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Received June 21, 1983 (1904)
New version September 12, 1984
Revised version March 3, 1986

STUDIA MATHEMATICA, T. LXXXVI. (1987)

Some results on the convergence of weighted sums of random elements in separable Banach spaces

by

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Abstract. Let X_n, $n \geqslant 1$, be a sequence of random elements taking values in a separable Banach space, A_n, $n \geqslant 1$, a sequence of real random variables and a_{nk}, $n \geqslant 1$, $k \geqslant 1$, a double array of real numbers. Under some conditions, we show that $\sum_{k\geqslant 1} a_{nk} A_k X_k$, $n \geqslant 1$, converges to 0 in the mean if and only if $\sum_{k\geqslant 1} a_{nk} f(A_k X_k)$, $n \geqslant 1$, converges to 0 in probability for every continuous linear functional f from the Banach space to the real line (Section 3). The main result in Section 3 unifies many results in the literature on the convergence of weighted sums of sequences of random elements. In Section 4, results on strong convergence are established. Marcinkiewicz–Zygmund–Kolmogorov's and Brunk–Chung's Strong Laws of Large Numbers are extended to separable Banach spaces. Using a certain stability theorem, a general result on strong convergence for weighted sums is proved from which many results in the literature follow as special cases under much less restrictive conditions.

1. Introduction. This paper is devoted to a study of limit theorems for weighted sums of sequences of random elements in separable Banach spaces. Section 2 presents some preliminaries needed in the subsequent sections. Section 3 concentrates on the convergence in probability and convergence in the mean of weighted sums of random elements. Let X_n, $n \geqslant 1$, be a sequence of random elements defined on some probability space $(\Omega, \mathscr{B}, P)$ taking values in a separable Banach space B, A_n, $n \geqslant 1$, a sequence of real random variables defined on Ω and a_{nk}, $n \geqslant 1$, $k \geqslant 1$, a double array of real numbers. Under some conditions, we show that $\sum_{k\geqslant 1} a_{nk} A_k X_k$, $n \geqslant 1$, converges to 0 in the mean if and only if $\sum_{k\geqslant 1} a_{nk} f(A_k X_k)$, $n \geqslant 1$, converges to 0 in probability for every continuous linear functional f from B to the real line $\boldsymbol{R}$ (Theorem 3.3). This result unifies many results in the literature on the underlying theme of Theorem 3.3. Moreover, the conditions imposed in

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AMS Subject Classification Scheme 1979: Primary 60B12, Secondary 60F15, 60F05.
Key words and phrases: random elements, Banach space, weighted sums, convergence in probability, in mean and almost surely, uniform integrability and uniform tightness.
* Part of the work of this author was supported by the Air Force Office of Scientific Research under contract F49620-82-K-0001. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Theorem 3.3 are weaker than those imposed in the results whose unification is the main goal of Theorem 3.3.

Section 4 deals with strong convergence. A result of Rohatgi [19, Theorem 2, p. 306] on the almost sure convergence of weighted sums of sequences of real random variables is extended to cover random variables taking values in separable Banach spaces. Marcinkiewicz–Zygmund–Kolmogorov's and Brunk–Chung's Strong Laws of Large Numbers are extended to separable Banach spaces. Finally, in this section we use a certain stability theorem to establish a strong limit theorem for weighted sums of random elements in separable normed linear spaces from which some results of Padgett and Taylor [17] and of Wei and Taylor [30] follow under much weaker conditions.

Hoffmann-Jørgensen and Pisier [10, Theorem 2.4, p. 592] proved a Weak Law of Large Numbers for a sequence X_n, $n \geqslant 1$, of Banach space valued random variables under the following condition: "Given $\varepsilon > 0$ there exists a compact subset C of the Banach space such that

$$\int_{\{X_n \in C^c\}} \|X_n\| \, dP < \varepsilon$$

for every $n \geqslant 1$." Taylor [24, Theorem 2] also imposed the above condition to prove a Strong Law of Large Numbers. He also observed that the above condition is implied by the conditions that X_n, $n \geqslant 1$, is uniformly tight and that $E\|X_n\|^p \leqslant K$, a constant, for every $n \geqslant 1$ for some $p > 1$. In Section 2, we show that the above condition is equivalent to the conditions that X_n, $n \geqslant 1$, is uniformly tight and that $\|X_n\|$, $n \geqslant 1$, is uniformly integrable. Further, in Section 3, we point out that the Weak Law of Large Numbers due to Hoffmann-Jørgensen and Pisier [10, Theorem 2.4, p. 592] is a special case of Theorem 3.3.

2. Preliminaries. Let B be a separable normed linear space equipped with a norm $\|\cdot\|$. The Borel σ-field on B is the smallest σ-field on B containing all closed subsets of B, and the sets in this σ-field are called Borel sets. Let $(\Omega, \mathscr{B})$ be a Borel structure, i.e., $\mathscr{B}$ is a σ-field on Ω. A map X from Ω to B is called a *random element* if it is measurable, i.e., $X^{-1}(G) \in \mathscr{B}$ for every Borel subset G of B. If $B = \boldsymbol{R}$, the real line, random elements are called *random variables*.

A subset C of B is said to be *totally bounded* if for every $r > 0$ there exists a finite number of points $x_1, x_2, \ldots, x_n$ in B such that $C \subset \bigcup_{i=1}^{n} O(x_i, r)$, where $O(x_i, r)$ is an open ball in B with centre at x_i and radius r. If B is a Banach space, a subset C of B is compact if and only if C is totally bounded and closed.

Let $(\Omega, \mathscr{B}, P)$ be a probability space. A sequence X_n, $n \geqslant 1$, of random elements defined on Ω taking values in B is said to be *uniformly tight* (*pre-tight*) if for every $\varepsilon > 0$ there exists a compact (totally bounded Borel) subset C of B such that $P\{X_n \in C^c\} < \varepsilon$ for every $n \geqslant 1$. A sequence X_n, $n \geqslant 1$, of random variables defined on Ω is said to be *uniformly absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_A |X_n| \, dP < \varepsilon$ for every $n \geqslant 1$ whenever $A \in \mathscr{B}$ and $P(A) < \delta$. X_n, $n \geqslant 1$, is said to be *uniformly integrable* if X_n, $n \geqslant 1$, is uniformly absolutely continuous and $\sup_{n \geqslant 1} E|X_n| < \infty$.

The following definition plays an important role in some of the limit theorems in Section 3.

Definition 2.1. Let X_n, $n \geqslant 1$, be a sequence of random elements defined on a probability space $(\Omega, \mathscr{B}, P)$ taking values in a separable normed linear space B and $r \geqslant 0$. X_n, $n \geqslant 1$, is said to be *compactly* (*pre-compactly*) *uniformly r-th-order integrable* if for every $\varepsilon > 0$ there exists a compact (totally bounded Borel) subset C of B such that

$$\int_{\{X_n \in C^c\}} \|X_n\|^r \, dP < \varepsilon \quad \text{for every } n \geqslant 1.$$

The following lemma relates the above notions with other known ideas.

Lemma 2.2. *Let X_n, $n \geqslant 1$, be a sequence of random elements defined on a probability space $(\Omega, \mathscr{B}, P)$ taking values in a separable Banach space B. Let $r > 0$. In the following,* (i) *and* (ii) *are equivalent, and* (ii) *implies* (iii).

(i) (a) *X_n, $n \geqslant 1$, is uniformly tight.*

(b) *$\|X_n\|^r$, $n \geqslant 1$, is uniformly absolutely continuous.*

(ii) *X_n, $n \geqslant 1$, is compactly uniformly r-th-order integrable.*

(iii) *$\|X_n\|^r$, $n \geqslant 1$, is uniformly integrable.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) and (ii) ⇒ (i)(b) are easy to establish. We show that (ii) ⇒ (i)(a). Let $\varepsilon > 0$.

