\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-t^2/\varepsilon} C(t)x\,dt$$

where the existence of the integral is due to the estimate (1). For each $\delta > 0$, we have

$$\begin{aligned}\|x_\varepsilon - x\| &\leqslant (\varepsilon\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-t^2/\varepsilon}\|C(t)x - x\|\,dt\\ &= (\varepsilon\pi)^{-1/2}\Big(\int_{|t|<\delta} e^{-t^2/\varepsilon}\|C(t)x-x\|\,dt + \int_{|t|\geqslant\delta} e^{-t^2/\varepsilon}\|C(t)x-x\|\,dt\Big)\\ &\leqslant \sup_{|t|<\delta}\|C(t)x-x\| + (\varepsilon\pi)^{-1/2}\int_{|t|\geqslant\delta} e^{-t^2/\varepsilon}(\|C(t)\|+1)\|x\|\,dt.\end{aligned}$$

Hence

$$\limsup_{\varepsilon\to 0+}\|x_\varepsilon - x\| \leqslant \inf_{\delta\to 0+}\sup_{|t|\leqslant\delta}\|C(t)x-x\| = 0$$

so that $\lim_{\varepsilon\to 0+} x_\varepsilon = x$.

On the other hand, one can easily verify that $x_\varepsilon \in C^\infty(A)$ and that

$$A^n x_\varepsilon = (\varepsilon\pi)^{-1/2}\int_{-\infty}^{+\infty} [e^{-t^2/\varepsilon}]^{(2n)} C(t)x\,dt.$$

But for the Hermite function

$$H_n(t) = (-1)^n(\sqrt{\pi}\,2^n n!)^{-1/2} e^{t^2/2}(e^{-t^2})^{(n)}$$

one has

$$\|H_n(\cdot)\|_1 = O(n^{1/4}) \qquad (\text{see [3], § 21.3}),$$

so that, for fixed ε,

$$\begin{aligned}\|A^n x_\varepsilon\| &\leqslant M(\varepsilon\pi)^{-1/2}\|x\|\Big|\int_{-\infty}^{+\infty} e^{\omega|t|}[e^{-t^2/\varepsilon}]^{(2n)}\,dt\Big|\\ &\leqslant M'\Big|\int_{-\infty}^{+\infty} e^{t^2/(2\varepsilon)}[e^{-t^2/\varepsilon}]^{(2n)}\,dt\Big|\\ &\leqslant M'' 2^n(2n!)^{1/2}(2n)^{1/4}\end{aligned}$$

for suitable constants M' and M''.

We see now that $\|A^n x_\varepsilon\|^{1/n} = O(n)$ and this ends the proof.

COROLLARY. *Let A be the generator of a regular cosine function. Then $D_{sa}^0(A) \cap D_a(A)$ is dense in X.*

Proof. Let $x \in C^\infty(A)$ with $\|A^n x\|^{1/n} = O(n)$. Then it is obvious that $x \in D_{sa}^0(A)$. On the other hand,

$$\|A^n x\|^{1/n} \leqslant M_0 n,\ n \in \boldsymbol{N} \Leftrightarrow \liminf_{n\to\infty} n\|A^n x\|^{-1/n} \geqslant M_0^{-1} > 0 \Leftrightarrow x \in D_a(A).$$

As $\|A^n x\|^{1/n} = O(n)$ on a dense set, the assertion follows.

Remark. Let us recall that Nelson [7] gave an example of a generator of a contraction semigroup on a Hilbert space for which $D_a = \{0\}$. The above result shows that this is no more possible when dealing with generators of regular cosine functions, which are, in particular, by Fattorini's result in [2], generators of holomorphic semigroups.

PROPOSITION 2. *Assume that for an operator A the subspace $D^0_{sa}(A)$ is dense in X and that there exists an operator B in X such that*

(i) $B^{-1} \in \mathscr{L}(X)$.

(ii) $B^2 = A$.

