STUDIA MATHEMATICA, T. LXXXVI. (1987)

Weighted norm inequalities for some classes of singular integrals

by

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Abstract. In the paper truncated Poisson kernels are considered and the classes of weight functions ψ for which the operators generated by these kernels are uniformly continuous in weighted $L^p(\psi)$ spaces, $1 \leqslant p < \infty$, are described. These results can be used to describe the classes of those weight functions ψ for which the system of functions resulting from the trigonometric system by removing finitely many members is an Abel basis in $L^p(\psi)$, $1 \leqslant p < \infty$, to solve the Dirichlet problem with weighted metric, and to study weighted $H^p(\psi)$ spaces (see [19]).

0. Introduction. In 1961 M. Rosenblum [22] obtained the following

THEOREM A. *Let $\psi(x)$ be a nonnegative 2π-periodic function, $P_r(x) = (1-r^2)\,[1-2r\cos x+r^2]^{-1}$ the Poisson kernel and $1 \leqslant p < \infty$. Then there is a constant $C_p > 0$ independent of f such that*

$$\Big\|\int_{-\pi}^{\pi} P_r(x-\cdot)f(x)\,dx\Big\|_{L^p_{[-\pi,\pi]}(\psi)} \overset{\text{def}}{=} \Big[\int_{-\pi}^{\pi}\Big|\int_{-\pi}^{\pi} P_r(x-t)f(x)\,dx\Big|^p\psi(t)\,dt\Big]^{1/p}$$
$$\leqslant C_p\|f\|_{L^p_{[-\pi,\pi]}(\psi)} \qquad (0<r<1)$$

if and only if ψ satisfies condition (A_p): for every interval $I \subset \mathbf{R}^1$,

$$|I|^{-1}\int_I \psi\,dt\,\Big[|I|^{-1}\int_I \psi^{-1/(p-1)}\,dt\Big]^{p-1} \leqslant B_p,$$

B_p is independent of I and

$$\Big[|I|^{-1}\int_I \psi^{-1/(p-1)}\,dt\Big]^{p-1} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{t\in I}[\psi(t)]^{-1} \qquad \textit{for } p=1.$$

Related work was done by Hardy and Littlewood [8], Babenko [1] Hirschman [10], Gaposhkin [5], Edwards [4], Chen [2], Helson and Szegö [9]. Condition (A_p) in the above form was first formulated in the famous paper of B. Muckenhoupt [20], where the following theorem was proved.

THEOREM B. *Let $\psi(x) \geqslant 0$ be a weight function defined on the real line and $1 \leqslant p < \infty$. If $M(f)(x)$ is the Hardy–Littlewood maximal function:*

$$M(f)(x) = \sup_{x\in I}|I|^{-1}\int_I |f|\,dt,$$

then there exists a constant $C_p > 0$ *independent of* f *such that*

$$\|M(f)\|_{L^p_{\boldsymbol{R}}(\psi)} \leqslant C_p \|f\|_{L^p_{\boldsymbol{R}}(\psi)}$$

if and only if ψ *satisfies condition* (A_p).

Then important weighted norm inequalities were proved by R. Hunt, B. Muckenhoupt, R. Wheeden, R. Gundy, R. Coifman, Ch. Fefferman, Wo-Sang Young ([11], [7], [3], [12]) and others. The measures considered in those papers have an essential restriction: if $f \in L^p(\psi)$, $p \geqslant 1$, then f is integrable with respect to the Lebesgue measure.

1. Formulation of the main results and notation.

Definition 1. We say that a weight function $\psi(x) \geqslant 0$ has a *p-th* $(1 \leqslant p < \infty)$ *power singularity* at a point x_0 if for every interval I with $x_0 \in I$, $1/\psi \notin L_I^{1/(p-1)}$ (if $p = 1$ then $1/(p-1) = \infty$).

Definition 2. We say that a weight function $\psi(x) \geqslant 0$ has a *p-th* $(1 \leqslant p < \infty)$ *power singularity of order* α $(\alpha = 1, 2, \ldots)$ if there exists $\delta_0 > 0$ such that

$$(x-x_0)^{p(\alpha-1)}/\psi(x) \notin L^{1/(p-1)}_{(x_0-\delta_0, x_0+\delta_0)},$$
$$(x-x_0)^{p\alpha}/\psi(x) \in L^{1/(p-1)}_{(x_0-\delta_0, x_0+\delta_0)}.$$

As usual, L^p_E $(L^p_E(d\mu))$ consists of all f for which $\int_E |f|^p\,dx < \infty$ $(\int_E |f|^p\,d\mu < \infty)$; if $d\mu(x) = \psi(x)\,dx$ we write $L^p_E(\psi)$. When the set on which the functions are defined is obvious from the context, we simply write $L^p(\psi)$.