By (ii), for each $i \geqslant 1$, there exists a compact subset K_i of B such that

$$\int_{K_i^c} \|X_n\|^r \, dP < \varepsilon/(i^r 2^i)$$

for every $n \geqslant 1$. We have $P\{X_n \in (K_i^c \cap [O(0, 1/i)]^c)\} < \varepsilon/2^i$ for every $n \geqslant 1$ and this inequality follows from

$$(1/i^r) P\{X_n \in (K_i^c \cap [O(0, 1/i)]^c)\} \leqslant \int_{K_i^c \cap [O(0,1/i)]^c} \|X_n\|^r \, dP \leqslant \int_{K_i^c} \|X_n\|^r \, dP,$$

where $O(0, 1/i)$ denotes the open ball in B with centre at 0 and radius $1/i$. Observe that, since K_i is a totally bounded subset of B, $K_i \cup O(0, 1/i)$ can be covered by a finite number of open balls in B each with radius $1/i$. Let

$$K = \bigcap_{i \geqslant 1} (K_i \cup O(0, 1/i)).$$

It now follows that K is a totally bounded subset of B. Since B is a complete metric space, $\bar{K}$, the closure of K in B, is compact. Note also that

$$P\{X_n \in \bar{K}^c\} \leqslant P\{X_n \in K^c\} \leqslant \sum_{i \geqslant 1} \varepsilon/2^i = \varepsilon$$

for every $n \geqslant 1$. This proves that X_n, $n \geqslant 1$, is uniformly tight.

Remarks. (i) If $B = \mathbf{R}$, the real line, then (i)–(iii) of Lemma 2.2 are equivalent. The uniform tightness of X_n, $n \geqslant 1$, follows from $\sup_{n \geqslant 1} E|X_n|^r < \infty$. (Use Chebyshev's inequality.)

(ii) In the case when B is a separable normed linear space, Lemma 2.2 is still valid if we replace uniform tightness in (i) (a) by uniform pre-tightness and compactness in (ii) by pre-compactness.

(iii) The implication (iii) ⇒ (i) is not true. A simple example is given by $B = l_1$, the space of all summable sequences $(x_1, x_2, \ldots)$ of real numbers with norm $\|(x_1, x_2, \ldots)\| = \sum_{n \geqslant 1} |x_n|$, and $X_n = (0, 0, \ldots, 1, 0, \ldots)$ with probability 1 for every $n \geqslant 1$, where in the above vector 1 occurs in the nth place. Trivially, $\|X_n\|$, $n \geqslant 1$, is uniformly integrable but X_n, $n \geqslant 1$, is not uniformly tight.

There are other notions related to the uniform integrability of a sequence of random variables. One such notion is: a sequence X_n, $n \geqslant 1$, of random variables is said to be *uniformly bounded* by a random variable X if

$$P\{|X_n| \geqslant a\} \leqslant P\{|X| \geqslant a\}$$

for every $n \geqslant 1$ and $a \geqslant 0$. The following lemma gives the connection between uniform boundedness and uniform integrability.

LEMMA 2.3. *Let X_n, $n \geqslant 1$, be a sequence of random variables. In the following,* (a) ⇒ (b) *and* (c) ⇒ (d).

(a) $\sup_{n \geqslant 1} E|X_n|^r < \infty$ *for some* $r > 0$.

(b) *There exists a nonnegative random variable X on Ω such that $EX^s < \infty$ for every $0 < s < r$ and X_n, $n \geqslant 1$, is uniformly bounded by X.*

(c) *X_n, $n \geqslant 1$, is uniformly bounded by a random variable X on Ω and $E|X|^r < \infty$ for some $r > 0$.*

(d) *$|X_n|^r$, $n \geqslant 1$, is uniformly integrable.*

Proof. See Wang and Bhaskara Rao [28].

Now we need some notions and results from functional analysis. Let B^* denote the dual of B, i.e., B^* is the space of all continuous linear functionals from B to $\mathbf{R}$. A subset B_1 of B^* is said to be *total* if $f(x) = 0$ for every f in B_1 for some x in B implies $x = 0$. The weak*-topology in B^* is described by defining convergence in B^* as follows: a net f_α, $\alpha \in \Delta$, in B^* is said to converge to an element f in B^* if f_α, $\alpha \in \Delta$, converges to f pointwise, i.e., $\lim_{\alpha \in \Delta} f_\alpha(x) = f(x)$ for every x in B. In this connection, we quote the following result from Kelley and Namioka [12, Theorem 16.5, p. 142].

LEMMA 2.4. *Let B be a separable Banach space and B_1 a total subset of B^*. Let $S(B_1)$ be the linear span of B_1. Then $S(B_1)$ is weak*-dense in B^*.*

LEMMA 2.5. *Let B be a separable Banach space. A convex subset of B^* is weak*-closed in B^* if and only if it is sequentially weak*-closed in B^*.*

See exercise 16 of Dunford and Schwartz [9, p. 437].

A Banach space B is said to admit a *Schauder basis* if there exists a sequence b_n, $n \geqslant 1$, in B with the following property: for every x in B, there exists a unique sequence t_n, $n \geqslant 1$, of real numbers such that $x = \sum_{n \geqslant 1} t_n b_n$. For each $n \geqslant 1$ let $f_n(x) = t_n$, $x \in B$. f_n is called the *n-th-coordinate functional* on B and is a continuous linear functional on B. For each $n \geqslant 1$, the *n-th partial sum operator* U_n from B to B is defined by

$$U_n(x) = \sum_{k=1}^{n} f_k(x) b_k, \quad x \in B.$$

U_n, $n \geqslant 1$, is a sequence of continuous linear operators from B to B satisfying $\lim_{n \to \infty} U_n(x) = x$ for every x in B. Also, for each $n \geqslant 1$, the *n-th residual operator* Q_n from B to B is defined by $Q_n(x) = x - U_n(x)$ for every x in B. Q_n, $n \geqslant 1$, is a sequence of continuous linear operators from B to B satisfying $\lim_{n \to \infty} Q_n(x) = 0$ for every x in B.

3. Convergence in probability and mean convergence. If X is a random element defined on a probability space $(\Omega, \mathscr{B}, P)$ taking values in a separable normed linear space B and A is a real random variable defined on Ω, one can show that AX is a random element (see Taylor [23, p. 24]). If X_n, $n \geqslant 1$, is a sequence of random elements, A_n, $n \geqslant 1$, is a sequence of random variables and a_{nk}, $n \geqslant 1$, $k \geqslant 1$, is a double array of real numbers, we examine, in this section, under what conditions the sequence

$$\sum_{k \geqslant 1} a_{nk} A_k X_k, \quad n \geqslant 1,$$

converges to 0 in probability. Specifically, we want to characterize the above convergence in terms of convergence to 0 in probability of the sequence

$$\sum_{k \geqslant 1} a_{nk} f(A_k X_k), \quad n \geqslant 1,$$

for every continuous linear functional f from B to $\mathbf{R}$. Theorem 3.3 provides such a characterization. We need the following results in the proof of Theorem 3.3.

LEMMA 3.1. *Let C be a compact subset of a Banach space B and T_n, $n \geqslant 1$, a sequence of continuous linear operators from B into a Banach space F converging pointwise to a continuous linear operator T from B to F. Then T_n, $n \geqslant 1$, converges to T uniformly on C.*

Taylor and Wei [27, Lemma 5, p. 154] established the above result in the special case when B is a Banach space admitting a Schauder basis b_n,

$n \geq 1$, and the sequence T_n, $n \geq 1$, is the sequence Q_n, $n \geq 1$, of residual operators associated with the basis b_n, $n \geq 1$. Their proof can be adapted to prove the above lemma.

LEMMA 3.2. *Let B be a Banach space admitting a Schauder basis b_n, $n \geq 1$. Let Q_t, $t \geq 1$, be the sequence of residual operators associated with the basis b_n, $n \geq 1$. Let X_n, $n \geq 1$, be a sequence of random elements defined on a probability space $(\Omega, \mathscr{B}, P)$ taking values in B such that X_n, $n \geq 1$, is compactly uniformly r-th-order integrable for some $r > 0$. Then*

$$\lim_{t\to\infty} \sup_{n\geq 1} E\|Q_t(X_n)\|^r = 0.$$

Proof. See Wang and Bhaskara Rao [29].

