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Nonexpansive maps in generalized Orlicz spaces

by

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Abstract. We give fixed point theorems for nonexpansive maps T operating in star-shaped subsets B of generalized Orlicz spaces $L^\varphi(\Omega, \mathcal{A}, \mu)$, under suitable compactness and boundedness conditions on B (compactness for the topology of pointwise convergence in the case of sequence spaces l^φ). The operator T is nonexpansive either for the modular $\varrho_\varphi(x) = \int \varphi(|x|)\,d\mu$, $x \in L^\varphi$, or for the associated Minkowski functional ν_φ (the Luxemburg norm when φ is convex). In both cases φ may be an arbitrary convex or concave strictly increasing Orlicz function (unbounded for ν_φ-nonexpansiveness), or even have a less regular behaviour; it may also depend on the integration variable (L^φ is then a "Musielak–Orlicz space").

1. Introduction. We give fixed point theorems for nonexpansive mappings T operating in star-shaped subsets B of Orlicz spaces l^φ of sequences or in Orlicz spaces $L^\varphi(\Omega, \mathcal{A}, \mu)$ of measurable functions (and even in more general Musielak–Orlicz spaces, where the function φ may depend on the integration variable). We do not limit our study to the case of locally convex spaces: φ may be far from being convex.

One can find in [L] a related work in the case of sequence spaces l^φ with φ convex. See also [Ki] for a recent "state of the art".

A subset B of a vector space E is said to be *star-shaped* when there exists a "center point" $u \in B$ such that $(1-\lambda)u + \lambda x \in B$ for every $x \in B$ and every real number $\lambda \in (0, 1)$.

We say that a mapping $T\colon B \to B$ is *nonexpansive* for some functional $F\colon B - B \to \mathbf{R}_+$ when T satisfies

$$F(Tx - Ty) \leqslant F(x - y)$$

for every pair (x, y) of points of B.

Our mapping T will be nonexpansive either for the modular ϱ_φ defined in the case of Orlicz spaces by

$$\varrho_\varphi(x) = \int_\Omega \varphi(|x(\omega)|)\,\mu(d\omega)$$

or for the Minkowski functional

$$\nu_\varphi(x) = \inf\{\lambda > 0\colon\ \varrho_\varphi(x/\lambda) \leqslant 1\}.$$

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When φ is convex, ν_φ is the Luxemburg norm of L^φ ([KR]).

When $\varphi(t) = t^p$, $0 < p < \infty$, the nonexpansiveness conditions for ϱ_φ or ν_φ are both equivalent to the nonexpansiveness for the usual norm (p-norm when $p < 1$) of $L^p(\Omega, \mathscr{A}, \mu)$.

In spaces l^φ of sequences, the set B where T operates is assumed to be compact for the topology of pointwise convergence. More generally, in spaces $L^\varphi(\Omega, \mathscr{A}, \mu)$ we assume that from every sequence of points of B we can extract a subsequence converging almost everywhere to a point of B (in other words, B is sequentially compact for the topology τ of "local convergence in measure"). Of course, B is sequentially τ-compact if it is compact for the natural topology of $L^\varphi(\Omega, \mathscr{A}, \mu)$.

In sequence spaces l^φ this τ-compactness condition on B seems to be quite satisfactory (it is e.g. fulfilled by the "balls" $B^\varphi(r) = \{x\colon \sum_\omega \varphi(|x(\omega)|) \leqslant r\}$, $0 < r < \sup_{t>0} \varphi(t)$).

In function spaces things are less simple. In $L^p(0, 1)$, $p > 0$, for example, τ-compact bounded sets are in fact compact in $L^r(0, 1)$ for every $r \in (0, p)$. When $1 < p < \infty$, we may also notice that they are compact for the weak topology of $L^p(0, 1)$; this is of course false for $p = 1$: the subsets of $L^1(0, 1)$ which are both τ-compact and weakly compact are norm-compact.

In $L^1(0, 1)$, weak compactness would not fit our purpose, since D.E. Alspach ([A]) has given an example of a weakly compact convex subset B of $L^1(0, 1)$ and a nonexpansive mapping $T\colon B \to B$ without fixed point (see, however, [Mau], [ELOS], [Ki], p. 129, for positive results). On the contrary, in $L^p(0, 1)$ with $1 < p < \infty$, by the classical theorem of Browder, Göhde and Kirk ([Ki], p. 123, or [G]) and the uniform convexity of the norm, a nonexpansive mapping leaving invariant a closed bounded convex set has a fixed point. So, in the case of convex subsets B of $L^p(0, 1)$, $1 < p < \infty$, our τ-compactness assumption is far too strong. But our sets B are star-shaped; they need not be convex.

The first author initiated this work in [LD1] and [LD2], considering Orlicz spaces of sequences l^φ; φ was convex and satisfied the condition Δ_2 at 0, or concave with l^φ locally bounded (a little more in fact). In the first case the mapping T was nonexpansive for the Luxemburg norm ν_φ, and for the modular ϱ_φ in the second case.

Here, besides the generalization to Musielak–Orlicz spaces $L^\varphi(\Omega, \mathscr{A}, \mu)$ with arbitrary measure spaces $(\Omega, \mathscr{A}, \mu)$, we extend this study to wide classes of functions φ, which include not only arbitrary concave or convex (or p-convex, $p > 0$) Orlicz functions but also more general ones; and this is done in both cases of ϱ_φ-nonexpansiveness and of ν_φ-nonexpansiveness. Let us remark that these functionals ϱ_φ, ν_φ are often not subadditive: then the nonexpansiveness condition does not refer to a metric.

These generalizations cover the case of non-locally bounded spaces $L^\varphi(\Omega, \mathscr{A}, \mu)$ (such as the space of sequences l^φ with $\varphi(t) = |\log t|^{-1}$ near 0), and also of spaces $L^\varphi(\Omega, \mathscr{A}, \mu)$ where φ does not satisfy the condition Δ_2. However, in this last case the star-shaped set B where T operates has to be contained in the closed subspace L_0^φ of L^φ generated by the set of simple functions and is subjected to a strong boundedness condition, chiefly in Theorem 10.1.

A curious phenomenon appears in non-locally bounded Orlicz sequence spaces l^φ: every strict ϱ_φ-contraction on a ball is constant (Theorem 12.2).

We shall mainly meet two types of growth conditions on φ:

– The hypothesis that φ has "non-null growth exponents", which is a little stronger than the necessary and sufficient condition of local boundedness for $L^\varphi(\Omega, \mathscr{A}, \mu)$, and which is fulfilled by every convex (or p-convex) function φ.

– The conditions "BL_k", drawn from a paper by H. Brézis and E. Lieb ([BL]), which generalize both subadditivity and convexity (and p-convexity).

Without giving too precise statements, we can summarize this paper in the following way.

The set B is as above. When $T\colon B \to B$ is nonexpansive for ϱ_φ, it has a fixed point if φ is strictly increasing and

– either φ is subadditive (Theorem 3.1);

– or (1) φ has non-null growth exponents and (2) φ satisfies the condition Δ_2 (Theorem 6.1) or the condition BL_k (Theorem 8.1); (1) can be omitted under some supplementary assumption on B (Theorem 9.3);

– or B is compact for the topology of $L^\varphi(\Omega, \mathscr{A}, \mu)$ (Proposition 9.1).

When T is nonexpansive for ν_φ, it has a fixed point if φ is strictly increasing, unbounded and satisfies the condition BL_k (Theorem 10.1).

Finally, under a strong nonexpansiveness hypothesis on T, φ may be arbitrary (Theorem 11.1).

As in [LD1], [LD2], we use asymptotic center techniques.

A key result (Lemma 6.3) asserts that, when a sequence of measurable functions x_n converges a.e. to a function $c \in L_0^\varphi(\Omega, \mathscr{A}, \mu)$, then the functional

$$R(x) = \sup_{0<s<1} \limsup_n \varrho_\varphi(s(x - x_n)), \qquad x \in L_0^\varphi(\Omega, \mathscr{A}, \mu),$$

attains its minimum value at the unique point c (unless $R(x) \equiv +\infty$). Here φ is arbitrary. Under conditions BL_k, we have a similar result letting s take the value 1, and also for the analogous functional defined with ν_φ (Lemmas 8.3, 9.5, 10.4).

The inequalities given in these "asymptotic center lemmas" appear already in [LD2] in special cases. Related estimates can also be found in [BL], in another context.

Our fixed points always come out after the following three steps.

First we find nonexpansive mappings T_n: $B \to B$ which converge pointwise to T and turn out to have fixed points x_n (or almost fixed in Theorem

11.1) converging a.e. to a point $c \in B$. Then we prove the inequality $R(Tc) \leqslant R(c)$, for R or for a similar functional, and we conclude in the third step, using the suitable asymptotic center lemma, that we must have $Tc = c$.

Contrary to the case of nonexpansive mappings in Banach spaces, the first two steps are not obvious. Indeed, by Theorem 12.2, in non-locally bounded spaces it may be not possible to approximate T by strict ϱ_φ-contractions T_n. Moreover, even when this is possible, the lack of the subadditivity of ϱ_φ and v_φ often creates new difficulties in the course of the proof.

2. A few definitions. Star-shaped sets and nonexpansive mappings have been defined in the introduction.

2.1. *Musielak–Orlicz functions.* A *Musielak–Orlicz function* (abbreviated to "M.O. function") on a (positive) measure space $(\Omega, \mathscr{A}, \mu)$ will be a mapping $\varphi\colon \boldsymbol{R}_+ \times \Omega \to \boldsymbol{R}_+$, where $\boldsymbol{R}_+ = [0, +\infty)$, enjoying the following properties. For every $t \in \boldsymbol{R}_+$, $\varphi(t, \cdot)\colon \Omega \to \boldsymbol{R}_+$ is measurable, and for every $\omega \in \Omega$ the function $\varphi(\cdot, \omega)\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is increasing, left-continuous and strictly positive on $(0, \infty)$, null and continuous at 0. We sometimes extend $\varphi(\cdot, \omega)$ to $+\infty$, letting

$$\varphi(\infty, \omega) = \sup\{\varphi(t, \omega)\colon t \in \boldsymbol{R}_+\}.$$

If $x\colon \Omega \to \boldsymbol{R}_+$ is measurable (for the σ-field $\mathscr{A}$) we denote by $\tilde{\varphi}(x)$ the function defined on Ω by

$$\tilde{\varphi}(x) = \varphi(x(\cdot), \cdot)\colon \omega \to \varphi(x(\omega), \omega),$$

which is still $\mathscr{A}$-measurable.

When φ does not depend on the variable ω in Ω, we say that φ is an *Orlicz function.*

2.2. *Musielak–Orlicz spaces.* To a M.O. function φ on a measure space $(\Omega, \mathscr{A}, \mu)$ we associate the modular ([MuO])

$$\varrho_\varphi(x) = \int_\Omega \tilde{\varphi}(|x|)\mu = \int_\Omega \varphi(|x(\omega)|, \omega)\mu(d\omega),$$

where x is a measurable complex-valued function on $(\Omega, \mathscr{A}, \mu)$, or a μ-class of such functions, and the spaces

$$L_0^\varphi(\Omega, \mathscr{A}, \mu), \qquad L^\varphi(\Omega, \mathscr{A}, \mu).$$

The first (resp. the second) space is the set of the μ-classes of measurable functions satisfying $\varrho_\varphi(tx) < \infty$ for every (resp. for some) real number $t > 0$. They are linear spaces, and are endowed with linear topologies as follows. The sets $rB^\varphi(r)$, $r > 0$, where

$$B^\varphi(r) = \{x \in L^\varphi(\Omega, \mathscr{A}, \mu)\colon \varrho_\varphi(x) \leqslant r\},$$

constitute a basis of zero-neighbourhoods in $L^\varphi(\Omega, \mathscr{A}, \mu)$, and the topology of $L_0^\varphi(\Omega, \mathscr{A}, \mu)$ is the induced topology. A sequence (x_n) in $L^\varphi(\Omega, \mathscr{A}, \mu)$ tends to 0 if and only if $\lim \varrho_\varphi(tx_n) = 0$ for every real number t.

When the measure space is the set N of nonnegative whole numbers with the counting measure, we get the Musielak–Orlicz spaces of sequences

$$l_0^\varphi, \quad l^\varphi$$

associated to the modular $\varrho_\varphi(x) = \sum_{\omega=0}^{\infty} \varphi(|x_\omega|, \omega)$, $x = (x_\omega)_{\omega=0}^{\gamma}$.

2.3. *The functions* $\tau_{r,A}$, $r > 0$, $A \in \mathscr{A}_f$. These functions are useful when the measure space $(\Omega, \mathscr{A}, \mu)$ has atoms (i.e. sets $a \in \mathscr{A}$ satisfying $\mu(a) > 0$ and $\mu(a') = 0$ or $\mu(a \backslash a') = 0$ if $a' \subset a$, $a' \in \mathscr{A}$, or μ-classes of such sets). Let

$$\mathscr{A}_f = \{A \in \mathscr{A}\colon 0 < \mu(A) < \infty\}.$$

For every real number $r > 0$ and every $A \in \mathscr{A}_f$ we define a function $\tau_{r,A}\colon A \to [0, +\infty]$ determined, up to a negligible function, by the following conditions. For every atom a of μ included in A,

$$\tau_{r,A}(a) = \sup\{t \in \boldsymbol{R}_+\colon \varphi(t, a)\mu(a) \leqslant r\}$$

where $\tau_{r,A}(a)$ and $\varphi(t, a)$ are respectively the essential values of $\tau_{r,A}$ and $\varphi(t, \cdot)$ on a. And

$$\tau_{r,A} = +\infty$$

a.e. (almost everywhere) on the $\mathscr{A}$-measurable subsets of A which contain no atoms of μ.

So if x is a measurable function on $(\Omega, \mathscr{A}, \mu)$ and $\varrho_\varphi(x) \leqslant r$ then, for every $A \in \mathscr{A}_f$, $|x| \leqslant \tau_{r,A}$ a.e. on A.