We prove inequalities which show that the truncated Poisson kernels in general play the same role in weighted L^p spaces with weights having pth power singularities as the Poisson kernel in weighted L^p spaces with weights without such singularities. Analogous results for other kernels follow previous papers of the author ([14], [15], [16], [18]).

For the definition of the truncated Poisson kernels suppose that we have fixed an arbitrary collection of distinct points $\mathfrak{X} = \{x_j\}_{j=1}^s$ in the interval $[-\pi, \pi)$ and natural numbers $\mathfrak{N} = \{\alpha_j\}_{j=1}^s$. We write $\Lambda = \sum_{j=1}^s \alpha_j$ and for the given $\mathfrak{X}$ and $\mathfrak{N}$ define the fundamental interpolation polynomials $T(x_j, \lambda, x)$ $(1 \leqslant j \leqslant s,\ 0 \leqslant \lambda \leqslant \alpha_j - 1)$:

If $\Lambda = 2N+1$ $(N = 0, 1, \ldots)$ then $T(x_j, \lambda, x)$ is a trigonometric polynomial of order at most N such that

(1.1) $T^{(h)}(x_j, \lambda, x_i) = \delta_{ij}\delta_{h\lambda}$, $\quad \delta_{ij}$ the Kronecker delta, $0 \leqslant h \leqslant \alpha_j - 1$,

where $T^{(h)}$ denotes the derivative of order h and $T^{(0)} = T$.

If $\Lambda = 2N$ $(N = 1, 2, \ldots)$ then besides (1.1) we also assume that the order of $T(x_j, \lambda, x)$ is less than N or the ratio of the coefficients of $\cos Nx$ and $\sin Nx$ in these polynomials is equal with opposite sign to the ratio of the coefficients of $\sin Nx$ and $\cos Nx$ in the trigonometric polynomial

$$\omega(x) = \prod_{1 \leqslant j \leqslant s} \sin^{\alpha_j} \tfrac{1}{2}(x - x_j). \tag{1.2}$$

When the coefficient of $\cos Nx$ or $\sin Nx$ in $\omega(x)$ is zero then accordingly the coefficients of $\sin Nx$ or $\cos Nx$ in the polynomials $T(x_j, \lambda, x)$ are zero.

In the case $\Lambda = 2N$ we have put the additional condition in order to obtain the uniqueness of the polynomials $T(x_j, \lambda, x)$ $(1 \leqslant j \leqslant s,\ 0 \leqslant \lambda \leqslant \alpha_j - 1)$.

We will study the following kernel:

$$P_{\mathfrak{X},\mathfrak{N},r}(x, t) = P_r(x-t) - \sum_{j=1}^{s} \sum_{\lambda=0}^{\alpha_j-1} P_r^{(\lambda)}(x_j - t)\, T(x_j, \lambda, t) \tag{1.3}$$

where

$$P_r^{(\lambda)}(\xi) = \frac{d^\lambda}{d\xi^\lambda} P_r(\xi), \qquad P_r^{(0)}(\xi) = P_r(\xi).$$

The following theorem is true:

Theorem 1. *For a* 2π*-periodic weight function* ψ *and* $1 < p < \infty$ *the following conditions are equivalent*:

(a) *There is a constant* $C_p > 0$ *independent of* $f \in L^p_{[-\pi,\pi]}(\psi)$ *such that*

$$\Big\| \int_{-\pi}^{\pi} f(x) P_{\mathfrak{X},\mathfrak{N},r}(x, \cdot)\,dx \Big\|_{L^p_{[-\pi,\pi]}(\psi)} \leqslant C_p \|f\|_{L^p_{[-\pi,\pi]}(\psi)} \quad (0 < r < 1). \tag{1.4}$$

(b) *There is a constant* $B_p > 0$ *independent of* I *and* γ $(0 < \gamma < 1)$ *such that*

(β_1) *for every interval* $I \subset \boldsymbol{R}^1$

$$\frac{1}{|\omega|(I)} \int_I \psi\,dt \left[\frac{1}{|\omega|(I)} \int_I \left[\frac{|\omega|^p}{\psi} \right]^{\frac{1}{p-1}} dt \right]^{p-1} \leqslant B_p, \tag{A_p^ω}$$

where $|\omega|(I) = \int_I |\omega|(t)\,dt$;