THEOREM 3.3. *Let X_n, $n \geq 1$, be a sequence of random elements taking values in a separable Banach space B such that X_n, $n \geq 1$, is compactly uniformly r-th-order integrable for some $r > 1$. Let a_{nk}, $n \geq 1$, $k \geq 1$, be a double array of real numbers and A_n, $n \geq 1$, a sequence of real random variables satisfying*

(3.1) $$\sum_{k\geq 1} |a_{nk}| (E|A_k|^{r/(r-1)})^{(r-1)/r} \leq \Gamma$$

for every $n \geq 1$ for some positive constant Γ. Let B_1 be any total subset of the dual space B^* of B. Then the following statements are equivalent:

(i) $\sum_{k\geq 1} a_{nk} g(A_k X_k)$, $n \geq 1$, *converges to 0 in probability for every g in B_1.*

(ii) $E|\sum_{k\geq 1} a_{nk} g(A_k X_k)|$, $n \geq 1$, *converges to 0 for every g in B_1.*

(iii) $\sum_{k\geq 1} a_{nk} f(A_k X_k)$, $n \geq 1$, *converges to 0 in probability for every f in B^*.*

(iv) $E|\sum_{k\geq 1} a_{nk} f(A_k X_k)|$, $n \geq 1$, *converges to 0 for every f in B^*.*

(v) $\sum_{k\geq 1} a_{nk} A_k X_k$, $n \geq 1$, *converges to 0 in probability.*

(vi) $E\|\sum_{k\geq 1} a_{nk} A_k X_k\|$, $n \geq 1$, *converges to 0.*

If A_k and $\|X_k\|$ are independently distributed for each $k \geq 1$, X_n, $n \geq 1$, is compactly uniformly first-order integrable and

(3.2) $$\sum_{k\geq 1} |a_{nk}| E|A_k| \leq \Gamma$$

for every $n \geq 1$ for some positive constant Γ, then the statements (i)–(vi) *above are also equivalent.*

Proof. The proof is carried out in the following steps.

1°. We prove the first part of the above theorem. We show that $\sum_{k\geq 1} a_{nk} A_k X_k$ converges a.e.[P] in B for every $n \geq 1$. For every $n \geq 1$, by Hölder's inequality, we have

(3.3) $$\begin{aligned}\sum_{k\geq 1} |a_{nk}| E\|A_k X_k\| &\leq \sum_{k\geq 1} |a_{nk}| (E\|X_k\|^r)^{1/r} (E|A_k|^{r/(r-1)})^{(r-1)/r} \\ &\leq \sup_{k\geq 1} (E\|X_k\|^r)^{1/r} \sum_{k\geq 1} |a_{nk}| (E|A_k|^{r/(r-1)})^{(r-1)/r} \\ &\leq \Gamma (\sup_{k\geq 1} E\|X_k\|^r)^{1/r} \\ &< \infty, \quad \text{by Lemma 2.2.}\end{aligned}$$

Since B is a complete metric space, it follows that $\sum_{k\geq 1} a_{nk} A_k X_k$ converges a.e.[P] in B for every $n \geq 1$. See Chung [8, (xii), p. 42].

2°. Let $Y_n = \sum_{k\geq 1} a_{nk} A_k X_k$, $n \geq 1$. We show that $\|Y_n\|$, $n \geq 1$, is uniformly integrable. From (3.3), it follows that $\sup_{n\geq 1} E\|Y_n\| < \infty$. Let $A \in \mathscr{B}$. By Hölder's inequality, for every $n \geq 1$

$$\begin{aligned}\int_A \|Y_n\| dP &\leq \int_A \sum_{k\geq 1} |a_{nk}| \|A_k X_k\| dP \\ &\leq \sum_{k\geq 1} |a_{nk}| \Big(\int_A \|X_k\|^r dP\Big)^{1/r} \Big(\int_A |A_k|^{r/(r-1)} dP\Big)^{(r-1)/r} \\ &\leq \sum_{k\geq 1} |a_{nk}| (E|A_k|^{r/(r-1)})^{(r-1)/r} \Big(\int_A \|X_k\|^r dP\Big)^{1/r}.\end{aligned}$$

Thus the uniform absolute continuity of $\|Y_n\|$, $n \geq 1$, follows from that of $\|X_n\|^r$, $n \geq 1$, and the assumption (3.1). The uniform integrability of $\|X_n\|^r$, $n \geq 1$, follows from Lemma 2.2.

3°. Since $\|Y_n\|$, $n \geq 1$, is uniformly integrable, Y_n, $n \geq 1$, converges to 0 in probability if and only if Y_n, $n \geq 1$, converges to 0 in the mean, i.e., $E\|Y_n\|$, $n \geq 1$, converges to 0 (see Chung [8, Theorem 4.5.4, p. 97]). The equivalence of (v) and (vi) is thus established. The equivalence of (i) and (ii) and that of (iii) and (iv) can be shown in a similar vein.

4°. We prove (ii) $\Rightarrow$ (iv). Let $S(B_1)$ be the linear span of B_1. Obviously, (ii) holds for every g in $S(B_1)$. Let

$$B_2 = \{f \in B^*;\ E|\sum_{k\geq 1} a_{nk} f(A_k X_k)|,\ n \geq 1, \text{ converges to } 0\}.$$

B_2 is a convex subset of B^*. We show that B_2 is sequentially weak*-closed in B^*. Let g_m, $m \geq 1$, be a sequence in B_2 converging to some f in B^* in the weak*-topology of B^*. We show that $f \in B_2$. By Theorem II.3.6 of Dunford and Schwartz [9, p. 60], there exists a positive constant Γ_1 such that $\|g_m\| \leq \Gamma_1$ for every $m \geq 1$ and $\|f\| \leq \Gamma_1$. Let $\varepsilon > 0$. Since X_n, $n \geq 1$, is compactly uniformly rth-order integrable, there exists a compact subset C of B such that

(3.4) $$\Big(\int_{(X_n \in C^c)} \|X_n\|^r dP\Big)^{1/r} < \varepsilon/(4\Gamma\Gamma_1)$$

for every $n \geqslant 1$. Without loss of generality, assume that $0 \in C$. Define for each $n \geqslant 1$

$$U_n = \begin{cases} X_n & \text{if } X_n \in C, \\ 0 & \text{if } X_n \in C^c, \end{cases} \qquad V_n = X_n - U_n.$$

From (3.4), for every $n \geqslant 1$, we have

$$(E\|V_n\|^r)^{1/r} = \Big(\int_{\{X_n \in C^c\}} \|X_n\|^r \, dP\Big)^{1/r} < \varepsilon/(4\Gamma\Gamma_1). \tag{3.5}$$

For the compact set C chosen above, by Lemma 3.1, there exists $m_0 \geqslant 1$ such that

$$|(f-g_m)(U_k(w))| < \varepsilon/(4\Gamma) \tag{3.6}$$

for every w in Ω and $k \geqslant 1$ whenever $m \geqslant m_0$. Also, by (ii), there exists $n_0 \geqslant 1$ such that

$$E\Big|\sum_{k\geqslant 1} a_{nk} g_{m_0}(A_k X_k)\Big| < \varepsilon/4 \tag{3.7}$$

whenever $n \geqslant n_0$. So, if $n \geqslant n_0$ then

$$\begin{aligned} E\Big|\sum_{k\geqslant 1} a_{nk} f(A_k X_k)\Big| &\leqslant E\Big|\sum_{k\geqslant 1} a_{nk}(f-g_{m_0})(A_k X_k)\Big| + E\Big|\sum_{k\geqslant 1} a_{nk} g_{m_0}(A_k X_k)\Big| \\ &< \sum_{k\geqslant 1} |a_{nk}| E|A_k(f-g_{m_0})(X_k)| + \varepsilon/4, \quad \text{by (3.7)} \\ &< \sum_{k\geqslant 1} |a_{nk}| E|A_k(f-g_{m_0})(U_k)| \\ &\quad + \sum_{k\geqslant 1} |a_{nk}| E|A_k(f-g_{m_0})(V_k)| + \varepsilon/4 \\ &< [\varepsilon/(4\Gamma)] \sum_{k\geqslant 1} |a_{nk}| E|A_k| + \varepsilon/4 \\ &\quad + \sum_{k\geqslant 1} |a_{nk}| \big(E|(f-g_{m_0})(V_k)|^r\big)^{1/r} \big(E|A_k|^{r/(r-1)}\big)^{(r-1)/r}, \\ &\qquad \text{by (3.6) and Hölder's inequality} \\ &< [\varepsilon/(4\Gamma)]\,\Gamma + \\ &\quad + \varepsilon/4 + \|f-g_{m_0}\| \sum_{k\geqslant 1} |a_{nk}| (E\|V_k\|^r)^{1/r} \big(E|A_k|^{r/(r-1)}\big)^{(r-1)/r}, \\ &\qquad \text{by (3.1)} \\ &< \varepsilon/4 + \varepsilon/4 \\ &\quad + 2\Gamma_1 [\varepsilon/(4\Gamma\Gamma_1)] \sum_{k\geqslant 1} |a_{nk}| \big(E|A_k|^{r/(r-1)}\big)^{(r-1)/r}, \quad \text{by (3.5)} \\ &< \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon, \quad \text{by (3.1).} \end{aligned}$$

This shows that $f \in B_2$ and that B_2 is sequentially weak*-closed in B^*. By Lemma 2.5, B_2 is weak*-closed in B^*. Since $B_2 \supset S(B_1)$, by Lemma 2.4, $B_2 = B^*$. This establishes (iv).