Of course, when μ is σ-finite, a measurable function τ_r defined in the same way on the whole of Ω could replace in the sequel the family of functions $\tau_{r,A}$, $A \in \mathscr{A}_f$.

2.4. *The topology of local convergence in measure on* $(\Omega, \mathscr{A}, \mu)$. This is the topology of convergence in measure on every set $A \in \mathscr{A}$ of finite measure. We shall consider subsets B of $L^\varphi(\Omega, \mathscr{A}, \mu)$ sequentially compact for that topology. Since every function in $L^\varphi(\Omega, \mathscr{A}, \mu)$ is null outside of some set of σ-finite measure, that means that every sequence of points of B contains a subsequence converging a.e. to some point of B.

2.5. *Bounded sets.* A subset B of $L^\varphi(\Omega, \mathscr{A}, \mu)$ is said to be *bounded* when $\sup\{\varrho_\varphi(\varepsilon x)\colon x \in B\}$ tends to 0 as $\varepsilon \in \boldsymbol{R}_+$ tends to 0 (i.e. when it is bounded in the sense of topological vector spaces).

We will rather use conditions like $\sup\{\varrho_\varphi(x-y)\colon (x, y) \in B \times B\} < \infty$, which are in general not equivalent to the boundedness of B, except when, for example, φ is convex (or p-convex) and satisfies the "strong condition Δ_2", defined in Section 5.

3. Fixed points for ϱ_φ-nonexpansive mappings with φ subadditive.

THEOREM 3.1. *Let φ be a Musielak–Orlicz function on a measure space $(\Omega, \mathscr{A}, \mu)$ and let B be a star-shaped subset of $L^\varphi(\Omega, \mathscr{A}, \mu)$ sequentially compact for the topology of local convergence in measure and satisfying*

$$\sup\{\varrho_\varphi(x):\ x\in B\} < \infty .$$

We assume that, for almost all $\omega\in\Omega$, $\varphi(\cdot, \omega)$: $t\to\varphi(t, \omega)$ is subadditive (i.e. $\varphi(s+t, \omega)\leqslant\varphi(s, \omega)+\varphi(t, \omega)$ for s, t in $\mathbf{R}_+$) and strictly increasing on $\mathbf{R}_+$.

Then a map T: $B\to B$ has a fixed point if it is nonexpansive for ϱ_φ, i.e. if

$$\varrho_\varphi(Tx-Ty)\leqslant\varrho_\varphi(x-y)$$

for every $(x, y)\in B\times B$.

Remark 3.2. Since φ is subadditive, the spaces $L^\varphi(\Omega, \mathscr{A}, \mu)$ and $L^\varphi_0(\Omega, \mathscr{A}, \mu)$ are both equal to $\{x:\ \varrho_\varphi(x)<\infty\}$.

Remark 3.3. $L^\varphi(\Omega, \mathscr{A}, \mu)$ need not be locally bounded, nor need B be bounded. This is for example the case for the Orlicz space of sequences l^φ, where φ is the subadditive Orlicz function defined by $\varphi(t) = -1/\log t$ for $0<t\leqslant e^{-1}$ and $\varphi(t) = te$ if $t>e^{-1}$, the star-shaped set B being a ball $B^\varphi(r)$, $r>0$.

LEMMA 3.4. *If φ is subadditive and $A\subset L^\varphi(\Omega, \mathscr{A}, \mu)$ is sequentially compact for the topology of local convergence in measure with $A\neq\emptyset$, then a map S: $A\to A$ has a fixed point if*

(1) $$\varrho_\varphi(Sx-Sy)<\varrho_\varphi(x-y)$$

provided $x\in A$, $y\in A$ and $x\neq y$.

Indeed, let

(2) $$a = \inf\{\varrho_\varphi(Sx-x):\ x\in B\}.$$

Using our remark in 2.4 we see that there exists a sequence (x_n) of points of A satisfying

(3) $$a = \lim\varrho_\varphi(Sx_n-x_n)$$

and such that the sequence Sx_n, $n\geqslant 0$, converges a.e. to a point $c\in A$.

Using the notation $\tilde\varphi$ of 2.1, we have, since φ is subadditive,

$$\left|\tilde\varphi(|Sc-Sx_n|)-\tilde\varphi(|c-Sx_n|)\right|\leqslant\tilde\varphi(|Sc-c|)$$

and the left-hand side tends a.e. to $\tilde\varphi(|Sc-c|)$ (the functions $\varphi(\cdot, \omega)$ are continuous since they are subadditive). So, applying the Lebesgue dominated convergence theorem, we get

(4) $$\varrho_\varphi(Sc-c) = \lim_n\big(\varrho_\varphi(Sc-Sx_n)-\varrho_\varphi(c-Sx_n)\big).$$

But since the mappings $\varphi(\cdot, \omega)$ are subadditive, the modular is also subadditive and we have

$$\varrho_\varphi(Sc-Sx_n)\leqslant\varrho_\varphi(c-x_n)\leqslant\varrho_\varphi(c-Sx_n)+\varrho_\varphi(Sx_n-x_n),$$

whence by (4), (3), (2)

$$\varrho_\varphi(Sc-c)\leqslant\lim_n\varrho_\varphi(Sx_n-x_n) = a\leqslant\varrho_\varphi(S^2c-Sc)$$

and in view of (1) the comparison of the first and last member gives $Sc = c$.

Proof of the theorem. The set B is star-shaped with respect to some center u.

If $0<\lambda<1$, the mapping T_λ: $B\to B$ defined by

$$T_\lambda x = u+\lambda(Tx-u), \qquad x\in B,$$

satisfies $\varrho_\varphi(T_\lambda x-T_\lambda y)<\varrho_\varphi(x-y)$ if $x\in B$, $y\in B$ and $x\neq y$. Indeed, we have

$$\varrho_\varphi(T_\lambda x-T_\lambda y) = \varrho_\varphi\big(\lambda(Tx-Ty)\big)\leqslant\varrho_\varphi(Tx-Ty)\leqslant\varrho_\varphi(x-y).$$

Since the functions $\varphi(\cdot, \omega)$ are strictly increasing, the second or first inequality is strict according as $Tx-Ty$ is null or not. Therefore T_λ has a fixed point and by compactness we find real numbers $\lambda_n\in(0, 1)$ tending to 1 and points $x_n\in B$ converging a.e. to some point $c\in B$ and satisfying $T_{\lambda_n}x_n = x_n$ for every n. The modular ϱ_φ being subadditive, we have

$$|\varrho_\varphi(Tc-x_n)-\varrho_\varphi(T_{\lambda_n}c-x_n)|\leqslant\varrho_\varphi(Tc-T_{\lambda_n}c) = \varrho_\varphi\big((1-\lambda_n)(Tc-u)\big)\to 0,$$

whence

$$\limsup_n\varrho_\varphi(Tc-x_n) = \limsup_n\varrho_\varphi(T_{\lambda_n}c-x_n)$$
$$= \limsup_n\varrho_\varphi(T_{\lambda_n}c-T_{\lambda_n}x_n)\leqslant\limsup_n\varrho_\varphi(c-x_n),$$

these upper limits being finite.

But as in the proof of Lemma 3.4 we have by the dominated convergence theorem

$$\varrho_\varphi(Tc-c) = \lim_n\big(\varrho_\varphi(Tc-x_n)-\varrho_\varphi(c-x_n)\big).$$

The preceding inequality shows that this limit must be null, so $Tc = c$ and c is a fixed point for T.

4. Functions φ with non-null growth exponents.

4.1. We say that a M.O. function φ on a measure space $(\Omega, \mathscr{A}, \mu)$ has *non-null growth exponents* when, for every $r>0$, $\varepsilon>0$ and every λ in the (open) interval $(0, 1)$, one can find a real number $\alpha<1$ and an $\mathscr{A}$-measurable function σ: $\Omega\to\mathbf{R}_+$ satisfying $\varrho_\varphi(\sigma)\leqslant\varepsilon$ and, for each $A\in\mathscr{A}_f$ (cf. 2.3) and for almost all $\omega \in A$,

(5) $$(\sigma(\omega) < \lambda t,\ t \leqslant \tau_{r,A}(\omega),\ t \in \boldsymbol{R}_+) \Rightarrow \varphi(\lambda t, \omega) \leqslant \alpha\varphi(t, \omega).$$

φ is said to have *non-null asymptotic growth exponents* when, for every $r > 0$, there exist real numbers λ and α in $(0, 1)$ and a function $\sigma\colon \Omega \to \boldsymbol{R}_+$ in $L^\varphi(\Omega, \mathscr{A}, \mu)$ satisfying (5) a.e. on every set $A \in \mathscr{A}_{\mathrm{f}}$.

Of course the first condition implies the second one.

We denote by $h_{\sigma,r}(\lambda)$ the least number $\alpha \geqslant 0$ satisfying (5) for every $A \in \mathscr{A}_{\mathrm{f}}$ and for almost all $\omega \in A$. In other words,

$$h_{\sigma,r}(\lambda) = \sup_{A \in \mathscr{A}_{\mathrm{f}}} \operatorname*{ess\,sup}_{\omega \in A} \sup \{\varphi(\lambda t, \omega)/\varphi(t, \omega)\colon \lambda^{-1}\sigma(\omega) < t \leqslant \tau_{r,A}(\omega)\}.$$

In order to justify the terminology, les us explain what we call "growth exponents" and "asymptotic growth exponents": if $h_{\sigma,r}(\lambda) = \lambda^{p(\sigma,r,\lambda)}$ they are respectively the nonnegative numbers

$$p_{\varepsilon,r,\lambda} = \sup\{p(\sigma, r, \lambda)\colon \sigma \in L^\varphi_+(\Omega, \mathscr{A}, \mu), \varrho_\varphi(\sigma) \leqslant \varepsilon\},$$

$$p_r = \sup\{p(\sigma, r, \lambda)\colon \sigma \in L^\varphi_+(\Omega, \mathscr{A}, \mu), \lambda \in (0, 1)\}.$$

The conditions on φ defined above express respectively that all the exponents $p_{\varepsilon,r,\lambda}$, $\varepsilon > 0$, $r > 0$, $\lambda \in (0, 1)$, or all the exponents p_r, $r > 0$, are not null. The word "asymptotic" refers to the fact that $p(\sigma, r, \lambda)$ tends to $\sup_\lambda p(\sigma, r, \lambda)$ as λ tends to 0 by the submultiplicativity of $h_{\sigma,r}$ (see below) and increases as σ increases.

The p_r's are known in standard Orlicz spaces (see 4.4 below).

In fact we should speak of lower exponents because they give a minorization of the "rate of growth" of φ.

4.2. For example, (5) is fulfilled if, for some real number $p > 0$, the mapping $t \to t^{-p}\varphi(t, \omega)$ is increasing on the interval $(\sigma(\omega), \tau_{r,A}(\omega))$, or if $\lambda\tau_{r,A}(\omega) \leqslant \sigma(\omega)$. We have indeed in this case $\varphi(\lambda t, \omega) \leqslant \lambda^p\varphi(t, \omega)$ when λt and t lie in this interval.

So φ has non-null growth exponents if, for some real number $p > 0$, and for every set $A \in \mathscr{A}_{\mathrm{f}}$, the function $\varphi(\cdot, \omega)$ is p-convex on $(0, \tau_{r,A}(\omega))$ (i.e. if $\varphi(t^{1/p}, \omega)$ is a convex function of t for $t^{1/p} \leqslant \tau_{r,A}(\omega)$) for almost all $\omega \in A$.

4.3. Let us notice that $\varphi(\infty, \omega) = +\infty$ for almost all ω in every set $A \in \mathscr{A}_{\mathrm{f}}$ if φ has non-null asymptotic growth exponents.

Indeed, on the atomless part of A, $\tau_{r,A}$ is a.e. infinite and $\varphi(\cdot, \omega)$ is a.e. unbounded by (5). In the same way, if $a \subset A$ is an atom, the boundedness of $\varphi(\cdot, a)$ would contradict (5) for large r since $\varphi(\infty, a) < \infty$ if and only if $\tau_{r,A}(a) = +\infty$ for some $r \in \boldsymbol{R}_+$.

4.4. Let us now assume that φ is an Orlicz function (i.e. independent of ω) and that μ is either the counting measure of the set Ω, with Ω infinite (case (a)), or an atomless measure with $\sup\{\mu(A)\colon A \in \mathscr{A}_{\mathrm{f}}\} = +\infty$ (case (b)), or an atomless measure with $0 < \sup\{\mu(A)\colon A \in \mathscr{A}_{\mathrm{f}}\} < \infty$ (case (c)).

Then the functions $\tau_{r,A}$ take essentially a constant value τ_r. In case (a), $\tau_r = \sup\{t\colon \varphi(t) \leqslant r\}$, in cases (b) and (c), $\tau_r = +\infty$.

And it is easily seen that (5) can be replaced by

$$\sup\{\varphi(\lambda t)/\varphi(t)\colon s/\lambda < t \leqslant \tau_r\} \leqslant \alpha$$

where $s = \inf\{\operatorname{ess\,inf}_{\omega \in A}\sigma(\omega)\colon A \in \mathscr{A}_{\mathrm{f}}\}$. The conditions on σ give $s = 0$ in cases (a), (b). In case (c), the condition "$\varrho_\varphi(\sigma)$ small" becomes "s small".

Finally, *for measure spaces of the above types, an Orlicz function φ has non-null growth exponents if and only if it is strictly increasing and unbounded on $\boldsymbol{R}_+$ and satisfies, for every $\lambda = (0, 1)$,*

(6) $$\limsup_t \varphi(\lambda t)/\varphi(t) < 1$$

where t tends to 0 *in case* (a), *to* 0 *or* $+\infty$ *in case* (b), *to* $+\infty$ *in case* (c).