(iii) $\pm B$ *are dissipative.*

Then A generates a cosine function of the form

$$C(t) = \tfrac{1}{2}(U(t)+U(-t)), \quad t \in \boldsymbol{R}, \tag{2}$$

where $\{U(t)\}_{t\in \boldsymbol{R}}$ is a group of contractions on X.

Proof. We shall first prove that $D^0_{sa}(A) = D^0_a(B)$.

It is easy to see that $D^0_a(B) \subset D^0_{sa}(A)$. Conversely, let $x \in D^0_{sa}(A)$. Then we have

$$\lim_{n\to\infty} \frac{\|B^{2n}\|^{1/(2n)}}{2n} = \tfrac{1}{2} \lim_{n\to\infty} \left(\frac{\|A^n x\|^{1/n}}{n^2}\right)^{1/2} = 0.$$

Further, we obtain

$$\lim_{n\to\infty} \frac{\|B^{2n-1} x\|^{1/(2n-1)}}{2n-1} \leqslant \|B^{-1}\| \lim_{n\to\infty} \frac{\|A^n x\|^{1/(2n-1)}}{n}$$
$$= \|B^{-1}\| \lim_{n\to\infty} \left(\frac{\|A^n x\|^{1/(2n)}}{n}\right)^{2n/(2n-1)} \frac{n^{2n/(2n-1)}}{n}$$
$$= \|B^{-1}\| \lim_{n\to\infty} \left(\frac{\|A^n x\|^{1/n}}{n^2}\right)^{n/(2n-1)} n^{1/(2n-1)} = 0.$$

Thus $x \in D^0_a(B)$ and by the Lumer–Phillips result mentioned in the introduction, $\pm B$ are generators of semigroups of contractions and thus B generates a group $\{U(t)\}_{t\in\boldsymbol{R}}$ of contractions. Then by formula (2) one defines a regular cosine function having A as generator (see [10], Th. 4.12).

Remark. In [2] it was established that if A is the generator of a regular cosine function then a translate of A has a square root containing zero in its resolvent set. It is also proved there that A is the generator of a regular cosine function iff $A_\alpha = A - \alpha^2 I$, $\alpha \in \boldsymbol{R}$, generates a regular cosine function. Moreover, one can easily prove that if $x \in D^0_{sa}(A)$ then $x \in D^0_{sa}(A_\alpha)$.

Thus, when dealing with generators of regular cosine functions all the hypotheses on A and B, except the last one, are fulfilled, at least for a translate of A. Condition (iii) ensures the validity of the representation (2).

In connection with these facts see also Assumption 6.2 from [2] and the equivalent conditions for it in [12], Propositions 2.6, 2.7 and 2.8.

3. Stieltjes vectors for generators of cosine functions. In [10] M. Sova proved that if A and $\tilde{A}$ are generators of two regular cosine functions and $A \subset \tilde{A}$, then $A = \tilde{A}$.

In the same direction we shall next give another result similar to P. Chernoff's one in [1], Th. 3.1, relative to generators of operator semigroups.

PROPOSITION 3. *Let A and $\tilde{A}$ be two operators on X such that $A \subset \tilde{A}$. Assume that A is closed with $D_s(A)$ total in X and that $\tilde{A}$ generates a regular cosine function. Then $A = \tilde{A}$.*

Proof. Let C be the regular cosine function generated by $\tilde{A}$ and let M, ω be as in (1). Let $\lambda > \omega$. Then by the spectral criterion for the generator $\tilde{A}$ ([10], Th. 3.2), λ^2 belongs to the resolvent set of $\tilde{A}$, in particular $\lambda^2 - \tilde{A}$ is injective and thus $\lambda^2 - A$ is also injective. Thus it is enough to prove that $\operatorname{range}(\lambda^2 I - A) = X$, because then $\lambda^2 \in \varrho(A)$ so that $\lambda^2 \in \varrho(A) \cap \varrho(\tilde{A})$ and this obviously implies $A = \tilde{A}$.