5°. Now, we prove (iv) $\Rightarrow$ (vi). Since every separable Banach space B can be embedded isometric isomorphically onto a closed subspace of $C[0, 1]$, the Banach space of all real-valued continuous functions defined on the unit interval $[0, 1]$ equipped with the supremum norm, we can assume, without loss of generality, that each X_n takes values in $C[0, 1]$ (see Semadeni [20, Theorem 8.7.2, p. 157]). $C[0, 1]$ admits a Schauder basis b_n, $n \geqslant 1$. Let f_t, $t \geqslant 1$, U_t, $t \geqslant 1$, and Q_t, $t \geqslant 1$, be the sequences of coordinate functionals, partial sum operators and residual operators respectively associated with the basis b_n, $n \geqslant 1$. Let $\varepsilon > 0$. By Lemma 3.2, there exists $t_0 \geqslant 1$ such that

$$(E\|Q_{t_0}(X_k)\|^r)^{1/r} < \varepsilon/(2\Gamma) \tag{3.8}$$

for every $k \geqslant 1$. By (iv), there exists $n_0 \geqslant 1$ such that

$$E\Big|\sum_{k\geqslant 1} a_{nk} f_t(A_k X_k)\Big| < \varepsilon/(2\|b_t\|\, t_0) \tag{3.9}$$

for $t = 1, 2, \ldots, t_0$, whenever $n \geqslant n_0$. So, if $n \geqslant n_0$ then

$$\begin{aligned} E\Big\|\sum_{k\geqslant 1} a_{nk} A_k X_k\Big\| &= E\Big\|U_{t_0}\Big(\sum_{k\geqslant 1} a_{nk} A_k X_k\Big) + Q_{t_0}\Big(\sum_{k\geqslant 1} a_{nk} A_k X_k\Big)\Big\| \\ &\leqslant E\Big\|\sum_{t=1}^{t_0} f_t\Big(\sum_{k\geqslant 1} a_{nk} A_k X_k\Big) b_t\Big\| + \sum_{k\geqslant 1} |a_{nk}| E\|Q_{t_0}(A_k X_k)\| \\ &\leqslant \sum_{t=1}^{t_0} E\Big|\sum_{k\geqslant 1} a_{nk} f_t(A_k X_k)\Big| \|b_t\| + \sum_{k\geqslant 1} |a_{nk}| E\|A_k Q_{t_0}(X_k)\| \\ &< \varepsilon/2 + \sum_{k\geqslant 1} |a_{nk}| (E\|Q_{t_0}(X_k)\|^r)^{1/r} \big(E|A_k|^{r/(r-1)}\big)^{(r-1)/r}, \\ &\qquad \text{by (3.9) and Hölder's inequality} \\ &< \varepsilon/2 + [\varepsilon/(2\Gamma)] \sum_{k\geqslant 1} |a_{nk}| \big(E|A_k|^{r/(r-1)}\big)^{(r-1)/r}, \quad \text{by (3.8)} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \text{by (3.1).} \end{aligned}$$

Hence $E\|\sum_{k\geqslant 1} a_{nk} A_k X_k\|$, $n \geqslant 1$, converges to 0.

6°. The implication (vi) $\Rightarrow$ (ii) is obvious.

7°. We now come to the second part of the theorem. If A_n and $\|X_n\|$ are independent random variables for each $n \geqslant 1$, the equivalence of (i)–(vi) can be shown essentially in the same way as above. We use the fact that $E\|A_n X_n\| = (E|A_n|)(E\|X_n\|)$ for every $n \geqslant 1$ instead of the inequality $E\|A_n X_n\| \leqslant (E\|X_n\|^r)^{1/r} (E|A_n|^{r/(r-1)})^{(r-1)/r}$ in the proof given above.

This completes the proof of the theorem.

Using the argument given above, one can establish the following result.

THEOREM 3.4. *The conclusion of Theorem* 3.3 *is valid with* $A_k X_k$ *replaced by* $A_k X_k - EA_k X_k$.

The above two theorems have the following analogues in the context of separable normed linear spaces.

COROLLARY 3.5. (a) *Let* X_n, $n \geq 1$, *be a sequence of random elements taking values in a separable normed linear space B such that* X_n, $n \geq 1$, *is pre-compactly uniformly r-th-order integrable for some* $r > 1$. *Let* a_{nk}, $n \geq 1$, $k \geq 1$, *be a double array of real numbers and* A_n, $n \geq 1$, *be a sequence of real random variables satisfying* (3.1) *of Theorem* 3.3. *Assume that* $\sum_{k \geq 1} a_{nk} A_k X_k$ *converges a.e.*[P] *in B for every* $n \geq 1$. *Then* (iii)–(vi) *of Theorem* 3.3 *are equivalent.*

(b) *If* A_k *and* $\|X_k\|$ *are independent random variables for each* $k \geq 1$, X_n, $n \geq 1$, *is pre-compactly uniformly first-order integrable,* $\sum_{k \geq 1} a_{nk} A_k X_k$ *converges a.e.*[P] *in B for every* $n \geq 1$ *and* (3.1) *of Theorem* 3.3 *is replaced by the condition that*

$$\sum_{k \geq 1} |a_{nk}|\, E|A_k| \leq \Gamma$$

for every $n \geq 1$ *for some positive constant* Γ, *then* (iii)–(vi) *of Theorem* 3.3 *are equivalent.*

Proof. Let $\tilde{B}$ be the completion of B. The sequence X_n, $n \geq 1$, can now be assumed taking values in $\tilde{B}$. Observe that X_n, $n \geq 1$, is compactly uniformly rth-order or first-order, as the case may be, integrable in $\tilde{B}$. Theorem 3.3 is now applicable to the sequence X_n, $n \geq 1$. Note also that B^* and $\tilde{B}^*$ are isometrically isomorphic.

Remarks. The above results generalize the following results in the literature.

(1) Taylor and Padgett [26, Theorem 2.5, p. 233] proved the equivalence of (i) and (v) of Theorem 3.3 under the following conditions. (a) B admits a Schauder basis. (b) X_n, $n \geq 1$, is identically distributed. (c) $E\|X_1\|^r < \infty$ for some $r > 1$. (d) a_{nk}, $1 \leq k \leq n$, $n \geq 1$, is a triangular array of real numbers satisfying $\lim_{n\to\infty} a_{nk} = 0$ for every $k \geq 1$ and $\sum_{k=1}^{n} |a_{nk}| \leq \Gamma^*$ for every $n \geq 1$ for some positive constant Γ^*. (e) (3.1) of Theorem 3.3 holds. (f) $EA_n X_n = EA_1 X_1$ for every $n \geq 1$. (e) B_1 consists of all coordinate functionals associated with the given Schauder basis of B. If X_n, $n \geq 1$, is identically distributed and $E\|X_1\|^r < \infty$, then, obviously, X_n, $n \geq 1$, is compactly uniformly rth-order integrable.

Taylor and Padgett [26, Theorem 2.6, p. 235] also proved the equivalence of (iii) and (v) of Theorem 3.3 under the above conditions but the condition that the separable Banach space B has a Schauder basis was dropped.