The necessity is clear (φ has to be strictly increasing on $\boldsymbol{R}_+$ even in case (a) because $\sup_r \tau_r = +\infty$).

For the converse, we have only to establish the inequality $\sup\{\varphi(\lambda t)/\varphi(t)\colon a \leqslant t \leqslant b\} < 1$ for $0 < a < b < +\infty$, $0 < \lambda < 1$ and φ strictly increasing (this is not obvious when φ is discontinuous): it is enough to show that we have $\limsup \varphi(\lambda t_n)/\varphi(t_n) < 1$ if (t_n) is a sequence in $[a, b]$ converging to a point t. For that purpose we remark that if $\lambda t < c < d < t$ then, for large n, c and d lie between λt_n and t_n, whence

$$\varphi(\lambda t_n)/\varphi(t_n) \leqslant \varphi(c)/\varphi(d) < 1.$$

One gets similar criteria for the nonvanishing of the asymptotic growth exponents p_r, $r > 0$, of an Orlicz function φ, in the above three cases: (6) has to be satisfied only for some $\lambda \in (0, 1)$, even small, and φ need not be strictly increasing.

Let us observe that, in these three cases, the asymptotic growth exponent p_r of an Orlicz function φ is independent of r (for $r < \varphi(\infty)$ in case (a)) and is a well-known index, denoted by σ_φ in [MaO] and by α_φ in [LT]: p_r is the limit as $\lambda \to 0_+$ of $\log k(\lambda)/\log\lambda$, where $k(\lambda)$ is the left-hand side of (6).

4.5. We now return to the general case.

These growth conditions will be used in the lemma below.

Let φ be a M.O. function on $(\Omega, \mathscr{A}, \mu)$. For every λ in $(0, 1)$ we define an increasing function $\theta(\lambda, \cdot)\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ by

(7) $$\theta(\lambda, r) = \sup\{\varrho_\varphi(\lambda x)\colon x \in L^\varphi(\Omega, \mathscr{A}, \mu), \varrho_\varphi(x) \leqslant r\}.$$

We have clearly for every measurable function x on $(\Omega, \mathscr{A}, \mu)$ with $\varrho_\varphi(x)$ finite,

(8) $$\varrho_\varphi(\lambda x) \leqslant \theta(\lambda, \varrho_\varphi(x)).$$

LEMMA 4.1. *If φ has non-null asymptotic growth exponents, then we have, for every $r \in \mathbf{R}_+$,*

(9) $$\lim_{\lambda\to 0} \theta(\lambda, r) = 0.$$

If φ has non-null growth exponents, then, for every $r \in \mathbf{R}_+$ and every $\lambda \in (0, 1)$, the sequence of the n-th iterates $\theta^n(\lambda, \cdot)$ of $\theta(\lambda, \cdot)$ satisfies

(10) $$\lim_n \theta^n(\lambda, r) = 0.$$

Proof. Let $r > 0$ and $\sigma \in L^\varphi(\Omega, \mathscr{A}, \mu)$, with $\sigma \geqslant 0$. We consider the function $h_{\sigma,r}(\lambda)$ defined in 4.1. Writing

$$\varphi(\lambda\lambda' t, \omega)/\varphi(t, \omega) = (\varphi(\lambda\lambda' t, \omega)/\varphi(\lambda' t, \omega))(\varphi(\lambda' t, \omega)/\varphi(t, \omega))$$

for $0 < \lambda, \lambda' < 1$ and $\sigma(\omega)/(\lambda\lambda') < t \leqslant \tau_{r,A}(\omega)$ we easily prove that $h_{\sigma,r}$ is submultiplicative on (0, 1).

If $0 < s \leqslant 1$ and $A \in \mathscr{A}_f$ we have a.e. on A

(11) $$\varphi(s\lambda t, \omega) \leqslant h_{\sigma,r}(\lambda)\varphi(t, \omega) + \varphi(s\sigma(\omega), \omega)$$

for $0 \leqslant t \leqslant \tau_{r,A}(\omega)$, $t \in \mathbf{R}_+$. If $x \in L^\varphi(\Omega, \mathscr{A}, \mu)$ and $\varrho_\varphi(x) \leqslant r$, then $|x| \leqslant \tau_{r,A}$ a.e. on every $A \in \mathscr{A}_f$, whence

$$\varrho_\varphi(s\lambda x) \leqslant h_{\sigma,r}(\lambda)\varrho_\varphi(x) + \varrho_\varphi(s\sigma) \leqslant h_{\sigma,r}(\lambda) r + \varrho_\varphi(s\sigma),$$

(12) $$\theta(s\lambda, r) \leqslant h_{\sigma,r}(\lambda) r + \varrho_\varphi(s\sigma).$$

Now, since φ has non-null asymptotic growth exponents, σ can be chosen in such a way that $h_{\sigma,r}(\lambda) < 1$ for some $\lambda \in (0, 1)$. This gives $\lim_{\lambda\to 0} h_{\sigma,r}(\lambda) = 0$ because $h_{\sigma,r}$ is submultiplicative. Moreover, $\lim_{s\to 0} \varrho_\varphi(s\sigma) = 0$ and we get (9).

Let us now assume that φ has non-null growth exponents. We first observe (10) can be readily derived from (12) when we can take $\sigma = 0$ in (5). Indeed, we now have $h_{\sigma,r}(\lambda) < 1$ for every $\lambda \in (0, 1)$. This remark can be applied for example when φ is an Orlicz function in the above cases (a) or (b) (but not (c)) of 4.4.

We now prove (10) in the general case. Since $\theta(\lambda, r) \leqslant r$ the sequence $\theta^n(\lambda, r)$, $n \geqslant 0$, is decreasing. We have to show that its limit $a \geqslant 0$ is null. Denoting $\theta(\lambda, s_+) = \inf_{t>s} \theta(\lambda, t)$, we have $\theta(\lambda, a_+) \geqslant a$. Indeed, if $t > a$ then $t > \theta^n(\lambda, r)$ for some n, whence $\theta(\lambda, t) \geqslant \theta^{n+1}(\lambda, r) \geqslant a$. So it suffices to prove that θ satisfies

$$\theta(\lambda, s_+) < s \quad \text{if } s > 0.$$

Let

$$q = \sup\{\varrho_\varphi(\lambda x)/\varrho_\varphi(x) \colon s/2 \leqslant \varrho_\varphi(x) \leqslant 2s\}$$

with $\sup\emptyset = 0$. Since φ has non-null growth exponents and $s > 0$, we have $\alpha = h_{\sigma,2s}(\lambda) < 1$ for some nonnegative function σ satisfying $\varrho_\varphi(\sigma) \leqslant s/4$. For every $A \in \mathscr{A}_f$ and for almost all $\omega \in A$, we have (obviously if $\lambda t \leqslant \sigma(\omega)$ or, else, by the definition of $h_{\sigma,2s}(\lambda)$)

$$\varphi(\lambda t, \omega) \leqslant \alpha\varphi(t, \omega) + (1-\alpha)\varphi(\sigma(\omega), \omega)$$

if $0 \leqslant t \leqslant \tau_{2s,A}(\omega)$, whence, for $s/2 \leqslant \varrho_\varphi(x) \leqslant 2s$ and $\beta = (1+\alpha)/2$,

$$\varrho_\varphi(\lambda x) \leqslant \alpha\varrho_\varphi(x) + (1-\alpha)s/4 \leqslant \beta\varrho_\varphi(x).$$

This gives $q \leqslant \beta$. Since $\beta < 1$ there exists t satisfying $s < t < s/\beta$. We have $\varrho_\varphi(\lambda x) \leqslant q\varrho_\varphi(x) \leqslant \beta t$ if $s/2 \leqslant \varrho_\varphi(x) \leqslant t$ and $\varrho_\varphi(\lambda x) \leqslant s/2 < \beta t$ for $\varrho_\varphi(x) \leqslant s/2$ since $\beta > 1/2$. So

$$\theta(\lambda, s_+) \leqslant \theta(\lambda, t) \leqslant \beta t < s.$$

This completes the proof of the lemma.

Remark 4.2. By (9) we see that when φ has non-null asymptotic growth exponents then the ball $B^\varphi(r)$ (cf. 2.2) is bounded for every $r \in \mathbf{R}_+$.

In particular, $L^\varphi(\Omega, \mathscr{A}, \mu)$ is locally bounded.

5. The condition Δ_2. A M.O. function φ on $(\Omega, \mathscr{A}, \mu)$ satisfies the so-called *condition* Δ_2 (resp. the *strong condition* Δ_2) when, for some (resp. for all) real numbers $r > 0$ there exist a finite constant $H > 0$ and an integrable function $g\colon (\Omega, \mathscr{A}, \mu) \to \mathbf{R}_+$ satisfying, for every $A \in \mathscr{A}_f$ and almost all $\omega \in A$,

(13) $$(t \leqslant \tau_{r,A}(\omega),\ t \in \mathbf{R}_+) \Rightarrow \varphi(2t, \omega) \leqslant H\varphi(t, \omega) + g(\omega).$$

Of course, the strong condition Δ_2 does not differ from the usual one when μ is atomless since in this case the functions $\tau_{r,A}$ are essentially infinite.

Moreover, let us assume that φ is an Orlicz function and, as above, that the measure μ is either the counting measure of Ω, Ω being infinite (case (a)), or atomless with $\sup\{\mu(A)\colon A \in \mathscr{A}_f\} = +\infty$ (case (b)), or atomless with this supremum finite (case (c)). Then it is easily seen that both the condition Δ_2 and the strong condition Δ_2 reduce to the usual condition

$$\limsup_t \varphi(2t)/\varphi(t) < \infty$$

where t tends to 0 in case (a), to 0 or $+\infty$ in case (b), to $+\infty$ in case (c).

Let us return to the general situation.

If φ satisfies the condition Δ_2 and $\varrho_\varphi(x) < \infty$, x measurable on $(\Omega, \mathscr{A}, \mu)$, then $\varrho_\varphi(Kx) < \infty$ for every scalar K. We have therefore

$$L_0^\varphi(\Omega, \mathscr{A}, \mu) = L^\varphi(\Omega, \mathscr{A}, \mu).$$

Indeed, if $r > 0$, there are only a finite number of atoms a such that $\varrho_\varphi(x1_a) > r$. On every set $A \in \mathscr{A}_f$ containing none of these atoms we have $|x| \leqslant \tau_{r,A}$ a.e. So the condition Δ_2 gives $\varrho_\varphi(2x) < \infty$, and therefore by induction $\varrho_\varphi(Kx) < \infty$ for every K.

If φ satisfies the strong condition Δ_2 we have, for every set B of measurable functions on $(\Omega, \mathscr{A}, \mu)$ and for every real number K;

$$\sup\{\varrho_\varphi(x):\ x\in B\}<\infty \Rightarrow \sup\{\varrho_\varphi(Kx):\ x\in B\}<\infty.$$

We also have the following statement.

LEMMA 5.1. *If φ satisfies the condition Δ_2 and if (x_n) is a sequence in $L^\varphi(\Omega, \mathscr{A}, \mu)$ satisfying $\lim_n \varrho_\varphi(x_n)=0$, then $\lim_n \varrho_\varphi(Kx_n)=0$ for every real number K (i.e. for the topology of $L^\varphi(\Omega, \mathscr{A}, \mu)$).*

It is enough to prove the result for $K=2$ and, for this, to prove that $\liminf \varrho_\varphi(2y_n)=0$ for every subsequence (y_n) of (x_n). Since $\varrho_\varphi(y_n)$ tends to 0 and $\varphi(t,\omega)>0$ for $t>0$, some subsequence (y'_n) of (y_n) tends to 0 μ-a.e. Let $r>0$ be given by the condition Δ_2. When n is large we have $\varrho_\varphi(y'_n)\leqslant r$ whence $|y'_n|\leqslant \tau_{r,A}$ a.e. on every $A\in\mathscr{A}_f$. This gives by (13)

$$\tilde\varphi(2|y'_n|)\leqslant H\tilde\varphi(|y'_n|)+g\wedge\tilde\varphi(2|y'_n|),$$

$$\varrho_\varphi(2y'_n)\leqslant H\varrho_\varphi(y'_n)+\textstyle\int g\wedge\tilde\varphi(2|y'_n|)\,\mu$$

where the last term tends to 0 in view of the dominated convergence theorem. So $\varrho_\varphi(2y'_n)$ tends to 0.

6. First fixed point theorem with non-null growth exponents condition.

THEOREM 6.1. *Let B be a star-shaped subset of a Musielak–Orlicz space $L_0^\varphi(\Omega, \mathscr{A}, \mu)$, sequentially compact for the topology of local convergence in measure and satisfying*

$$(\beta)\qquad \sup\{\varrho_\varphi(x-y):\ (x,y)\in B\times B\}<\infty.$$

We assume that the M.O. function φ has non-null growth exponents (for example, φ convex, or p-convex, $0<p\leqslant 1$) and satisfies the condition Δ_2.

Then a mapping $T\colon B\to B$ has a fixed point if it is nonexpansive for ϱ_φ.

The condition Δ_2 may be replaced by the more general assumption that T satisfies the following property (of uniform continuity type): (x_n), (y_n) being sequences in B, and (ε_n) a scalar sequence, the condition

$$(*)\qquad x_n-y_n\in\varepsilon_n(B-B)\quad \text{with}\quad \lim_n \varepsilon_n=0$$

implies

$$(**)\qquad \forall K\in\boldsymbol{R}_+,\quad \lim_n \varrho_\varphi(K(Tx_n-Ty_n))=0.$$

Proof. We first observe that $(*)$ implies $(**)$ if φ satisfies the condition Δ_2. Indeed, since φ has non-null growth exponents $B-B$ is bounded (Remark 4.2) so $\varrho_\varphi(x_n-y_n)$ tends to 0 if $(*)$ is true. Then $\varrho_\varphi(Tx_n-Ty_n)$ tends to 0 and the condition Δ_2 gives $(**)$ (Lemma 5.1).