(β_2) *for every* γ $(0 < \gamma < 1)$ *and natural* j $(1 \leqslant j \leqslant s)$

$$\gamma^{-p\alpha_j} \int_{O_{j\gamma}} \psi\,dt \left[\int_{O_{j\gamma}^c} \left[\left| \left(\sin\frac{t-x_j}{2} \right)^{-1} \omega(t) \right|^p [\psi(t)]^{-1} \right]^{\frac{1}{p-1}} dt \right]^{p-1} \leqslant B_p,$$

where $O_{j\gamma} = \{t\colon |t - x_j| < \gamma\}$, $O_{j\gamma}^c = \{t\colon |t - x_j| \geqslant \gamma\} \cap [-\pi, \pi)$.

(c) *The following representation is true*: $\psi(x) = |\omega(x)|^p w(x)$. *where* $w(x)$ *is a weight function which satisfies condition* (A_p) *and* $\omega(x)$ *is defined by* (1.2).

Theorem 2. *For a* 2π*-periodic weight function* $\psi(x) \geqslant 0$ *the following conditions are equivalent*:

(a′) *There is a constant* $C_1 > 0$ *independent of* $f \in L_{[-\pi,\pi]}(\psi)$ *such that* (1.4) *is true for* $p = 1$.

(b′) *There is a constant* $B_1 > 0$ *independent of* I *and* γ $(0<\gamma<1)$ *such that*

(β_1') *for every interval* $I \subset \mathbf{R}^1$

$$(A_1^{\omega}) \qquad \frac{1}{|\omega|(I)} \int_I \psi\, dt \cdot \operatorname{ess\,sup}_{x\in I} [|\omega(x)|/\psi(x)] \leqslant B_1,$$

(β_2') *for every* γ $(0<\gamma<1)$ *and natural* j $(1\leqslant j\leqslant s)$

$$\gamma^{-\alpha_j} \int_{O_{j\gamma}} \psi\, dt \cdot \operatorname{ess\,sup}_{x\in O_{j\gamma}^c} \left[|\omega(x)| \left| \sin\frac{x-x_j}{2} \right|^{-1} [\psi(x)]^{-1} \right] \leqslant B_1,$$

where $O_{j\gamma}$ *and* $O_{j\gamma}^c$ *are the same as in* (β_2).

(c′) *The following representation is true:*

$$\psi(x) = \prod_{j=1}^{s} \left| \sin^{\alpha_j - 1} \frac{x-x_j}{2} \right| \theta_j(x)\, w(x)$$

where $w(x)$ *is a weight function which satisfies condition* (A_1), *and for every interval* I *with center at* x_j $(1\leqslant j\leqslant s)$, $|I|^{-1}\int_I w\,dt \leqslant C$; *for every natural* j $(1\leqslant j\leqslant s)$ θ_j *is a nonnegative bounded* 2π*-periodic function and there exists* $C_j > 0$ *such that for every* $x_j < x' < x'' \leqslant x_j+\pi$ *and* $x_j - \pi \leqslant x'' < x' < x_j$

$$[\theta_j(x')]^{-1} |\sin\tfrac{1}{2}(x'-x_j)| \leqslant C_j [\theta_j(x'')]^{-1} |\sin\tfrac{1}{2}(x''-x_j)|.$$

With the help of Theorems 1 and 2 one can prove the basicness of some concrete subsystems of the trigonometric system in weighted L^p spaces. Also, one can study weighted H^p spaces and solve the Dirichlet problem in weighted metric (see [16], [17], [18], [19]). We note that V. F. Gaposhkin [6] and R. L. Wheeden and J.-O. Strömberg [23] have also obtained results concerning weights with pth power singularities.

C_p, B_p will be absolute constants. We omit subscripts or use the same subscript in different contexts when we believe that no confusion can arise. We also write

$$\Big[\int_E |f|^{1/0} dx\Big]^0 \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{x\in E} |f(x)| \quad \text{and} \quad \xi_r(x) = 1 - 2r\cos x + r^2.$$

2. Some auxiliary facts. We say that $w(x) \geqslant 0$ satisfies the *doubling condition* if $\int_{I^*} w(x)\,dx \leqslant C\int_I w(x)\,dx$ for any interval I. (I^* denotes the double of I and C is a constant independent of I.) Consider the maximal function

$$(2.1) \qquad M^w(f)(t) = \sup_{t\in I} \Big(\int_I w(x)\,dx\Big)^{-1} \int_I |f(x)|\, w(x)\,dx,$$

where the supremum is taken over all intervals $I \ni t$. The following theorem is a modification of Theorem B (see [20]).