Theorem 2.3 of Taylor and Padgett [26, p. 231] is a special case of Theorem 3.3 (second part) if we take $A_n = 1$ for every $n \geq 1$. Theorem 2.4 of [26, p. 232] also follows from the second part of Theorem 3.3.

(2) Wei and Taylor [31, Theorem 4, p. 285] proved the equivalence of (iii) and (v) of Theorem 3.3 under the following conditions. (a) X_n, $n \geq 1$, is uniformly tight each with mean zero. (b) $A_n = 1$ for every $n \geq 1$. (c) $\sup_{n \geq 1} E\|X_n\|^r < \infty$ for some $r > 1$. (d) $\sum_{k \geq 1} |a_{nk}| \leq \Gamma$ for all $n \geq 1$ for some positive constant Γ. Note that, by Lemmas 2.2 and 2.3, X_n, $n \geq 1$, is compactly uniformly first-order integrable.

(3) Taylor and Wei [27, Theorem 4, p. 153] proved the equivalence of (i) and (iii) of Corollary 3.5 under the following conditions. (a) X_n, $n \geq 1$, is uniformly tight. (b) $\sup_{n \geq 1} E\|X_n\|^r < \infty$ for some $r > 1$. (c) $a_{nk} = 1/n$ if $1 \leq k \leq n$, $= 0$ if $k > n$, for all $n \geq 1$. (d) $A_n = 1$ for all $n \geq 1$. This result can be deduced from Corollary 3.5.

Theorem 3.3 can be used to derive some Weak Laws of Large Numbers for sequences of random elements taking values in a separable Banach space. As an example, we give a result (Corollary 3.8) which generalizes Theorem 2.4 of Hoffmann-Jørgensen and Pisier [10, p. 592] from which a Weak Law of Large Numbers follows. Before that, we observe the following.

PROPOSITION 3.6. *Let* X_n, $n \geq 1$, *be a sequence of random elements taking values in a separable Banach space B such that* X_n, $n \geq 1$, *is uniformly tight and* $\|X_n\|$, $n \geq 1$, *is uniformly absolutely continuous. Then* $X_n - EX_n$, $n \geq 1$, *is uniformly tight.*

Proof. Taylor [23, Lemma 5.2.1, p. 121] established this result under the stronger assumption that $\sup_{n \geq 1} E\|X_n\|^r < \infty$ for some $r > 1$. This result is also valid under the weaker assumption that $\|X_n\|$, $n \geq 1$, is uniformly absolutely continuous and essentially, the same proof works.

PROPOSITION 3.7. *Let* X_n, $n \geq 1$, *be a sequence of random elements taking values in a separable Banach space B. If* X_n, $n \geq 1$, *is compactly uniformly r-th-order integrable for some* $r \geq 1$, *then* $X_n - EX_n$, $n \geq 1$, *is compactly uniformly r-th-order integrable.*

Proof. By Proposition 3.6 and Lemma 2.2, $X_n - EX_n$, $n \geq 1$, is uniformly tight. It is not difficult to see that $\|X_n - EX_n\|$, $n \geq 1$, is uniformly absolutely continuous. By Lemma 2.2, $X_n - EX_n$, $n \geq 1$, is compactly uniformly rth-order integrable.

COROLLARY 3.8. *Let* X_n, $n \geq 1$, *be a sequence of pairwise independent random elements taking values in a separable Banach space B such that* X_n, $n \geq 1$, *is compactly uniformly first-order integrable. Then*

$$\frac{1}{n}\sum_{k=1}^{n} (X_k - EX_k), \quad n \geq 1, \quad \textit{converges to } 0 \textit{ in the mean.}$$

Proof. By Proposition 3.7 and Lemma 2.2, $f(X_n - EX_n)$, $n \geq 1$, is uniformly integrable for every f in B^*. By Theorem 3 of Wang and Bhaskara Rao [28],

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k - EX_k),\ n \geqslant 1, \quad \text{converges to 0 in the mean.}$$

By the second part of Theorem 3.3, the assertion now follows.

The following result of Wang and Bhaskara Rao [29, Theorem 2.3] is also a consequence of Theorem 3.3 above.

COROLLARY 3.9. *Let X_n, $n \geqslant 1$, be a sequence of random elements taking values in a separable Banach space B such that X_n, $n \geqslant 1$, is compactly uniformly r-th-order integrable for some $r \geqslant 1$. Let a_{nk}, $n \geqslant 1$, $k \geqslant 1$, be a double array of real numbers satisfying*

$$\sum_{k \geqslant 1} |a_{nk}| \leqslant \Gamma$$

for every $n \geqslant 1$ for some positive constant Γ. Let B_1 be any total subset of the dual space B^* of B. Then the following statements are equivalent:

(i) $E\left|g\left(\sum_{k \geqslant 1} a_{nk} X_k\right)\right|^r$, $n \geqslant 1$, *converges to 0 for every g in B_1.*

(ii) $g\left(\sum_{k \geqslant 1} a_{nk} X_k\right)$, $n \geqslant 1$, *converges to 0 in probability for every g in B_1.*

(iii) $E\left|f\left(\sum_{k \geqslant 1} a_{nk} X_k\right)\right|^r$, $n \geqslant 1$, *converges to 0 for every f in B^*.*

(iv) $f\left(\sum_{k \geqslant 1} a_{nk} X_k\right)$, $n \geqslant 1$, *converges to 0 in probability for every f in B^*.*

(v) $E\left\|\sum_{k \geqslant 1} a_{nk} X_k\right\|^r$, $n \geqslant 1$, *converges to 0.*

(vi) $\sum_{k \geqslant 1} a_{nk} X_k$, $n \geqslant 1$, *converges to 0 in probability.*

Proof. From Theorem 3.3 it follows that (ii), (iv) and (vi) are equivalent. Using the uniform integrability of the sequence $\left\|\sum_{k \geqslant 1} a_{nk} X_k\right\|^r$, $n \geqslant 1$, one can show that (i) and (ii), (iii) and (iv), and (v) and (vi) are equivalent as in steps 2° and 3° of Theorem 3.3.

3. Strong convergence. In this section, we establish some strong limit theorems for weighted sums of random elements in separable Banach spaces.

Generalizing a result of Pruitt [18, Theorem 2, p. 769], Rohatgi [19, Theorem 2, p. 306] proved the following result.

THEOREM 4.1. *Let X_n, $n \geqslant 1$, be a sequence of independent real random variables uniformly bounded by a random variable X with $E|X|^{1+1/r} < \infty$ for some $r > 0$. Let a_{nk}, $n \geqslant 1$, $k \geqslant 1$, be a double array of real numbers satisfying*

(a) $\sum_{k \geqslant 1} |a_{nk}| \leqslant \Gamma$ *for every $n \geqslant 1$ for some positive constant Γ, and*

(b) $\max_{k \geqslant 1} |a_{nk}| = O(n^{-r})$ *as $n \to \infty$.*

Then $\sum_{k \geqslant 1} a_{nk}(X_k - EX_k)$, $n \geqslant 1$, converges to 0 a.e.[P].

Some comments are in order on the above result. Rohatgi imposed an additional condition that each $EX_n = 0$. This is not necessary. We note that $X_n - EX_n$, $n \geqslant 1$, is uniformly bounded by $|X| + E|X|$, and $E(|X| + E|X|)^{1+1/r} < \infty$. Further, we observe that the above theorem for $r \geqslant 1$ is true under the weaker assumption that X_n, $n \geqslant 1$, is pairwise independent. Lemma 3 of Pruitt [18, p. 773] is valid under the assumption of the pairwise independence of X_n, $n \geqslant 1$, if $r \geqslant 1$ and this ensures the validity of the above theorem under the assumption of the pairwise independence of X_n, $n \geqslant 1$, in this case.

The following result is an extension of the above result to separable Banach spaces.