In the rest of the proof, φ is no more supposed to satisfy this condition Δ_2.

As in the preceding theorem, we start by proving the existence of fixed points for the mappings $T_\lambda\colon B\to B$ defined for $0<\lambda<1$ by

$$T_\lambda x=u+\lambda(Tx-u),\qquad x\in B,$$

B being star-shaped with respect to the point $u\in B$.

This will be given by the following lemma.

LEMMA 6.2. *Let X be a Hausdorff topological space endowed with a mapping $j\colon X\times X\to[0,+\infty]$. We assume that, for every $y\in X$, the mapping $x\to j(x,y)$ is lower semi-continuous. We also assume that j satisfies*

$$R=\sup\{j(x,x_0):\ x\in X\}<\infty$$

for some $x_0\in X$ and that a sequence (x_n) converges in X provided $\lim j(x_p,x_n)=0$ as n and p tend to ∞.

Then a mapping $S\colon X\to X$ has a fixed point if we have

$$(14)\qquad j(Sx,Sy)\leqslant\theta(j(x,y))$$

for every $(x,y)\in X\times X$ and for some increasing function $\theta\colon \boldsymbol{R}_+\to\boldsymbol{R}_+$ satisfying $\lim_n\theta^n(r)=0$ for every $r\in\boldsymbol{R}_+$, where θ^n is the n-th iterate of θ for every positive integer n.

Proof. If S^n is the nth iterate of S we get by induction on n, for any whole numbers $n\geqslant 0$ and $k\geqslant 0$,

$$j(S^{k+n}x_0, S^nx_0)\leqslant\theta^n(j(S^kx_0,x_0))\leqslant\theta^n(R).$$

So $\lim_n j(S^{k+n}x_0,S^nx_0)=0$ uniformly in $k\geqslant 0$ and the sequence S^nx_0, $n\geqslant 0$, has a limit $x\in X$ satisfying by lower semi-continuity

$$j(x,S^nx_0)\leqslant\theta^n(R),$$

whence $\lim_n j(x,S^nx_0)=0$ and

$$j(Sx,S^nx_0)\leqslant\theta(j(x,S^{n-1}x_0))\leqslant j(x,S^{n-1}x_0)$$

tends to 0 (the assumptions on θ give $\theta(r)<r$ for $0<r<\infty$). Consequently $\lim_n S^nx_0=Sx$ (because the sequence $y_{2n}=S^nx_0$, $y_{2n+1}=Sx$ fulfils the convergence condition of the lemma), whence $Sx=x$.

Now we apply the lemma, taking for S the mappings T_λ, for X the set B endowed with the topology τ of local convergence in measure and for j the mapping $(x,y)\to\varrho_\varphi(x-y)$ of $B\times B$ into $\boldsymbol{R}_+$. Let us check that the hypotheses of the lemma are fulfilled.

Since φ is left-continuous $\varrho_\varphi(x-y)$ is lower semi-continuous with respect to x.

If $\lim_n\varrho_\varphi(x_n)=0$ then x_n tends to 0 for τ. Indeed, it suffices to show that 0 is a cluster point for τ of any subsequence (y_n) of (x_n). But (y_n) has a subsequence (z_n) such that $\varphi(|z_n(\omega)|,\omega)$ converges to 0 a.e. and this implies that z_n tends to 0 a.e., and therefore for τ.

Consequently, if $\lim_{n,p}\varrho_\varphi(x_p-x_n)=0$, (x_n) is a Cauchy sequence in (B,τ) and converges since (B,τ) is sequentially compact.

Finally, as $T_\lambda x-T_\lambda y=\lambda(Tx-Ty)$, the functions $\theta(\lambda,\cdot)$ of Lemma 4.1 satisfy

$$\varrho_\varphi(T_\lambda x-T_\lambda y)\leqslant\theta(\lambda,\varrho_\varphi(Tx-Ty))\leqslant\theta(\lambda,\varrho_\varphi(x-y))$$

and play the role of the function θ of the above lemma since φ has non-null growth exponents.

Hence there exists $x_\lambda\in B$ satisfying $T_\lambda x_\lambda=x_\lambda$, for each λ in $(0,1)$.

Now let us pick a sequence of positive real numbers $\lambda_n<1$ tending to 1. If $T_n=T_{\lambda_n}$ and $x_n=x_{\lambda_n}$, the mappings $T_n\colon B\to B$ satisfy, for every (x,y) in $B\times B$ and K in $\boldsymbol{R}_+$,

$$T_nx_n=x_n, \tag{15}$$

$$\varrho_\varphi(T_nx-T_ny)\leqslant\varrho_\varphi(x-y), \tag{16}$$

$$\varrho_\varphi(K(T_nx-T_ny))\leqslant\varrho_\varphi(K(Tx-Ty)), \tag{17}$$

$$\lim_n\varrho_\varphi(K(Tx-T_nx))=0. \tag{18}$$

In (17) the left-hand side, equal to $\varrho_\varphi(K\lambda_n(Tx-Ty))$, is majorized by $\varrho_\varphi(K(Tx-Ty))$ since λ_n lies in $(0,1)$. If $K=1$ the nonexpansiveness of T gives (16). And $Tx-T_nx=(1-\lambda_n)(Tx-u)$ satisfies (18).

We claim that the conditions (15)–(18) entail the following inequality, for every $c\in B$:

$$\sup_{0<s<1}\limsup_n\varrho_\varphi(s(Tc-x_n))\leqslant\sup_{0<s<1}\limsup_n\varrho_\varphi(s(c-x_n)). \tag{19}$$

This will be proved by using the following modular inequality. For strictly positive real numbers s_i, s and finite-valued measurable functions y, y_i, $i=1,\ldots,k$, we have

$$\varrho_\varphi(sy)\leqslant\sum_{i=1}^k\varrho_\varphi(s_i|y_i|)\quad\text{if}\quad y=\sum_{i=1}^k y_i,\ s^{-1}=\sum_{i=1}^k s_i^{-1}. \tag{20}$$

Indeed, we have $s|y|\leqslant\sup_i s_i|y_i|$ since sy is the barycenter of the s_iy_i's with the weights s/s_i.

In order to obtain (19) from (15)–(18) let us pick numbers r, s in the interval $(0,1)$. For every $x\in B$ let

$$x_r=u+r(x-u).$$

If $K=3s/(1-s)$ then $s^{-1}=1+3K^{-1}$. Hence, using (15) and (20),

$$\varrho_\varphi(s(Tc-x_n))\leqslant\varrho_\varphi(K(Tc-Tc_r))+\varrho_\varphi(K(Tc_r-T_nc_r))$$
$$+\varrho_\varphi(T_nc_r-T_nx_{n,r})+\varrho_\varphi(K(T_nx_{n,r}-T_nx_n)).$$

Since $c_r-x_{n,r}=r(c-x_n)$ we derive from (16), (17), (18)

$$\limsup_n\varrho_\varphi(s(Tc-x_n))$$
$$\leqslant\varrho_\varphi(K(Tc-Tc_r))+\limsup_n\varrho_\varphi(r(c-x_n))+\sup_n\varrho_\varphi(K(Tx_{n,r}-Tx_n)).$$

Now we let r tend to 1. From the "uniform continuity" property of T it follows that the last term tends to 0, since $x_{n,r}-x_n=(r-1)(x_n-u)$ is in $(r-1)(B-B)$. And the same is true for the first term. Hence,

$$\limsup_n\varrho_\varphi(s(Tc-x_n))\leqslant\sup_{0<r<1}\limsup_n\varrho_\varphi(r(c-x_n))$$

which gives (19).

Finally, in view of the compactness assumption on B, the sequence (λ_n) can be chosen in such a way that the corresponding fixed points x_n of T_{λ_n} converge μ-a.e. to a point $c\in B$.

Then the following lemma shows that the functional

$$x\to\sup_{0<s<1}\limsup_n\varrho_\varphi(s(x-x_n))$$

attains its minimum value at the unique point c. By (19) we must have $Tc=c$.

LEMMA 6.3 (first asymptotic center lemma). *Let φ be a M.O. function on $(\Omega,\mathscr{A},\mu)$ and (x_n) be a sequence in $L^\varphi(\Omega,\mathscr{A},\mu)$ converging μ-a.e. to a point $c\in L_0^\varphi(\Omega,\mathscr{A},\mu)$. Then, for every $x\in L_0^\varphi(\Omega,\mathscr{A},\mu)$, we have*

$$\sup_{0<s<1}\limsup_n\varrho_\varphi(s(x-x_n))\geqslant\varrho_\varphi(x-c)+\sup_{0<s<1}\limsup_n\varrho_\varphi(s(c-x_n)).$$

Proof. A number $s\in(\frac{1}{2},1)$ being chosen, it suffices to prove the inequality

$$\limsup_n\varrho_\varphi(x-x_n)\geqslant\varrho_\varphi(x-c)+\limsup_n\varrho_\varphi(s(c-x_n)). \tag{21}$$

Indeed, this will give for $0<s<r<1$

$$\limsup_n\varrho_\varphi(r(x-x_n))\geqslant\varrho_\varphi(r(x-c))+\limsup_n\varrho_\varphi(s(c-x_n))$$

and when r and s tend to 1 we get the lemma because, φ being left-continuous, $\varrho_\varphi(r(x-c))$ tends to $\varrho_\varphi(x-c)$.

Thus let $K=s/(1-s)$. We consider the decomposition

$$\tilde\varphi(|x-x_n|)=f_n+g_n,\quad f_n=\tilde\varphi(|x-x_n|\wedge K|x-c|).$$

Since φ is left-continuous and $K\geqslant 1$, $\liminf_n f_n\geqslant\tilde\varphi(|x-c|)$. Hence

$$\liminf_n\int f_n\mu\geqslant\varrho_\varphi(x-c). \tag{22}$$

On the other hand we have

$$g_n = \tilde{\varphi}(|x-x_n| \vee K|x-c|) - \tilde{\varphi}(K|x-c|).$$

From $s^{-1} = 1+K^{-1}$ we obtain $s|c-x_n| \leqslant |x-x_n| \vee K|x-c|$ (as in (20)). So,

$$g_n \geqslant (\tilde{\varphi}(s|c-x_n|) - \tilde{\varphi}(K|x-c|))_+$$
$$= \tilde{\varphi}(s|c-x_n|) - \tilde{\varphi}(K|x-c|) \wedge \tilde{\varphi}(s|c-x_n|).$$

But $\tilde{\varphi}(K|x-c|)$ is summable and $\tilde{\varphi}(s|c-x_n|)$ converges μ-a.e. to 0. So, by the Lebesgue dominated convergence theorem, we have

$$\lim_n \int \tilde{\varphi}(K|x-c|) \wedge \tilde{\varphi}(s|c-x_n|)\,\mu = 0$$

and

(23) $$\limsup_n \int g_n \mu \geqslant \limsup_n \varrho_\varphi(s(c-x_n)).$$

From (22) and (23) follows (21).

7. The conditions BL_k. These conditions are derived from a similar one introduced by H. Brézis and E. Lieb ([BL]).

7.1. We first define the strong condition BL_k.

For a given real number $k \geqslant 1$ we say that a M.O. function φ on $(\Omega, \mathscr{A}, \mu)$ satisfies the *strong condition* BL_k when, for every $r > 0$ and $\varepsilon > 0$, there exist a positive real number $\lambda < 1$ and an integrable function $g: (\Omega, \mathscr{A}, \mu) \to \boldsymbol{R}_+$ satisfying, a.e. on every set $A \in \mathscr{A}_{\mathrm{f}}$,

(24) $$0 \leqslant kt \leqslant \tau_{r,A}(\omega) \Rightarrow \varphi(t, \omega) \leqslant \varphi(\lambda t, \omega) + \varepsilon\varphi(kt, \omega) + g(\omega).$$

Of course, the strong condition BL_h implies the strong condition BL_k if $h < k$.

7.2. If φ fulfils the strong condition Δ_2 then the strong conditions BL_k, $k > 1$, are all equivalent to the strong condition BL_1.

Indeed, the strong condition Δ_2 implies that for every $r > 0$ there exists a real number $s > 0$ satisfying, for every $A \in \mathscr{A}_{\mathrm{f}}$, $2\tau_{r,A} \leqslant \tau_{s,A}$ a.e. on A. Using this, it is readily verified that the strong condition BL_k implies the strong condition BL_h for every $h \geqslant k/2$ and, iterating, for $h = 1$.

7.3. *A M.O. function φ on $(\Omega, \mathscr{A}, \mu)$ satisfies the strong condition BL_k for every $k > 1$ if for some measurable $\boldsymbol{R}_+$-valued function σ on $(\Omega, \mathscr{A}, \mu)$ with $\varrho_\varphi(\sigma) < \infty$ and for every $A \in \mathscr{A}_{\mathrm{f}}$ the mapping $\varphi(\cdot, \omega)$ is convex on $[\sigma(\omega), +\infty)$ for almost all $\omega \in A$.*

Indeed, let $k > 1$, $\varepsilon \in (0, k^{-1})$ and let $\lambda = (1-\varepsilon k)/(1-\varepsilon)$. Then $\lambda \in (0, 1)$ and if $t \in \boldsymbol{R}_+$ and $A \in \mathscr{A}_{\mathrm{f}}$ we have a.e. on A

$$\varphi(t, \omega) \leqslant \frac{k-1}{k-\lambda}\varphi((\lambda t) \vee \sigma(\omega), \omega) + \frac{1-\lambda}{k-\lambda}\varphi(kt, \omega)$$
$$\leqslant \varphi(\lambda t, \omega) + \varepsilon\varphi(kt, \omega) + \varphi(\sigma(\omega), \omega)$$

where $g(\omega) = \varphi(\sigma(\omega), \omega)$ is integrable.

From this we deduce that for $0 < p \leqslant 1$, p-convex Musielak–Orlicz functions satisfy the strong conditions BL_k for every $k > 1$. More generally, if a M.O. function φ fulfils the strong condition BL_k then, for $0 < p < \infty$, the M.O. function $\psi(t, \omega) = \varphi(t^p, \omega)$ satisfies the strong condition BL_h with $h = k^{1/p}$.