Let us first remark that $\operatorname{range}(\lambda^2 I - A)$ is closed; indeed, this is a consequence of the fact that A is closed and $\lambda^2 I - A$ is bounded below. It thus suffices to prove that

$$\overline{\operatorname{range}(\lambda^2 I - A)} = X.$$

Suppose that there is an $x^* \in X^*$, $x^* \neq 0$, such that

$$\langle x^*, Ax\rangle = \lambda^2 \langle x^*, x\rangle, \quad \forall x \in D(A).$$

Let $x \in D_s(A)$ and define

$$f(t) = \langle x^*, C(t)x\rangle - \cosh \lambda t \cdot \langle x^*, x\rangle, \quad t \in \boldsymbol{R}.$$

It is clear that f is of class C^∞ on $\boldsymbol{R}$ and for $n \in \boldsymbol{N}$ and $t \in \boldsymbol{R}$ we have

$$f^{(2n)}(t) = \langle x^*, C(t)A^n x\rangle - \lambda^{2n} \cosh \lambda t \cdot \langle x^*, x\rangle,$$
$$f^{(2n-1)}(t) = \langle x^*, S(t)A^n x\rangle - \lambda^{2n-1} \sinh \lambda t \cdot \langle x^*, x\rangle,$$

where $S(t)x = \int_0^t C(s)x\,ds$. Hence $f^{(n)}(0) = 0$ for each $n \in \boldsymbol{N}$.

Moreover, for $s > 0$ fixed, we get

$$\sup_{t\in[-s,s]} |f^{(2n)}(t)| \leqslant c(\|A^n x\| + \lambda^{2n})$$

for a suitable constant $c = c_s$.

Using the fact that $\sum_{n=1}^{\infty} \|A^n x\|^{-1/(2n)} = +\infty$ implies that also

$$\sum_{n=1}^{\infty} [c(\|A^n x\| + \lambda^{2n})]^{-1/(2n)} = +\infty$$

(this can be proved exactly as in Lemma 3.2 in [1]) we obtain

$$\sum_{n=1}^{\infty} K_n^{-1/n} \geqslant \sum_{n=1}^{\infty} K_{2n}^{-1/(2n)} = +\infty, \quad \text{where} \quad K_n = \sup_{t\in[-s,s]} |f^{(n)}(t)|.$$

Now, by the Denjoy–Carleman theorem ([9], Th. 19.11), we get $f \equiv 0$ on $[-s, s]$. As $s > 0$ was arbitrary, it follows that $f \equiv 0$ on $\boldsymbol{R}$. Therefore

$$\langle x^*, C(t)x\rangle = \cosh \lambda t \cdot \langle x^*, x\rangle \quad \text{for all } x \in D_s(A),\ t \in \boldsymbol{R}.$$

But as $D_s(A)$ is total in X, this implies

$$C^*(t)x^* = \cosh \lambda t \cdot x^*.$$

Finally, we obtain

$$\|C(t)\| = \|C^*(t)\| \geqslant \frac{\|C^*(t)x^*\|}{\|x^*\|} = \cosh \lambda t$$

and this contradicts (1). Thus $\lambda^2 I - A$ has a dense range.

As a consequence we finally give a new proof of the following result due to Nussbaum [8] and Masson and McClary [6]:

COROLLARY. *Let A be a closed, symmetric and semibounded operator in a Hilbert space H. Then A is selfadjoint iff $D_s(A)$ is total in H.*

Proof. The necessity of the condition follows from Proposition 1.

Let us prove that the condition is also sufficient. We can suppose that $A \geqslant I$. Denote by $\tilde{A}$ the Friedrichs extension of A which is selfadjoint and $\geqslant I$. Then $-\tilde{A}$ generates a regular cosine function. Indeed, consider the operator $\tilde{A}^{1/2}$ and the corresponding group of unitary operators $\{e^{it\tilde{A}^{1/2}}\}$, $t \in \boldsymbol{R}$. Then $C(t) = \frac{1}{2}(e^{it\tilde{A}^{1/2}} + e^{-it\tilde{A}^{1/2}})$ defines a regular cosine function with generator $(i\tilde{A}^{1/2})^2 = -\tilde{A}$. By the above proposition, $A = \tilde{A}$.

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