THEOREM 4.2. *Let X_n, $n \geqslant 1$, be a sequence of uniformly tight random elements taking values in a separable Banach space B such that $\|X_n\|$, $n \geqslant 1$, is uniformly bounded by a real random variable X on Ω satisfying $E|X|^{1+1/r} < \infty$ for some $r > 0$. Let a_{nk}, $n \geqslant 1$, $k \geqslant 1$, be a double array of real numbers satisfying (a) and (b) of Theorem 4.1.*

(a) *If $0 < r < 1$ and X_n, $n \geqslant 1$, is independent, then*

$$\lim_{n \to \infty} \sum_{k \geqslant 1} a_{nk}(X_k - EX_k) = 0 \quad \text{a.e.}[P].$$

(b) *If $r \geqslant 1$ and X_n, $n \geqslant 1$, is pairwise independent, then*

$$\lim_{n \to \infty} \sum_{k \geqslant 1} a_{nk}(X_k - EX_k) = 0 \quad \text{a.e.}[P].$$

Proof. Assume, without loss of generality, that B admits a Schauder basis b_n, $n \geqslant 1$. (Since every separable Banach space is a closed linear subspace of $C[0, 1]$, the Banach space of all real-valued continuous functions on $[0, 1]$, and $C[0, 1]$ admits a Schauder basis, we can assume that each X_n takes values in $C[0, 1]$.) Let Q_n, $n \geqslant 1$, be the sequence of residual operators associated with the basis b_n, $n \geqslant 1$. There exists a positive constant Γ_1 such that $\|Q_n\| \leqslant \Gamma_1$ for every $n \geqslant 1$ since Q_n, $n \geqslant 1$, converges to 0 pointwise on B (see Dunford and Schwartz [9, Theorem II.3.6, p. 60]). Let $\varepsilon > 0$. Since X_n, $n \geqslant 1$, is uniformly tight and $\|X_n\|$, $n \geqslant 1$, is uniformly bounded by an integrable random variable, by Lemmas 2.2 and 2.3, X_n, $n \geqslant 1$, is compactly uniformly first-order integrable.

By Lemma 3.2 there exists $t_0 \geqslant 1$ such that

$$E\|Q_t(X_n)\| \leqslant \varepsilon/(2\Gamma) \tag{4.1}$$

for every $n \geq 1$ whenever $t \geq t_0$. From (4.1), we note that

$$(4.2)\quad \sum_{k\geq 1} |a_{nk}| E\|Q_t(X_k)-Q_t(EX_k)\| \leq \sum_{k\geq 1} |a_{nk}| [E\|Q_t(X_k)\|+E\|Q_t(EX_k)\|]$$
$$\leq 2\sum_{k\geq 1} |a_{nk}| E\|Q_t(X_k)\|$$
$$< 2[\varepsilon/(2\Gamma)]\Gamma = \varepsilon$$

for every $n \geq 1$ whenever $t \geq t_0$.

Let f_n, $n \geq 1$, be the sequence of coordinate functionals associated with the basis b_n, $n \geq 1$. Let U_n, $n \geq 1$, be the sequence of partial sum operators associated with that basis. Note that $\|X_n - EX_n\|$, $n \geq 1$, is uniformly bounded by $|X|+E|X|$ with $E(|X|+E|X|)^{1+1/r} < \infty$. For each $k \geq 1$, $f_k(X_n - EX_n)$, $n \geq 1$, is uniformly bounded by $\|Q_k\|(|X|+E|X|)$ with $E[\|Q_k\|(|X|+E|X|)]^{1+1/r} < \infty$. By Theorem 4.1

$$(4.3)\quad \sum_{i=1}^{t_0} |\sum_{k\geq 1} a_{nk} f_i(X_k - EX_k)| \|b_i\|,\ n \geq 1, \text{ converges to 0 a.e.}[P], \text{ and}$$

$$(4.4)\quad \sum_{k\geq 1} |a_{nk}| (\|Q_{t_0}(X_k-EX_k)\| - E\|Q_{t_0}(X_k-EX_k)\|),\ n \geq 1,$$

converges to 0 a.e.$[P]$.

Note that

$$\|\sum_{k\geq 1} a_{nk}(X_k-EX_k)\| = \|\sum_{k\geq 1} a_{nk} U_{t_0}(X_k-EX_k) + \sum_{k\geq 1} a_{nk} Q_{t_0}(X_k-EX_k)\|$$
$$= \|\sum_{k\geq 1} a_{nk} \sum_{i=1}^{t_0} f_i(X_k-EX_k) b_i + \sum_{k\geq 1} a_{nk} Q_{t_0}(X_k-EX_k)\|$$
$$\leq \sum_{i=1}^{t_0} |\sum_{k\geq 1} a_{nk} f_i(X_k-EX_k)| \|b_i\| + \sum_{k\geq 1} |a_{nk}| (\|Q_{t_0}(X_k-EX_k)\| - E\|Q_{t_0}(X_k-EX_k)\|) + \sum_{k\geq 1} |a_{nk}| E\|Q_{t_0}(X_k-EX_k)\|.$$

From (4.2)–(4.4), it follows that

$$\limsup_{n\to\infty} \|\sum_{k\geq 1} a_{nk}(X_k-EX_k)\| \leq \varepsilon \quad \text{a.e.}[P].$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n\to\infty} \sum_{k\geq 1} a_{nk}(X_k-EX_k) = 0$ a.e.$[P]$. This completes the proof.

Remarks. (1) In case B is only a separable normed linear space, the above theorem is still valid under the additional assumptions that EX_n exists for every $n \geq 1$ and that $\sum_{k\geq 1} a_{nk}(X_k - EX_k)$ converges a.e.$[P]$ for every $n \geq 1$.

(2) Padgett and Taylor [16, Theorem 3, p. 395] established the conclusion of the above theorem under the stronger conditions that X_n, $n \geq 1$, is independently identically distributed and that $EX_n = 0$ for every $n \geq 1$.

(3) Theorem 5.1.3 of Taylor [23, p. 112] in the context of separable normed linear spaces can be deduced from Theorem 4.2 above.

(4) Theorem 7 of Wei and Taylor [31, p. 288] is a special case of Theorem 4.2 above (use Lemma 2.3).

(5) Theorem 3.2 of Bozorgnia and Bhaskara Rao [3, p. 433] also follows from Theorem 4.2 above.

The following result used in the proof of Theorem 4.4 is a trivial corollary of Theorem 4.2 above. See also Taylor and Wei [27, Theorem 1, p. 151].

COROLLARY 4.3. *Let X_n, $n \geq 1$, be a sequence of pairwise independent random elements taking values in a compact subset of a separable Banach space. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - EX_i) = 0 \quad a.e.[P].$$

Now, we extend Marcinkiewicz–Zygmund–Kolmogorov's Strong Law of Large Numbers and Brunk–Chung's Strong Law of Large Numbers to separable Banach spaces. For these results on the real line, see Chung [8, p. 125], Chow and Teicher [6, Theorem 3, p. 333], Chung [8, p. 348] and Brunk [4].

The following is an extension of these results.

THEOREM 4.4. *Let X_n, $n \geq 1$, be a sequence of independent uniformly tight random elements taking values in a separable Banach space B such that $\|X_n\|$, $n \geq 1$, is uniformly absolutely continuous.*

(a) *If $1 \leq r \leq 2$ and $\sum_{n\geq 1} \frac{1}{n^r} E\|X_n\|^r < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - EX_i) = 0 \quad a.e.[P].$$

(b) *If $r \geq 2$ and $\sum_{n\geq 1} \frac{1}{n^{1+r/2}} E\|X_n\|^r < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - EX_i) = 0 \quad a.e.[P].$$

Proof. First, we remark that, by Lemma 2.2, the uniform tightness and uniform absolute continuity of $\|X_n\|$, $n \geq 1$, are jointly equivalent to the condition that X_n, $n \geq 1$, is compactly uniformly first-order integrable. Let $\varepsilon > 0$. There exists a compact, convex and symmetric subset C of B such that

$$\int_{\{X_n \in C^c\}} \|X_n\| \, dP < \varepsilon \tag{4.5}$$

for every $n \geq 1$. For each $n \geq 1$, let

$$Y_n = \begin{cases} X_n & \text{if } X_n \in C, \\ 0 & \text{if } X_n \in C^c, \end{cases} \quad Z_n = X_n - Y_n.$$