7.4. On the other hand, a M.O. function φ on $(\Omega, \mathscr{A}, \mu)$ satisfies the strong condition BL_1 (hence BL_k for every $k \geqslant 1$) if for some $\boldsymbol{R}_+$-valued measurable function σ on $(\Omega, \mathscr{A}, \mu)$ with $\varrho_\varphi(\sigma) < \infty$ and some positive real number p (and a.e. on every $A \in \mathscr{A}_{\mathrm{f}}$), the mapping $t \to t^{-p}\varphi(t, \omega)$ is decreasing for $t \geqslant \sigma(\omega)$ (for $t > 0$ when $\sigma(\omega) = 0$), in particular when $t \to \varphi(t^{1/p}, \omega)$ is concave on $[\sigma(\omega)^p, +\infty)$.

We have indeed for $\lambda \in (0, 1)$

$$\varphi(t, \omega) \leqslant \varphi(t \vee (\sigma(\omega)/\lambda), \omega) \leqslant \lambda^{-p}\varphi((\lambda t) \vee \sigma(\omega), \omega)$$
$$\leqslant \varphi(\lambda t, \omega) + (\lambda^{-p}-1)\varphi(t, \omega) + \lambda^{-p}\varphi(\sigma(\omega), \omega)$$

so that (24) is satisfied with $k = 1$ when λ is close to 1.

Let us also observe that φ obviously satisfies the strong condition BL_1 if the function $\varphi(\infty, \cdot)$ is integrable on $(\Omega, \mathscr{A}, \mu)$.

7.5. We now suppose that φ is an Orlicz function and that μ is as in one of the cases (a), (b), (c) of Section 4.4, i.e. respectively the counting measure with Ω infinite, atomless and (roughly speaking) unbounded, atomless and bounded.

Then the functions $\tau_{r,A}$ take essentially a constant value τ_r (infinite when μ is atomless) depending only on r and the function g in (24) can be assumed to be identically null in cases (a) and (b).

We therefore easily get the following statements.

For a given $k \geqslant 1$, φ satisfies the strong condition BL_k (for μ) if and only if we have

(25) $$\lim_{\lambda \nearrow 1} \limsup_t (\varphi(t) - \varphi(\lambda t))/\varphi(kt) = 0, \qquad \lambda \in (0, 1),$$

where t tends to 0 in case (a), to 0 or $+\infty$ in case (b), to $+\infty$ in case (c), φ being furthermore continuous on $\boldsymbol{R}_+$ in cases (a) and (b) (φ need not be continuous in case (c)).

One sees in particular that φ satisfies the strong condition BL_1 (and therefore BL_k for any k) if and only if we have

$$\lim_{\lambda \nearrow 1} \liminf_{t} \varphi(\lambda t)/\varphi(t) = 1, \quad \lambda \in (0, 1), \tag{26}$$

where as above t tends to 0, to $\overset{0}{+\infty}$, or to $+\infty$ according to the case, φ being continuous on $\boldsymbol{R}_+$ in cases (a) and (b).

It is easily checked as above that (25) holds for any number $k > 1$ if φ is p-convex for some $p > 0$ in a neighborhood of 0 in case (a), of 0 and $+\infty$ in case (b), of $+\infty$ in case (c); and that we have the stronger condition (26) if the mapping $t \to t^{-p}\varphi(t)$ is decreasing for some $p \in \boldsymbol{R}_+$ in a neighborhood of 0, 0 and $+\infty$, or $+\infty$ as above.

Let us observe that for a p-convex Orlicz function, in cases (a), (b), (c), the strong condition BL_1 is equivalent to the condition Δ_2 (this is a consequence of 7.2 and of the fact that (26) implies the condition Δ_2). So, the condition (CV) of [LD2] means in fact that φ is convex and satisfies Δ_2 at 0.

We may also remark that (26) holds if $\lim_t \varphi(\lambda t)/\varphi(t)$ exists and is non-null for every $\lambda \in (0, 1)$, in other words if φ is regularly varying (in the sense of Karamata ([Ka])) at 0, 0 and $+\infty$ or $+\infty$ according to the case.

Indeed, this limit will be equal to λ^p for some $p \geq 0$, being an increasing multiplicative function of λ.

If $\lim_t \varphi(\lambda t)/\varphi(t) = 0$ for some $\lambda \in (0, 1)$, then φ clearly satisfies (25), i.e. the strong condition BL_k with $k = 1/\lambda$.

7.6. We now define the weak condition BL_k.

We say that a M.O. function φ on $(\Omega, \mathscr{A}, \mu)$ satisfies the *weak condition* BL_k for some real number $k \geq 1$ when, for any real numbers $r > 0$ and $\varepsilon > 0$, there exist finite positive constants H, K and an integrable function g: $(\Omega, \mathscr{A}, \mu) \to \boldsymbol{R}_+$ satisfying, a.e. on every $A \in \mathscr{A}_r$,

$$0 \leq s \leq t \leq \frac{1}{k}\tau_{r,A}(\omega) \tag{27}$$

$$\Rightarrow \varphi(t, \omega) \leq \varphi(s, \omega) + \varepsilon\varphi(kt, \omega) + H\varphi\big(K(t-s), \omega\big) + g(\omega).$$

For example, φ obviously satisfies the weak condition BL_k if, for every $A \in \mathscr{A}_r$, $\varphi(\cdot, \omega)$ is subadditive on $\big(0, k^{-1}\tau_{r,A}(\omega)\big)$ for almost all $\omega \in A$.

7.7. *The strong condition* BL_k *implies the weak condition* BL_k.

The converse is true for a M.O. function with non-null asymptotic growth exponents: in this case we say, more simply, "the condition BL_k".

Indeed, let us assume that (24) holds (with $0 < \lambda < 1$ and $\omega \in A$) and that we have $0 \leq s \leq t \leq k^{-1}\tau_{r,A}(\omega)$. Since $t < (t-s)/(1-\lambda)$ if $\lambda t > s$, we have

$$\varphi(\lambda t, \omega) \leq \varphi(s, \omega) + \varphi\left(\frac{\lambda}{1-\lambda}(t-s), \omega\right)$$

and (27) is fulfilled with $H = 1$ and $K = \lambda/(1-\lambda)$.

Conversely, let us assume that φ has non-null asymptotic growth exponents and satisfies the weak condition BL_k. With $0 \leq kt \leq \tau_{r,A}(\omega)$ and $s = \lambda t$, $\lambda \in (0, 1)$, (27) yields

$$\varphi(t, \omega) \leq \varphi(\lambda t, \omega) + \varepsilon\varphi(kt, \omega) + H\varphi\big(K(1-\lambda)t, \omega\big) + g(\omega).$$

Applying (11) in Section 4, where the function $h_{\sigma,r}$ satisfies $h_{\sigma,r}(0+) = 0$, we have for λ close to 1

$$H\varphi\big(K(1-\lambda)t, \omega\big) \leq \varepsilon\varphi(t, \omega) + \varphi\big(\sigma(\omega), \omega\big)$$

with $\varphi\big(\sigma(\cdot), \cdot\big)$ integrable. So (27) implies a relation similar to (24).

8. Second fixed point theorem with non-null growth exponents condition.

Theorem 8.1. *Let φ be a M.O. function on $(\Omega, \mathscr{A}, \mu)$ with non-null growth exponents and let $B \subset L^{\varphi}_0(\Omega, \mathscr{A}, \mu)$ be star-shaped and sequentially compact for the topology of local convergence in measure.*

Moreover, for some real number $k \geq 1$, we suppose that φ satisfies the condition BL_k *and that we have*

$$\sup\{\varrho_\varphi(k(x-y)) : (x, y) \in B \times B\} < \infty. \tag{β_k}$$

Then a mapping T: $B \to B$ has a fixed point if it is nonexpansive for ϱ_φ.

By 4.2 and 7.3 we have the following special case.

Corollary 8.2. *If φ is convex (or p-convex, $p > 0$) and $B \subset L^{\varphi}_0(\Omega, \mathscr{A}, \mu)$ is as above, satisfying (β_k) for some $k > 1$, then a map T: $B \to B$ has a fixed point when it is nonexpansive for ϱ_φ.*

If we compare this with Theorem 6.1 when φ is convex and fails to satisfy the condition Δ_2, we see that the uniform continuity assumption on T in Theorem 6.1 has been dropped, but the boundedness condition (β) of that theorem has been strengthened: we need (β_k) for some $k > 1$.

Proof. As in the proof of Theorem 6.1 we construct mappings T_n: $B \to B$ and points $x_n \in B$ satisfying (15)–(18) and converging μ-a.e. to a point $c \in B$.

If $0 < s < 1$ and $K = s/(1-s)$ we have as in (20)

$$\varrho_\varphi\big(s(Tc - x_n)\big) \leq \varrho_\varphi\big(K(Tc - T_n c)\big) + \varrho_\varphi(T_n c - T_n x_n)$$

and (16), (18) give

$$\sup_{0<s<1} \limsup_n \varrho_\varphi\big(s(Tc - x_n)\big) \leq \limsup_n \varrho_\varphi(c - x_n).$$

But B satisfies (β_k) and, by 7.7, φ satisfies the strong condition BL_k. Hence the left-continuity statement of Lemma 8.3 below gives

$$\sup_{0<s<1} \limsup_n \varrho_\varphi\big(s(Tc - x_n)\big) \leq \sup_{0<s<1} \limsup_n \varrho_\varphi\big(s(c - x_n)\big)$$

and, by Lemma 6.3, we therefore get $Tc = c$.

LEMMA 8.3. *We assume that, for some real number $k \geqslant 1$, a M.O. function φ on $(\Omega, \mathscr{A}, \mu)$ satisfies the strong condition* BL_k. *Let (x_n) be a sequence of points of $L^\varphi(\Omega, \mathscr{A}, \mu)$ tending to 0 μ-a.e.*

Then the mapping $s \to \limsup_n \varrho_\varphi(sx_n)$, defined on $\boldsymbol{R}$, *is left-continuous at 1 if $\sup_n \varrho_\varphi(kx_n) < \infty$. It is also right-continuous at 1 if $\sup_n \varrho_\varphi(k'x_n) < \infty$ for some real number $k' > k$.*

If φ satisfies only the weak condition BL_k, *we have the same conclusions under the further requirement that (x_n) be bounded for the topology of $L^\varphi(\Omega, \mathscr{A}, \mu)$.*

Proof. Let $\varepsilon > 0$. If $r = \sup_n \varrho_\varphi(kx_n)$, we have $k|x_n| \leqslant \tau_{r,A}$ a.e. on every $A \in \mathscr{A}_f$. So the strong condition BL_k yields a number $\lambda \in (0, 1)$ and a nonnegative integrable function g on $(\Omega, \mathscr{A}, \mu)$ satisfying

$$\tilde{\varphi}(|x_n|) \leqslant \tilde{\varphi}(\lambda|x_n|) + \varepsilon\tilde{\varphi}(k|x_n|) + \tilde{\varphi}(|x_n|) \wedge g,$$

$$\varrho_\varphi(x_n) \leqslant \varrho_\varphi(\lambda x_n) + \varepsilon\varrho_\varphi(kx_n) + \int \tilde{\varphi}(|x_n|) \wedge g\mu.$$

In view of the dominated convergence theorem, the last term tends to 0 and we get

$$\limsup_n \varrho_\varphi(x_n) \leqslant \limsup_n \varrho_\varphi(\lambda x_n) + \varepsilon \sup_n \varrho_\varphi(kx_n).$$

This gives the left-continuity. For the right-continuity we find in the same way that for $k/\lambda \leqslant k'$

$$\limsup_n \varrho_\varphi(x_n/\lambda) \leqslant \limsup_n \varrho_\varphi(x_n) + \varepsilon \sup_n \varrho_\varphi(k'x_n).$$

Under the weak condition BL_k, we have for some constants H, K and for every $s \in (0, 1)$

$$\tilde{\varphi}(|x_n|) \leqslant \tilde{\varphi}(s|x_n|) + \varepsilon\tilde{\varphi}(k|x_n|) + H\tilde{\varphi}(K(1-s)|x_n|) + g \wedge \tilde{\varphi}(|x_n|),$$

$$\limsup_n \varrho_\varphi(x_n) \leqslant \limsup_n \varrho_\varphi(sx_n) + \varepsilon \sup_n \varrho_\varphi(kx_n) + H \sup_n \varrho_\varphi(K(1-s)x_n).$$

If (x_n) is bounded the last term tends to 0 as $s \to 1$ and we get the left-continuity. The right-continuity is proved in an analogous manner.

9. Mappings with a nonexpansive compact approximation property. As in Theorem 3.1 the spaces considered in this section need not be locally bounded. But here, the functions φ are no more subadditive. They are submitted to much weaker conditions. Even in locally bounded spaces, we need not the restrictive assumption on growth exponents of Theorems 6.1 and 8.1. The counterpart is that B has to fulfil some compact approximation property, easily satisfied in spaces of sequences (Theorem 9.3).

The main problem is to approximate T with nonexpansive mappings T_n possessing fixed points.

In Theorems 6.1 and 8.1, the condition on the growth exponents chiefly served to yield mappings T_n satisfying the "θ-contraction" condition (14) of Lemma 6.2, and therefore endowed with fixed points.

Theorem 12.2 below will show that, in non-locally bounded spaces, there is generally little hope to obtain mappings T_n with this "θ-contraction" property, especially in spaces of sequences.

The difficulty was solved in Theorem 3.1 using the subadditivity of φ. Here, we find mappings T_n with compact range (for the topology of the space L^φ), and consequently possessing fixed points, by Proposition 9.1 below.