By (4.5), $E\|Z_n\| < \varepsilon$ for all $n \geq 1$. The sequence $Y_n - EY_n$, $n \geq 1$, of random elements takes values in the set $C+C = \{x+y;\ x,\ y \in C\}$. $C+C$ is a compact subset of B. By Corollary 4.3 above,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (Y_i - EY_i) = 0 \quad \text{a.e.}[P].$$

We now prove (a). By C_r-inequality and Jensen's inequality,

$$\begin{aligned} \sum_{n\geq 1} (E\,|\,\|Z_n\| - E\|Z_n\|\,|^r)/n^r &\leq \sum_{n\geq 1} 2^{r-1} [E\|Z_n\|^r + (E\|Z_n\|)^r]/n^r \\ &\leq \sum_{n\geq 1} 2^{r-1} (E\|Z_n\|^r + E\|Z_n\|^r)/n^r \\ &\leq 2^r \sum_{n\geq 1} (E\|Z_n\|^r)/n^r \\ &\leq 2^r \sum_{n\geq 1} (E\|X_n\|^r)/n^r < \infty. \end{aligned}$$

By Marcinkiewicz–Zygmund–Kolmogorov's theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\|Z_i\| - E\|Z_i\|) = 0 \quad \text{a.e.}[P].$$

Observe that for every $n \geq 1$

$$\begin{aligned} \Big\|n^{-1}\sum_{k=1}^{n} (X_k - EX_k)\Big\| &\leq \Big\|n^{-1}\sum_{k=1}^{n} (Y_k - EY_k)\Big\| + \Big\|n^{-1}\sum_{k=1}^{n} (Z_k - EZ_k)\Big\| \\ &\leq \Big\|n^{-1}\sum_{k=1}^{n} (Y_k - EY_k)\Big\| + n^{-1}\sum_{k=1}^{n} \|Z_k\| + n^{-1}\sum_{k=1}^{n} E\|Z_k\| \\ &\leq \Big\|n^{-1}\sum_{k=1}^{n} (Y_k - EY_k)\Big\| + 2n^{-1}\sum_{k=1}^{n} E\|Z_k\| \\ &\quad + n^{-1}\sum_{k=1}^{n} (\|Z_k\| - E\|Z_k\|) \\ &< \Big\|n^{-1}\sum_{k=1}^{n} (Y_k - EY_k)\Big\| + 2\varepsilon + n^{-1}\sum_{k=1}^{n} (\|Z_k\| - E\|Z_k\|). \end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) \leq 0 + 2\varepsilon + 0 \quad \text{a.e.}[P].$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - EX_k) = 0 \quad \text{a.e.}[P].$$

The proof of (b) is analogous to that of (a).

Remark. The above theorem can also be proved using results of Wang and Bhaskara Rao [29] and de Acosta [1]. By Theorem 2.4 of Wang and Bhaskara Rao [29], $n^{-1}\sum_{k=1}^{n}(X_k - EX_k)$, $n \geq 1$, converges to 0 in probability. By Theorem 3.2 of de Acosta [1, p. 159], $n^{-1}\sum_{k=1}^{n}(X_k - EX_k)$, $n \geq 1$, converges to 0 a.e.$[P]$.

Also, compare Theorem 4.4 above with Theorem 5.4 of Hoffmann-Jørgensen [11, p. 210].

Hoffmann-Jørgensen [11, Theorem 5.4, p. 210] presented a result characterizing Banach spaces for which Theorem 4.4(a) is true. More precisely, he showed that B is of type p for some $1 \leq p \leq 2$ if and only if $n^{-1}\sum_{i=1}^{n}(X_i - EX_i)$, $n \geq 1$, converges to 0 a.e.$[P]$ whenever $X_n, n \geq 1$, is a sequence of independent random elements taking values in B and satisfying $\sum_{n\geq 1} n^{-p} E\|X_n\|^p < \infty$. Theorem 4.4 above relaxes the geometric condition imposed on the Banach space B in Theorem 5.4 of Hoffmann-Jørgensen and imposes an additional condition on the sequence $X_n, n \geq 1$, in that it is compactly uniformly first-order integrable for the same conclusion to be valid.

As a consequence of Theorem 4.4, we obtain the following result of Taylor and Wei [27, Theorem 2, p. 152]. See also Kuelbs and Zinn [13, Corollary 2, p. 80].

Corollary 4.5. *Let $X_n, n \geq 1$, be a sequence of independent uniformly tight random elements taking values in a separable Banach space B such that $\sup_{n\geq 1} E\|X_n\|^r < \infty$ for some $r > 1$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - EX_i) = 0 \quad \text{a.e.}[P].$$

Using a certain stability theorem, we now establish a limit theorem for sequences of weighted sums of random elements. This limit theorem generalizes quite a number of limit theorems in the literature under much weaker conditions.

Theorem 4.6. *Let X_n, $n \geq 1$, be a sequence of random elements taking values in a separable normed linear space B such that $\|X_n\|$, $n \geq 1$, is uniformly bounded by a real random variable X satisfying $E|X|^r < \infty$ for some $r > 0$. Let $p > \max\{r, 1\}$ and $q > 1$ satisfy $1/p + 1/q = 1$. Let a_{nk}, $1 \leq k \leq n$, $n \geq 1$, be a* *triangular array of real numbers satisfying*

$$\limsup_{n\to\infty}\sum_{k=1}^{n}|a_{nk}|^q<\infty.$$

Then

$$\lim_{n\to\infty}\frac{1}{n^{1/r}}\sum_{k=1}^{n}a_{nk}X_k=0\quad a.e.[P].$$

Proof. Observe that, by Hölder's inequality, for every $n\geqslant 1$

$$\left\|\frac{1}{n^{1/r}}\sum_{k=1}^{n}a_{nk}X_k\right\|\leqslant\sum_{k=1}^{n}|a_{nk}|\frac{\|X_k\|}{n^{1/r}}$$

$$\leqslant\Big(\sum_{k=1}^{n}|a_{nk}|^q\Big)^{1/q}\left(\sum_{k=1}^{n}\frac{\|X_k\|^p}{n^{p/r}}\right)^{1/p}.$$

Also, we note that $0<r/p<1$ and $\|X_k\|^p$, $k\geqslant 1$, is uniformly bounded by $|X|^p$ with $E(|X|^p)^{r/p}<\infty$. By the Stability Theorem [14, E, p. 387],

$$\lim_{n\to\infty}\frac{1}{n^{p/r}}\sum_{k=1}^{n}\|X_k\|^p=0\quad \text{a.e.}[P].$$

Since $\limsup_{n\to\infty}\sum_{k=1}^{n}|a_{nk}|^q<\infty$, it follows that

$$\lim_{n\to\infty}\frac{1}{n^{1/r}}\sum_{k=1}^{n}a_{nk}X_k=0\quad \text{a.e.}[P].$$

The following result was established by Padgett and Taylor [17, Theorem 3, p. 192] under the additional assumptions that X_n, $n\geqslant 1$, is independently identically distributed with $E\|X_1\|<\infty$ and $EX_1=0$.

COROLLARY 4.7. *Let X_n, $n\geqslant 1$, be a sequence of random elements taking values in a separable normed linear space B such that $\|X_n\|$, $n\geqslant 1$, is uniformly bounded by a real random variable X with $E|X|<\infty$. Let a_{nk}, $1\leqslant k\leqslant n$, $n\geqslant 1$, be a triangular array of real numbers satisfying*

$$\limsup_{n\to\infty}\sum_{k=1}^{n}a_{nk}^2<\infty.$$

Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}a_{nk}X_k=0\quad a.e.[P].$$

Proof. In Theorem 4.6, take $r=1$ and $p=2$.

The following result was established by Padgett and Taylor [17, Theorem 5, p. 194] under the additional assumptions that X_n, $n\geqslant 1$, is independent, EX_n exists and equals zero for every $n\geqslant 1$, and $\beta>\alpha$, using Theorem 3(ii) of Stout [21].

COROLLARY 4.8. *Let X_n, $n\geqslant 1$, be a sequence of random elements taking values in a separable normed linear space B such that $\sup_{n\geqslant 1}E\|X_n\|^2<\infty$. Let a_{nk}, $1\leqslant k\leqslant n$, $n\geqslant 1$ be a triangular array of real numbers such that for $\alpha>0$ and $\beta>1$*

(i) $|a_{nk}|\leqslant\Gamma_1 n^{-\alpha}$ *for all* $1\leqslant k\leqslant n$ *and* $n\geqslant 1$,

(ii) $\sum_{k=1}^{n}|a_{nk}|\leqslant\Gamma_2 n^{\alpha-\beta}$ *for every* $n\geqslant 1$

for some positive constants Γ_1 and Γ_2. *Then*

$$\lim_{n\to\infty}\sum_{k=1}^{n}a_{nk}X_k=0\quad a.e.[P].$$

Proof. By Lemma 2.3, there exists a real random variable X such that $\|X_n\|$, $n\geqslant 1$, is uniformly bounded by X and $E|X|^r<\infty$ for $0\leqslant r<2$, since $\sup_{n\geqslant 1}E\|X_n\|^2<\infty$. Choose $p>0$ and $1<r<2$ satisfying

$$r<p<2\quad\text{and}\quad\frac{p}{r}\cdot\frac{1}{p-1}<\beta.$$

If $q>0$ satisfies $1/p+1/q=1$, then $q>2$. Let $d_{nk}=n^{1/r}a_{nk}$ for every $1\leqslant k\leqslant n$ and $n\geqslant 1$. We rewrite

$$\sum_{k=1}^{n}a_{nk}X_k=\frac{1}{n^{1/r}}\sum_{k=1}^{n}d_{nk}X_k\quad\text{for all } n\geqslant 1.$$

We check for every $n\geqslant 1$

$$\sum_{k=1}^{n}|d_{nk}|^q=\sum_{k=1}^{n}|a_{nk}|^q n^{q/r}\leqslant\max_{1\leqslant k\leqslant n}|a_{nk}|^{q-1}\sum_{k=1}^{n}|a_{nk}|\,n^{q/r}$$

$$\leqslant\Gamma_1^{q-1}n^{-\alpha(q-1)}\Gamma_2 n^{\alpha-\beta}n^{q/r}$$

$$=\Gamma_1^{q-1}\Gamma_2 n^{-\alpha q+2\alpha}n^{q/r-\beta}.$$

Since $q>2$, $-\alpha q+2\alpha<0$. Since $1/q+1/p=1$, $q=p/(p-1)$ and so, $q/r=(p/r)[1/(p-1)]<\beta$. Consequently, $q/r-\beta<0$. Therefore,

$$\lim_{n\to\infty}\sum_{k=1}^{n}|d_{nk}|^q=0.$$

The conditions of Theorem 4.6 are met with respect to the triangular array d_{nk}, $1\leqslant k\leqslant n$, $n\geqslant 1$, of real numbers. Hence

$$\lim_{n\to\infty}\frac{1}{n^{1/r}}\sum_{k=1}^{n}d_{nk}X_k=0\quad \text{a.e.}[P].$$

But $\frac{1}{n^{1/r}}\sum_{k=1}^{n} d_{nk}X_k = \sum_{k=1}^{n} a_{nk}X_k$ for every $n \geqslant 1$. This completes the proof.

COROLLARY 4.9. *Let X_n, $n \geqslant 1$, be a sequence of random elements taking values in a separable normed linear space B such that $\|X_n\|$, $n \geqslant 1$, is uniformly bounded by a random variable X such that $E|X|^r < \infty$ for some $0 < r < 2$. Let a_{nk}, $1 \leqslant k \leqslant n$, $n \geqslant 1$, be a triangular array of real random variables satisfying*

$$\limsup_{n\to\infty} \sum_{k=1}^{n} a_{nk}^2 = \Gamma < \infty \quad a.e.[P].$$

Then

$$\lim_{n\to\infty} \frac{1}{n^{1/r}} \sum_{k=1}^{n} a_{nk}X_k = 0 \quad a.e.[P].$$

Proof. The proof of Theorem 4.6 can be adapted to prove this result by taking $p = q = 2$.

Wei and Taylor [30, Theorem 3, p. 55] established the conclusion of the above result under the stronger condition that X_n, $n \geqslant 1$, is independently identically distributed with $E\|X_1\|^r < \infty$ for some $1 \leqslant r < 2$. (In fact, Theorem 4.6 is valid when a_{nk}, $1 \leqslant k \leqslant n$, $n \geqslant 1$, is a triangular array of random variables satisfying $\limsup_{n\to\infty}\sum_{k=1}^{n}|a_{nk}|^q = \Gamma < \infty$ a.e.$[P]$.) Incidentally, the above corollary extends in one direction Theorem 9 of Chow and Lai [5, p. 823] to separable normed linear spaces for the case $0 < r < 2$. Chow and Lai assume that X_n, $n \geqslant 1$, is a sequence of independently identically distributed real random variables with $E|X_1|^r < \infty$ for some $1 \leqslant r \leqslant 2$. For the case $0 < r < 2$, the above assumption is dropped and we merely assume that X_n, $n \geqslant 1$, is uniformly bounded by a real random variable X with $E|X|^r < \infty$ for some $0 < r < 2$. When $1 \leqslant r < 2$, Chow and Lai centre their random variables at their means but this is not necessary.

COROLLARY 4.10. *Let X_n, $n \geqslant 1$, be a sequence of random elements taking values in a separable normed linear space B such that $\|X_n\|$, $n \geqslant 1$, is uniformly bounded by a random variable X satisfying $E|X|^r < \infty$ for some $1 < r \leqslant 2$. Let a_{nk}, $1 \leqslant k \leqslant n$, $n \geqslant 1$, be a triangular array of real numbers satisfying*

$$\max_{1\leqslant k\leqslant n} |a_{nk}| = O(n^{-\alpha}) \quad as\ n \to \infty$$

for some $0 < 1/\alpha < r-1$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}X_k = 0 \quad a.e.[P].$$

Proof. For each $1 \leqslant k \leqslant n$ and $n \geqslant 1$, let $d_{nk} = a_{nk}n^{1/r}$. Then

$$\sum_{k=1}^{n} d_{nk}^2 \leqslant \max_{1\leqslant k\leqslant n} a_{nk}^2 \sum_{k=1}^{n} (n^{1/r})^2 \leqslant Bn^{-2\alpha}n^{2/r}n$$
$$\leqslant Bn^{-2/(r-1)}nn^{2/r} = Bn^{1-2/[r(r-1)]} \quad \text{as } n \to \infty,$$

for some positive constant B. If $1 < r < 2$, we have $r^2 - r - 2 < 0$ so that $2/[r(r-1)] > 1$. Consequently,

$$\limsup_{n\to\infty} \sum_{k=1}^{n} d_{nk}^2 < \infty.$$

Now Corollary 4.9 is applicable. So,

$$\lim_{n\to\infty} \frac{1}{n^{1/r}} \sum_{k=1}^{n} d_{nk}X_k = 0 \quad \text{a.e.}[P]$$

and the conclusion follows. If $r = 2$, choose p and q such that $p > r = 2$, $1/p + 1/q = 1$ and $\alpha q > 2$. This is possible because $\lim_{q\uparrow 2}\alpha q = 2\alpha > 2$. Then

$$\sum_{k=1}^{n} |d_{nk}|^q \leqslant \max_{1\leqslant k\leqslant n} |a_{nk}|^q \sum_{k=1}^{n} (n^{1/2})^q$$
$$\leqslant Bn^{-\alpha q}n^2 \quad \text{as } n \to \infty.$$

Consequently,

$$\limsup_{n\to\infty} \sum_{k=1}^{n} |d_{nk}|^q < \infty.$$

An application of Theorem 4.6 covers the case $r = 2$. This completes the proof.

Taylor [23, Theorem 5.3.1, p. 137] established the conclusion of the above corollary under additional assumptions that B is Beck-convex, X_n, $n \geqslant 1$, is independent, $EX_n = 0$ for all $n \geqslant 1$, $\sup_{n\geqslant 1} E\|X_n\|^r < \infty$ for some $r > 1$, a_{nk}'s are nonnegative with sum over k less than or equal to unity for all $n \geqslant 1$ and

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} a_{nk} - n \min_{1\leqslant k\leqslant n} a_{nk}\right) = 0.$$

For the definition of Beck-convexity, see Beck [2].

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Received February 4, 1985 (2030)