PROPOSITION 9.1. *Let B be a star-shaped subset of $L^\varphi(\Omega, \mathscr{A}, \mu)$ compact for the topology of $L^\varphi(\Omega, \mathscr{A}, \mu)$ and satisfying $\varrho_\varphi(x-y) < \infty$ for every pair x, y in B.*

We also assume that, for almost all $\omega \in \Omega$, the function $\varphi(\cdot, \omega)$ is strictly increasing.

Then a mapping $T\colon B \to B$ has a fixed point if it is nonexpansive for ϱ_φ.

Remark 9.2. It is known ([RS]) that if B is a compact star-shaped subset of an (F)-normed space $(E, \|\cdot\|)$ satisfying $\|tx\| < \|x\|$ when $x \neq 0$ and $t \in (0, 1)$, then every mapping $T\colon B \to B$ nonexpansive for $\|\cdot\|$ has a fixed point. This gives the above statement when φ is subadditive (and strictly increasing), because in this case ϱ_φ is an (F)-norm with the required property.

Proof. As in Section 3, the mappings $T_\lambda\colon B \to B$ defined for $\lambda \in (0, 1)$ by $T_\lambda x = u + \lambda(Tx - u)$ satisfy for x, y in B

$$(*) \qquad \varrho_\varphi(T_\lambda x - T_\lambda y) < \varrho_\varphi(x - y) \quad \text{if } x \neq y.$$

The operator T_λ is continuous for the topology τ_φ induced by $L^\varphi(\Omega, \mathscr{A}, \mu)$. Indeed, if x, x_n are in B and $\lim_n \varrho_\varphi(x - x_n) = 0$, then $\varrho_\varphi(T_\lambda x - T_\lambda x_n)$ tends to 0. From this it follows as in the proof of Theorem 6.1 that $T_\lambda x - T_\lambda x_n$ tends to 0 for the topology of local convergence in measure, and hence for the topology τ_φ by compactness.

Consequently, by Fatou's lemma and the left-continuity of φ, the mapping $x \to \varrho_\varphi(T_\lambda x - x)$ is lower semi-continuous on (B, τ_φ). Since (B, τ_φ) is compact this mapping attains its minimum at some point $x_\lambda \in B$. So,

$$\varrho_\varphi(T_\lambda(T_\lambda x_\lambda) - T_\lambda x_\lambda) \geqslant \varrho_\varphi(T_\lambda x_\lambda - x_\lambda)$$

whence $T_\lambda x_\lambda = x_\lambda$ by $(*)$.

We deduce from this that T has a fixed point.

Indeed, by compactness we can find a sequence (λ_n), $0 < \lambda_n < 1$, converging to 1 and a sequence (x_n) converging to a point $c \in B$ for the topology τ_φ, with $T_{\lambda_n} x_n = x_n$ for all n. But, just as T_λ, the mapping T is continuous for the topology τ_φ. So,

$$Tc - c = \lim_n (Tx_n - T_{\lambda_n} x_n) = \lim_n (1 - \lambda_n)(Tx_n - u) = 0.$$

Now we consider a Musielak–Orlicz space $l^{\varphi}(\mu) = L^{\varphi}(N, \mathscr{P}(N), \mu)$ of sequences on the set N of nonnegative integers, for the weight $\mu = (\mu_\omega)_{\omega \in N}$.

For every integer $n \geqslant 0$, let P_n be the finite rank projection of $l^{\varphi}(\mu)$ defined by

$$P_n x = (x_0, x_1, \ldots, x_n, 0, 0, \ldots), \qquad x \in l^{\varphi}(\mu).$$

We recall that a subset B of $l^{\varphi}(\mu)$ is said to be *bounded* when $\varrho_\varphi(\varepsilon x)$ tends to 0 as $\varepsilon \to 0$ uniformly for $x \in B$.

THEOREM 9.3. *Let B be a subset of $l_0^{\varphi}(\mu)$ compact for the topology of pointwise convergence, star-shaped with respect to some "center-point" u and satisfying*

$$u + P_n(x-u) \in B, \qquad n = 0, 1, \ldots,$$

for every $x \in B$. We assume that, for every $\omega \in N$, the function $\varphi(\cdot, \omega)$ is strictly increasing.

Then a mapping $T: B \to B$ has a fixed point if it is nonexpansive for ϱ_φ and if either the condition (c_1) *or* (c_2) *below is fulfilled*:

(c_1) *B is bounded in $l^{\varphi}(\mu)$ and φ satisfies the strong condition Δ_2 (for μ).*

(c_2) *For some real number $k \geqslant 1$, φ satisfies the weak condition BL_k (for μ) and we have*

$$\sup\{\varrho_\varphi(k(x-y)) : (x, y) \in B \times B\} < \infty.$$

COROLLARY 9.4. *Let us assume that the functions $\varphi(\cdot, \omega)$, $\omega \in N$, are strictly increasing and unbounded, and that φ satisfies the strong condition Δ_2.*

Then every mapping $T: B \to B$ nonexpansive for ϱ_φ has a fixed point if B is a ball (cf. Section 2) *$B^{\varphi}(r)$, $r \in \boldsymbol{R}_+$, and if φ satisfies the weak condition BL_1, or if B is a ball $B^{\psi}(r)$ for some M.O. function ψ on N for which $B^{\psi}(r)$ is bounded in $l^{\varphi}(\mu)$.*

Indeed, these balls fulfil the compactness and stability requirements and they are contained in $l_0^{\varphi}(\mu)$ since $l_0^{\varphi}(\mu) = l^{\varphi}(\mu)$ by the condition Δ_2.

Let us add that the ball $B^{\psi}(r)$ is bounded in $l^{\varphi}(\mu)$ if, for every $\alpha > 0$, one can find a real number $\lambda > 0$ and a nonnegative element σ of $l^{\varphi}(\mu)$ satisfying $\varphi(\lambda t, \omega) \leqslant \alpha\psi(t, \omega)$ provided $\omega \in N$, $t > \sigma(\omega)$ and $\psi(t, \omega)\mu_\omega \leqslant r$. Indeed, if $(x_n) \subset B^{\psi}(r)$ and $\alpha > 0$, then, for n large, we have

$$\varrho_\varphi(x_n/n) \leqslant \alpha\varrho_\psi(x_n) + \varrho_\varphi(\sigma/n) \leqslant 2\alpha r$$

and $\varrho_\varphi(x_n/n)$ tends to 0.

In particular, if the ball $B^{\varphi}(r)$ itself is bounded in $l^{\varphi}(\mu)$, i.e. if the asymptotic growth exponent p_r of Section 4.1 is not null, it has the fixed point property for ϱ_φ-nonexpansive mappings, without need of the stronger assumption of Theorem 6.1, nor of the condition BL_1.

When φ (strictly increasing and unbounded at each point ω) satisfies the weak condition BL_k, $k \geqslant 1$, but not the strong condition Δ_2, B will be a suitable set if

$$B = \{x \in \boldsymbol{C}^N : \varrho_\varphi(nx) \leqslant M_n, n = 1, 2, \ldots\}$$

where $\lim_n M_n = +\infty$. Indeed, B is included in $l_0^{\varphi}(\mu)$ and compact in $\boldsymbol{C}^N$.

Proof of the theorem. For every integer $n \geqslant 0$ the set $B_n = u + P_n(B-u)$ is contained in B, star-shaped (with respect to u), compact for the topology of $l^{\varphi}(\mu)$ and the mapping

$$T_n: x \to u + P_n(Tx-u), \qquad x \in B,$$

is nonexpansive for ϱ_φ and maps B_n into B_n. By the previous proposition, T_n has a fixed point $x_n \in B_n$. Since $B \subset l_0^{\varphi}$, $\lim_n T_n x = x$ for all $x \in B$, for the topology τ_φ.

Under the condition (c_1) we continue exactly as in the proof of Theorem 6.1 since the conditions (15)–(18) hold and B is bounded (which, together with the condition Δ_2, gives the uniform continuity requirement for T). We also need that ϱ_φ be bounded on $B-B$: this follows from the strong condition Δ_2 and the boundedness of B.

Let us now assume the condition (c_2) is satisfied.

Passing to a subsequence (simultaneously in (x_n) and (T_n)) we can assume that (x_n) tends for pointwise convergence on N to some point $c \in B$ and we show that c is a fixed point for T.

First we have

$$\limsup_n \varrho_\varphi(Tc-x_n) \leqslant \limsup_n \varrho_\varphi(c-x_n). \tag{28}$$

Indeed,

$$\varrho_\varphi(Tc-x_n) = \varrho_\varphi(Tc-T_n c) + \varrho_\varphi(T_n c - T_n x_n)$$

because $x_n = T_n x_n$ and, by the definition of T_n, $Tc - T_n c$ and $T_n c - T_n x_n$ have disjoint supports. The first term tends to 0 and the second is majorized by $\varrho_\varphi(c-x_n)$, whence (28).

Now the following lemma gives $Tc = c$ if we apply it to $x = Tc$.

LEMMA 9.5 (second asymptotic center lemma). *Let φ be a M.O. function on a measure space $(\Omega, \mathscr{A}, \mu)$ satisfying the weak condition BL_k for some real number $k \geqslant 1$.*

Then if x and c are points of $L_0^{\varphi}(\Omega, \mathscr{A}, \mu)$ and (x_n) is a sequence in $L^{\varphi}(\Omega, \mathscr{A}, \mu)$ converging a.e. to c, we have

$$\limsup_n \varrho_\varphi(x-x_n) \geqslant \varrho_\varphi(x-c) + \limsup_n \varrho_\varphi(c-x_n) \tag{29}$$

if $\sup_n \varrho_\varphi(k(c-x_n))$ and $\sup_n \varrho_\varphi(k(x-x_n))$ are finite.

Proof. When (x_n) is bounded or φ satisfies the strong condition BL_k, (29) is a consequence of Lemma 6.3. and Lemma 8.3.

When $\varphi(t, \omega)$ is continuous with respect to t and the function g of the weak condition BL_k is null, (29) can be derived from Theorem 2 in [BL].

Let us prove (29) in the general case. We have

$$(30) \qquad \tilde{\varphi}(|x-x_n|) = \tilde{\varphi}(|x-x_n| \wedge 2|x-c|) + (\tilde{\varphi}(|x-x_n|) - \tilde{\varphi}(2|x-c|))_+ .$$

Since φ is left-continuous and x_n tends to c a.e., we have by Fatou's lemma

$$(31) \qquad \liminf_n \int \tilde{\varphi}(|x-x_n| \wedge 2|x-c|)\mu \geqslant \int \tilde{\varphi}(|x-c|)\mu = \varrho_\varphi(x-c).$$

On the other hand, the function

$$f_n = |(\tilde{\varphi}(|x-x_n|) - \tilde{\varphi}(2|x-c|))_+ - \tilde{\varphi}(|c-x_n|)|$$

tends to 0 a.e. (if $x(\omega) \neq c(\omega)$ we have for n large $|x-x_n|(\omega) < 2|x-c|(\omega)$, whence $\varphi(|x-x_n|(\omega), \omega) \leqslant \varphi(2|x-c|(\omega), \omega)$) and

$$f_n \leqslant |\tilde{\varphi}(|x-x_n|) - \tilde{\varphi}(|c-x_n|)| + \tilde{\varphi}(2|x-c|).$$

Let $\varepsilon > 0$ be given and let r be the lower upper bound of the $\varrho_\varphi(k(c-x_n))$'s and $\varrho_\varphi(k(x-x_n))$'s, $n \geqslant 0$. By the condition BL_k there exist constants H, K and an integrable nonnegative function g satisfying

$$|\tilde{\varphi}(|t|) - \tilde{\varphi}(|s|)| \leqslant \varepsilon\tilde{\varphi}(k(|s| \vee |t|)) + H\tilde{\varphi}(K|s-t|) + g$$

if t, s are measurable functions on $(\Omega, \mathscr{A}, \mu)$ with $\varrho_\varphi(s) \vee \varrho_\varphi(t) \leqslant r$. So

$$f_n \leqslant \varepsilon\tilde{\varphi}(k(|x-x_n| \vee |c-x_n|)) + f_n \wedge [H\tilde{\varphi}(K|x-c|) + g + \tilde{\varphi}(2|x-c|)].$$

On the right-hand side, the function in brackets is integrable because x and c are in $L_0^\varphi(\Omega, \mathscr{A}, \mu)$, so the integral of the second term tends to 0 as $n \to \infty$ by dominated convergence. We therefore have $\limsup_n \int f_n \mu \leqslant 2\varepsilon r$. Finally, $\lim_n \int f_n \mu = 0$, we have

$$(32) \qquad \limsup_n \int (\tilde{\varphi}(|x-x_n|) - \tilde{\varphi}(2|x-c|))_+ \mu = \limsup_n \int \tilde{\varphi}(|c-x_n|)\mu$$

and (29) follows from (30), (31), (32).

The above method can be generalized.

If $B \subset L^\varphi(\Omega, \mathscr{A}, \mu)$ let us say that a sequence of mappings τ_n: $B \to B$ is a *nonexpansive compact approximation of the identity* for ϱ_φ when the τ_n's are nonexpansive for ϱ_φ, the sets $\tau_n(B)$ being relatively compact in B for the topology of $L^\varphi(\Omega, \mathscr{A}, \mu)$, and if, for this topology, $\tau_n x$ tends to x for each x in B.

PROPOSITION 9.6. *Let φ be a M.O. function on $(\Omega, \mathscr{A}, \mu)$, the mapping $\varphi(\cdot, \omega)$ being strictly increasing and continuous for almost all $\omega \in \Omega$.*

Let $B \subset L_0^\varphi(\Omega, \mathscr{A}, \mu)$ be star-shaped and sequentially compact for the topology of local convergence in measure. We also suppose that B has a nonexpansive compact approximation of the identity (τ_n) for ϱ_φ.

Then a mapping T: $B \to B$, nonexpansive for ϱ_φ, has a fixed point under either the condition (c_1) *or* (c_2) *of the previous theorem.*

Proof. The sets $\tau_n(B)$ are included in star-shaped subsets B_n of B, compact for the topology of $L^\varphi(\Omega, \mathscr{A}, \mu)$. The mappings $T_n = \tau_n \circ T$ have the same properties as those of Theorem 9.3, except the condition (17) (in the proof of Theorem 6.1); but (16) suffices, using the condition Δ_2 of the hypothesis (c_1). The only change occurs in the proof of the inequality

$$\limsup_n \varrho_\varphi(Tc-x_n) \leqslant \limsup_n \varrho_\varphi(c-x_n)$$

where $x_n = T_n x_n$ tends a.e. to c, under the hypothesis (c_2).

Let $u_n = Tc - x_n$ and $v_n = T_n c - x_n$. If $\varepsilon > 0$ is given, the condition BL_k yields a summable nonnegative function g and constants H, K satisfying

$$|\tilde{\varphi}(|u_n|) - \tilde{\varphi}(|v_n|)| \leqslant \varepsilon\tilde{\varphi}(k|u_n| \vee k|v_n|) + H\tilde{\varphi}(K(|u_n - v_n|)) + g \wedge |\tilde{\varphi}(|u_n|) - \tilde{\varphi}(|v_n|)|.$$

We integrate and let n tend to $+\infty$. On the right-hand side $\varrho_\varphi(K(u_n - v_n))$ tends to 0 because $u_n - v_n = Tc - T_n c$. The integral of the last term tends to 0 by dominated convergence. Indeed, both u_n and v_n converge a.e. to $Tc - c$, so $\tilde{\varphi}(|u_n|) - \tilde{\varphi}(|v_n|)$ converges a.e. to 0: the continuity assumption on φ is used here. Finally,

$$\limsup_n \int |\tilde{\varphi}(|u_n|) - \tilde{\varphi}(|v_n|)|\mu \leqslant \varepsilon \sup_n \varrho_\varphi(k|u_n| \vee k|v_n|).$$

Hence the left-hand side is null and

$$\limsup_n \varrho_\varphi(Tc-x_n) = \limsup_n \varrho_\varphi(T_n c - x_n) \leqslant \limsup_n \varrho_\varphi(c-x_n)$$

since $x_n = T_n x_n$ and T_n is nonexpansive.

EXAMPLE. The τ_n's may be truncations $\tau_n(x) = \mathrm{sg}(x)(f_n \wedge |x|)$, where $x = \mathrm{sg}(x)|x|$, if we can find a sequence of functions $f_n \geqslant 0$ in $L_0^\varphi(\Omega, \mathscr{A}, \mu)$ tending a.e. to $+\infty$ such that $\tau_n(B) \subset B$. The required conditions are easily checked, B being as in the proposition. For instance, $\tau_n(B)$ is compact in $L^\varphi(\Omega, \mathscr{A}, \mu)$ by the dominated convergence theorem.

10. Mappings nonexpansive for the Minkowski functional ν_φ. We now consider the functional

$$\nu_\varphi(x) = \inf\{a > 0:\ \varrho_\varphi(x/a) \leqslant 1\}, \quad x \in L^\varphi(\Omega, \mathscr{A}, \mu),$$

and examine mappings T: $B \to B$ nonexpansive for ν_φ, i.e. satisfying

$$\nu_\varphi(Tx - Ty) \leqslant \nu_\varphi(x-y), \quad x, y \text{ in } B.$$

The function $\nu_\varphi\colon L^\varphi(\Omega, \mathscr{A}, \mu) \to \boldsymbol{R}_+$ is the Minkowski functional of the ball $B^\varphi(1)$ (cf. 2.2) of $L^\varphi(\Omega, \mathscr{A}, \mu)$: it is positively homogeneous and $\nu_\varphi(x) \leqslant 1 \Leftrightarrow \varrho_\varphi(x) \leqslant 1$. When φ is convex ν_φ is the Luxemburg norm of $L^\varphi(\Omega, \mathscr{A}, \mu)$.

THEOREM 10.1. *Let φ be a M.O. function on $(\Omega, \mathscr{A}, \mu)$, the function $\varphi(\cdot, \omega)$ being strictly increasing and unbounded for almost all $\omega \in \Omega$.*

Let B be a star-shaped subset of $L_0^\varphi(\Omega, \mathscr{A}, \mu)$, sequentially compact for the topology of local convergence in measure and satisfying the conditions

(i) $$\sup\{\nu_\varphi(x-y)\colon (x, y) \in B\times B\} < \infty,$$

(ii) $$\sup\{\varrho_\varphi(K(x-y))\colon (x, y) \in B\times B\} < \infty \quad \textit{for all } K \in \boldsymbol{R}_+.$$

We also assume that φ satisfies the strong condition BL_k for some $k \geqslant 1$ (if B is bounded the weak condition BL_k suffices).

Then a mapping $T\colon B \to B$ has a fixed point if it is nonexpansive for ν_φ.

This theorem covers a wide range of cases (Section 7). It applies for example to every function φ convex, or p-convex $(0 < p \leqslant 1)$, or concave $-$subadditive when B is bounded$-$unbounded and strictly increasing. The space $L^\varphi(\Omega, \mathscr{A}, \mu)$ need not be locally bounded.

The inequality (i) is equivalent to

(i′) $$\sup\{\varrho_\varphi(\varepsilon(x-y))\colon (x, y) \in B\times B\} \leqslant 1 \quad \text{for some } \varepsilon > 0.$$

If φ satisfies the strong condition Δ_2 then (i) implies (ii).

When φ has non-null asymptotic growth exponents, then (i) is equivalent to the boundedness of B (Remark 4.2).

Remark 10.2. On an atom a of μ, φ may be bounded, provided we have $\varphi(\infty, a)\mu(a) > 1$, and we need that $\varphi(t, a)$ be strictly increasing in t only when $\varphi(t, a)\mu(a) \leqslant 1$.

Proof of the theorem. First we establish that if $\lambda \in (0, 1)$ the mapping $T_\lambda\colon B \to B$ defined as previously has a fixed point.

It suffices to check that Lemma 6.2 can be applied, with $S = T_\lambda$, $j(x, y) = \nu_\varphi(x-y)$ and $X = B$ endowed with the topology τ of local convergence in measure.

The functional $x \to \nu_\varphi(x-y)$ is lower semi-continuous for τ since by Fatou's lemma and the left-continuity of φ this is true for the modular ϱ_φ.

For a sequence (x_n) in $L^\varphi(\Omega, \mathscr{A}, \mu)$ the condition $\lim \nu_\varphi(x_n) = 0$, i.e. $\varrho_\varphi(x_n/a_n) \leqslant 1$ for some null sequence of numbers $a_n > 0$, implies $\lim x_n = 0$ for τ (in other words, $B^\varphi(1)$ is bounded for τ). Indeed, if $\varepsilon > 0$, $A \in \mathscr{A}_f$ and $A_n = \{\omega \in A\colon |x_n(\omega)| > \varepsilon\}$, then $\int_{A_n} \varphi(\varepsilon/a_n, \cdot)\mu \leqslant 1$ for every n, which by Egorov's theorem implies $\lim_n \mu(A_n) = 0$.

So, if (x_n) is in B and $\nu_\varphi(x_p - x_n)$ tends to 0 as n, p tend to ∞, the sequence (x_n) is a Cauchy sequence of (B, τ), hence convergent in (B, τ) since B is sequentially compact for τ.

And the condition (14) of Lemma 6.2 is satisfied since

$$\nu_\varphi(T_\lambda x - T_\lambda y) = \lambda \nu_\varphi(Tx - Ty) \leqslant \lambda \nu_\varphi(x-y).$$

So, by this lemma, T_λ has a fixed point.

Using the sequential compactness of B for τ we find a sequence of positive real numbers $\lambda_n < 1$ converging to 1 and a sequence of points $x_n \in B$ converging a.e. to a point $c \in B$ and satisfying $T_{\lambda_n} x_n = x_n$.

We want to show that c is a fixed point for T.

We first prove the inequality

(33) $$\limsup_n \nu_\varphi(Tc - x_n) \leqslant \limsup_n \nu_\varphi(c - x_n).$$

Since we have

$$\nu_\varphi(T_{\lambda_n} c - x_n) = \nu_\varphi(T_{\lambda_n} c - T_{\lambda_n} x_n) = \lambda_n \nu_\varphi(Tc - Tx_n) \leqslant \nu_\varphi(c - x_n)$$

it suffices to show that, if $Tc \neq c$, the following holds:

$$\lim_n \big(\nu_\varphi(Tc - x_n) - \nu_\varphi(T_{\lambda_n} c - x_n)\big) = 0.$$

But both sequences $Tc - x_n$ and $T_{\lambda_n} c - x_n$ converge a.e. to $Tc - c$, which is assumed non-null, and their difference $(1-\lambda_n)(Tc - u)$ converges to 0 for the topology τ_φ of $L^\varphi(\Omega, \mathscr{A}, \mu)$. So it is enough to apply the following lemma.

LEMMA 10.3. *If $0 \notin X \subset L^\varphi(\Omega, \mathscr{A}, \mu)$, if X is sequentially compact for the topology τ of local convergence in measure, with $\sup\{\nu_\varphi(x)\colon x \in X\} < \infty$, and if the function $\varphi(\cdot, \omega)$ is strictly increasing for almost all $\omega \in \Omega$, then the mapping $\nu_\varphi\colon X \to \boldsymbol{R}_+$ is uniformly continuous for the uniform structure induced on X by $L^\varphi(\Omega, \mathscr{A}, \mu)$.*

Proof. Let $b = \sup\{\nu_\varphi(x)\colon x \in X\}$, $\varepsilon > 0$, and

$$Z = \{\lambda x\colon (\varepsilon + b)^{-1} \leqslant \lambda \leqslant \varepsilon^{-1},\ x \in X,\ \varrho_\varphi(\lambda x) \leqslant 1\}.$$

Clearly, Z is sequentially τ-compact and $0 \notin Z$. We also have

(34) $$a(r) > 0 \quad \text{if} \quad a(r) = \inf\{\varrho_\varphi(z) - \varrho_\varphi(rz)\colon z \in Z\} \text{ and } r \in (0, 1).$$

Indeed, we can find a sequence (z_n) in Z converging a.e. to a point $z \in Z$ (whence $z \neq 0$) satisfying

$$a(r) = \lim_n \int \big(\tilde{\varphi}(|z_n|) - \tilde{\varphi}(r|z_n|)\big)\mu.$$

Since φ is strictly increasing and left-continuous, Fatou's lemma gives for $r < s < 1$

$$a(r) \geqslant \int \liminf_n \big(\tilde{\varphi}(|z_n|) - \tilde{\varphi}(r|z_n|)\big)\mu \geqslant \int \big(\tilde{\varphi}(|z|) - \tilde{\varphi}(s|z|)\big)\mu > 0.$$

Now, let x, y in X satisfy

$$\varrho_\varphi\left(\frac{r}{\eta\varepsilon}(y-x)\right) \leqslant a(r),$$

where $r \in (0, 1)$ and $\eta > 0$ will be chosen later.

If $\lambda(x) = (\varepsilon + v_\varphi(x))^{-1}$ then $\lambda(x)x \in Z$ and (34) gives

$$\varrho_\varphi(r\lambda(x)x) \leqslant \varrho_\varphi(\lambda(x)x) - a(r) \leqslant 1 - a(r)$$

whence, using the modular inequality (20),

$$\varrho_\varphi\left(\frac{r}{1+\eta}\lambda(x)y\right) \leqslant \varrho_\varphi(r\lambda(x)x) + \varrho_\varphi\left(\frac{r}{\eta}\lambda(x)(y-x)\right) \leqslant 1$$

since $\lambda(x) \leqslant \varepsilon^{-1}$. So, if r and η have been chosen in such a way that $1+\eta \leqslant r(1+\varepsilon)$ we get

$$v_\varphi(y) \leqslant \frac{1}{r}(1+\eta)(\varepsilon + v_\varphi(x)) \leqslant v_\varphi(x) + \varepsilon(1+\varepsilon+b)$$

whence

$$|v_\varphi(y) - v_\varphi(x)| \leqslant \varepsilon(1+\varepsilon+b)$$

and the lemma is proved.

So the inequality (33) holds. But this implies $Tc = c$ by the following lemma, because φ being a.e. unbounded, $v_\varphi(Tc - c) > 0$ if $Tc - c \neq 0$.

Consequently, c is a fixed point for T.

LEMMA 10.4 (third asymptotic center lemma). *Let x and c be two points of a Musielak–Orlicz space $L_0^\varphi(\Omega, \mathscr{A}, \mu)$ satisfying $v_\varphi(x-c) > 0$ and let (x_n) be a sequence in $L_0^\varphi(\Omega, \mathscr{A}, \mu)$ converging a.e. to c and satisfying $\sup_n \varrho_\varphi(K(c-x_n)) < \infty$ for every $K \in \boldsymbol{R}_+$.*

Then, if φ satisfies the strong condition BL_k for some $k \geqslant 1$, we have

$$\limsup_n v_\varphi(x-x_n) > \limsup_n v_\varphi(c-x_n)$$

if the left-hand side is finite (when (x_n) is bounded, the weak condition BL_k suffices).

Proof. Let $\alpha = \limsup_n v_\varphi(x-x_n)$, which is assumed to be finite. By Lemma 6.3 or Lemma 9.5 we have

$$1 \geqslant \sup_{a>\alpha} \limsup_n \varrho_\varphi\left(\frac{1}{a}(x-x_n)\right)$$
$$\geqslant \sup_{a>\alpha}\left[\varrho_\varphi\left(\frac{x-c}{a}\right) + \limsup_n \varrho_\varphi\left(\frac{1}{a}(c-x_n)\right)\right].$$

This shows first that we have $1 \geqslant \sup_{a>\alpha} \varrho_\varphi((x-c)/a)$, whence $\alpha \geqslant v_\varphi(x-c)$, and therefore $\alpha > 0$. Lemma 8.3 gives therefore

$$1 \geqslant \varrho_\varphi((x-c)/\alpha) + \inf_{b<\alpha} \limsup_n \varrho_\varphi((c-x_n)/b).$$

Since $\varrho_\varphi((x-c)/\alpha) > 0$ we have for some $b < \alpha$

$$\limsup_n \varrho_\varphi((c-x_n)/b) < 1$$

whence

$$\limsup_n v_\varphi(c-x_n) \leqslant b < \alpha.$$

Remark 10.5. When φ does not satisfy the strong condition Δ_2, we cannot omit in Lemma 10.4 the hypothesis that $\sup_n \varrho_\varphi(K(c-x_n))$ is finite for every $K \in \boldsymbol{R}_+$, as shown in [L], p. 533 (under the strong condition Δ_2, this assumption is a consequence of the finiteness of $\limsup_n v_\varphi(x-x_n)$).

Of course, when B is compact for the topology τ_φ of $L^\varphi(\Omega, \mathscr{A}, \mu)$, Lemma 10.4 is immediate, without need of that hypothesis, nor of the condition BL_k (because $x_n \to c$ for τ_φ). So we can conclude as follows.

Remark 10.6. When B is compact for the topology τ_φ of $L^\varphi(\Omega, \mathscr{A}, \mu)$, Theorem 10.1 is true under the sole assumption that $\varphi(\cdot, \omega)$ is strictly increasing and unbounded for almost all $\omega \in \Omega$.

If φ is furthermore convex, then v_φ is a norm, and the result is well known: in a normed space, every compact star-shaped set has the fixed point property for nonexpansive mappings (see a generalization in [RS]).

Now, under the hypotheses of Theorem 10.1, B is automatically τ_φ-compact if $\mu(\Omega) < \infty$ and if φ is an Orlicz function satisfying, for some number $k > 1$, $\lim \varphi(t)/\varphi(kt) = 0$ as $t \to +\infty$ (which is of course incompatible with the condition Δ_2); the function $\varphi(t) = e^t - 1$ is an example. This follows from the equi-integrability of the sets $\{\tilde{\varphi}(K(x-y)) \colon (x, y) \in B \times B\}$, $K \in \boldsymbol{R}_+$, given by (ii) and the assumption on φ.

So, Theorem 10.1 is of interest chiefly under the condition Δ_2, or in the general case, for unbounded measures μ (e.g. in sequence spaces l^φ).

11. A strong condition of nonexpansiveness. Strengthening the nonexpansiveness assumption of the preceding section, we get a fixed point theorem for arbitrary M.O. functions φ.

For $r > 0$, let v_φ^r be the Minkowski functional of the ball $B^\varphi(r)$ of $L^\varphi(\Omega, \mathscr{A}, \mu)$:

$$v_\varphi^r(x) = \inf\{a > 0 \colon \varrho_\varphi(x/a) \leqslant r\}, \quad x \in L^\varphi(\Omega, \mathscr{A}, \mu).$$

THEOREM 11.1. *Let B be a star-shaped subset of a Musielak–Orlicz space $L_0^\varphi(\Omega, \mathscr{A}, \mu)$, sequentially compact for the topology of local convergence in* *measure and satisfying, for some real number* $R > 0$,

(35) $$\sup\{v_\varphi^R(x-y)\colon (x, y)\in B\times B\} < \infty.$$

Then a mapping $T\colon B\to B$ *has a fixed point if it is nonexpansive for every functional* v_φ^r, $0 < r \leqslant R$.

The condition (35) signifies that $B-B$ is absorbed by $B^\varphi(R)$.

It is easily checked that T fulfils the nonexpansiveness condition of the theorem if and only if we have, for x, y in B and $t\in \boldsymbol{R}_+$,

(36) $$\varrho_\varphi(t(x-y))\leqslant R \Rightarrow \varrho_\varphi(t(Tx-Ty))\leqslant \varrho_\varphi(t(x-y)).$$

Proof of the theorem. We again consider the mappings (see Section 3) $T_\lambda\colon B\to B$, $0<\lambda<1$, and their iterates T_λ^n, n positive integer. We have for $r\in(0, R)$ and x, y in B

$$v_\varphi^r(T_\lambda x - T_\lambda y) = \lambda v_\varphi^r(Tx-Ty)\leqslant \lambda v_\varphi^r(x-y)$$

whence, for every whole number $n\geqslant 0$,

$$v_\varphi^r(T_\lambda^{1+n}x - T_\lambda^n x)\leqslant \lambda^n v_\varphi^r(T_\lambda x - x).$$

So, choosing a point x in B and putting $y_{\lambda,n} = T_\lambda^n x$, we have

$$\lim_n v_\varphi^r(T_\lambda y_{\lambda,n}-y_{\lambda,n}) = 0, \quad r>0,$$

whence

$$\lim_n \varrho_\varphi(K(T_\lambda y_{\lambda,n}-y_{\lambda,n})) = 0, \quad K\in\boldsymbol{R}_+.$$

By compactness we find a sequence (λ_n) in $(0, 1)$ tending to 1 and a sequence (x_n) in B converging a.e. to some point c in B and satisfying

$$\lim_n \varrho_\varphi(K(T_{\lambda_n}x_n - x_n)) = 0, \quad K\in\boldsymbol{R}_+.$$

Let us prove that c is a fixed point for T. If $m\in\boldsymbol{R}_+$ is the supremum in (35), let $\varepsilon\in(0,\infty)$ with $\varepsilon^{-1}\geqslant m$ and let $0<s<t<\varepsilon$. If K is defined by $s^{-1} = t^{-1}+2K^{-1}$ the modular inequality (20) gives

$$\varrho_\varphi(s(Tc-x_n))\leqslant \varrho_\varphi(K(Tc-T_{\lambda_n}c))+\varrho_\varphi(t(T_{\lambda_n}c - T_{\lambda_n}x_n)) + \varrho_\varphi(K(T_{\lambda_n}x_n - x_n)).$$

The first term on the right-hand side, equal to $\varrho_\varphi((1-\lambda_n)K(Tc-u))$, converges to 0, and so does the last term by construction of the x_n's. From $t\in(0,\varepsilon)$ follows $\varrho_\varphi(t(c-x_n))\leqslant R$ and by (36) the second term is majorized by $\varrho_\varphi(t(c-x_n))$. So

$$\limsup_n \varrho_\varphi(s(Tc-x_n))\leqslant \limsup_n \varrho_\varphi(t(c-x_n)), \quad 0<s<t<\varepsilon,$$

whence

$$\sup_{s<\varepsilon}\limsup_n \varrho_\varphi(s(Tc-x_n))\leqslant \sup_{s<\varepsilon}\limsup_n \varrho_\varphi(s(c-x_n)),$$

the left-hand side being finite (at most equal to R). So Lemma 6.3 gives $\varrho_\varphi(\varepsilon(Tc-c)) = 0$, and therefore $Tc = c$.

12. Strict contractions in non-locally bounded Orlicz spaces. If B is a subset of a Musielak–Orlicz space $L^\varphi(\Omega, \mathscr{A}, \mu)$ we say that a map $T\colon B\to L^\varphi(\Omega,\mathscr{A},\mu)$ is a *strict contraction* for the modular ϱ_φ when we have

$$\varrho_\varphi(Tx-Ty)\leqslant k\varrho_\varphi(x-y), \quad (x, y)\in B\times B,$$

for some positive constant $k<1$.

First, in the following proposition we observe that the Banach fixed point principle applies to these contractions, even when ϱ_φ is not subadditive: this is a special case of Lemma 6.2, where we take for X the set B endowed with the topology τ or τ_φ. We have a similar result for strict contractions for the Minkowski functional v_φ, with $\varphi(\infty,\omega) = \infty$ a.e. and B sequentially τ-complete (or τ_φ-complete when $B^\varphi(1)$ is bounded). It was used for the T_λ's in Section 10.

PROPOSITION 12.1. *Let* $B\subset L^\varphi(\Omega,\mathscr{A},\mu)$ *be sequentially complete for the topology* τ *of local convergence in measure (or the topology* τ_φ *of* $L^\varphi(\Omega,\mathscr{A},\mu)$ *when* φ *fulfils the condition* Δ_2*) and assume that*

$$\sup\{\varrho_\varphi(x-x_0)\colon x\in B\}<\infty$$

for some point $x_0\in B$.

Then every strict contraction $T\colon B\to B$ *for* ϱ_φ *has a fixed point.*

We want to show that there exist few examples of strict contractions for ϱ_φ in non-locally bounded spaces, chiefly in Orlicz sequence spaces l^φ (for instance when $\varphi(t) = -1/\log t$ on some interval $(0, a)$, $a<1$). So, as announced in Section 9, it is generally not possible in these spaces to use the Banach principle (or Proposition 12.1 above) to get approximating fixed points for a mapping nonexpansive for ϱ_φ.

Following [HOS] we say that a set $B\subset l^\varphi$ is *coordinatewise star-shaped* with respect to a center $u = (u_i)_{i=0}^\infty$ when it fulfils the following condition: For every element $x = (x_i)_{i=0}^\infty$ of B and for every sequence of real numbers t_i with $0\leqslant t_i\leqslant 1$, the sequence $(u_i + t_i(x_i-u_i))_{i=0}^\infty$ is an element of B.

THEOREM 12.2. *Let* l^φ *be a non-locally bounded Orlicz space of sequences (for the counting measure on* $\boldsymbol{N}$*), and* B *a subset of* l^φ *coordinatewise star-shaped with respect to some center* u *and satisfying* $\varrho_\varphi(x-u)<\infty$ *for every* $x\in B$.

Then a mapping $T\colon B\to l^\varphi$ *is constant if it is a strict contraction for* ϱ_φ.

We recall ([Ro]) that l^φ is not locally bounded if and only if the Orlicz function φ satisfies

(37) $$\limsup_{t\to 0+} \varphi(\lambda t)/\varphi(t) = 1, \quad 0 < \lambda < 1$$

(i.e. if and only if (6) in Section 4 holds for no λ in (0, 1)).

Proof. By translation invariance we may assume that the "center" u of B is 0. Let $x = (x_0, x_1, \ldots)$ be a point of B. We want to prove the equality $T(x) = T(0)$.

Let us show that the function

$$F(t_0, t_1, \ldots) = T(t_0 x_0, t_1 x_1, \ldots),$$

where $0 \leqslant t_i \leqslant 1$ for every $i \in \mathbf{N}$, is constant with respect to each variable t_i separately. The variables t_j being fixed for $j \neq i$, let $F^i(t) = F(t_0, \ldots, t_{i-1}, t, t_{i+1}, \ldots)$, $0 \leqslant t \leqslant 1$, and let F^i_n be the nth component of F^i, for every $n \in \mathbf{N}$. If t, t' are in [0, 1], we have

$$\varphi(|F^i_n(t') - F^i_n(t)|) \leqslant \varrho_\varphi(F^i(t') - F^i(t)) \leqslant k\varphi(|t'-t|\,|x_i|)$$

whence $|F^i_n(t') - F^i_n(t)| \leqslant |x_i|\,|t'-t|$ since φ is increasing and $0 \leqslant k < 1$. Hence F^i_n is absolutely continuous on [0, 1], in particular differentiable almost everywhere. Let s be a point of [0, 1) at which F^i_n is differentiable, with derivative λ. For $t > 0$ small enough we have

$$\varphi(|\lambda|\, t/2) \leqslant \varphi(|F^i_n(s+t) - F^i_n(s)|) \leqslant k\varphi(t\,|x_i|).$$

If $x_i = 0$ then $\lambda = 0$. If $x_i \neq 0$ and $\lambda' = |\lambda|/(2|x_i|)$ we have

$$\limsup_{t\to 0+} \varphi(\lambda' t)/\varphi(t) \leqslant k < 1.$$

From (37) follows $\lambda' = 0$ and therefore $\lambda = 0$.

So F^i_n is constant on [0, 1].

As n is arbitrary, F^i is constant on [0, 1], for every $i \in \mathbf{N}$. So $F(0, 0, \ldots) = F(1, 0, \ldots) = F(1, 1, 0, \ldots) = \ldots$, i.e. $T(P_n x) = T(0)$ for every integer $n \geqslant 0$ if $P_n x = (x_0, \ldots, x_n, 0, \ldots)$. But

$$\varrho_\varphi(Tx - TP_n x) \leqslant k\varrho_\varphi(x - P_n x) = k \sum_{i>n} \varphi(|x_i|)$$

tends to 0 as $n \to \infty$ since $\varrho_\varphi(x) < \infty$.

So $\varrho_\varphi(T(x) - T(0)) = 0$ and $T(x) = T(0)$.

Corollary 12.3. *If U is a connected open subset of a non-locally bounded Orlicz sequence space l^φ, every strict contraction $T\colon U \to l^\varphi$ for ϱ_φ is constant.*

Indeed, T is locally constant since the neighbourhoods $u + rB^\varphi(r)$, $r > 0$, at every point u are coordinatewise star-shaped (with center u).

Remark 12.4. This corollary is generally false in function spaces.

For instance, let φ be an Orlicz function. If $\mathbf{R}_+$ is endowed with the Lebesgue measure and $0 < k < 1$ the mapping $T_k\colon x(\omega) \to x(\omega/k)$ of the Orlicz space $L^\varphi(\mathbf{R}_+)$ into itself is a strict contraction for ϱ_φ: it satisfies

$$\varrho_\varphi(T_k x - T_k y) = k\varrho_\varphi(x-y).$$

However, if φ is an Orlicz function satisfying (37), $L^\varphi(\mathbf{R}_+)$ contains nontrivial star-shaped subsets B without nonconstant strict ϱ_φ-contractions $B \to B$: it suffices to embed l^φ in $L^\varphi(\mathbf{R}_+)$ by a linear injection f preserving the modular ϱ_φ (for instance, $f(x) = \sum x_i 1_{(i,i+1)}$ if $x = (x_i)_{i=0}^\infty$).

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