THEOREM C. *Let* w *and* ψ *be nonnegative functions with* $w, \psi \in L_{\mathrm{loc}}(\mathbf{R}^1)$ *and* w *satisfying the doubling condition. Let* $p > 1$. *Then there is a constant* C *independent of* f *such that*

$$\int_{\mathbf{R}} [M^w(f)]^p \psi\, dt \leqslant C \int_{\mathbf{R}} |f|^p \psi\, dt$$

if and only if there is a constant $B_p > 0$ *such that for every interval* I

$$(A_p^w) \qquad \Big(\int_I w\,dt\Big)^{-1} \int_I \psi\,dt \left[\Big(\int_I w\,dt\Big)^{-1} \int_I \Big(\frac{w^p}{\psi}\Big)^{\frac{1}{p-1}} dt \right]^{p-1} \leqslant B_p.$$

Using the ideas of R. Coifman and Ch. Fefferman [3] it is possible to give an easy proof of this theorem (see [16]).

G. Tomaselli [25], G. Talenti [24], and M. Artola (we quote following [21]) obtained the following result.

THEOREM D. *If* $1 \leqslant p \leqslant \infty$, *there is a finite* $B > 0$ *such that*

$$(2.2) \qquad \Big[\int_0^\infty \Big|U(x)\int_0^x f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant B\Big[\int_0^\infty |V(x)f(x)|^p dx\Big]^{1/p}$$

if and only if

$$(H_p) \qquad D = \sup_{a>0} \Big[\int_a^\infty |U(x)|^p dx\Big]^{1/p} \Big[\int_0^a |V(x)|^{-q} dx\Big]^{1/q} < \infty,$$

where $p^{-1} + q^{-1} = 1$. *Furthermore, if* B *is the least constant for which* (2.2) *holds, then* $D \leqslant B \leqslant p^{1/p} q^{1/q} D$ *for* $1 < p < \infty$, *and* $B = D$ *if* $p = 1$ *or* ∞.

The following theorem, dual to Theorem D, was proved by B. Muckenhoupt [21].

THEOREM E. *If* $1 \leqslant p \leqslant \infty$, *there is a finite* $B > 0$ *such that*

$$(2.3) \qquad \Big[\int_0^\infty \Big|U(x)\int_x^\infty f(t)\,dt\Big|^p dx\Big]^{1/p} \leqslant B\Big[\int_0^\infty |V(x)f(x)|^p dx\Big]^{1/p}$$

if and only if

$$(H_p') \qquad D = \sup_{a>0} \Big[\int_0^a |U(x)|^p dx\Big]^{1/p} \Big[\int_a^\infty |V(x)|^{-q} dx\Big]^{1/q} < \infty,$$

where $p^{-1} + q^{-1} = 1$. *Furthermore, if* B *is the least constant for which* (2.3) *is true, then* $D \leqslant B \leqslant p^{1/p} q^{1/q} D$.

Write

$$(2.4) \qquad B_{rj}(x,t) = P_r(x-t)\,T(x_j,0,x) - \sum_{\lambda=0}^{\alpha_j-1} P_r^{(\lambda)}(x_j-t)\,T(x_j,\lambda,x).$$

We have the equality

$$(2.5) \qquad \sum_{j=1}^{s} T(x_j,0,x) = 1,$$

which for $\Lambda = 2N+1$ immediately follows from the theorem of uniqueness of interpolating trigonometric polynomials, and for $\Lambda = 2N$ we may also observe that $\sum_{j=1}^{s} T(x_j, 0, x)-1$ cannot coincide with $C\omega(x)$. From (2.4), (2.5) and (1.3) we obtain

$$P_{\mathfrak{X},\mathfrak{N},r}(x, t) = \sum_{j=1}^{s} B_{rj}(x, t). \tag{2.6}$$

This equality allows us to consider only the kernels $B_{rj}(x, t)$ $(1 \leqslant j \leqslant s)$ in the proof of Theorems 1 and 2.

3. Factorization of the kernels $B_{rj}(x, t)$ $(1 \leqslant j \leqslant s)$. For brevity we will only consider the case $\Lambda = 2N+1$. The proof for the case $\Lambda = 2N$ is analogous and differs only by some technical modifications. Now we settle several basic lemmas.

LEMMA 1. *Let $\Lambda = 2N+1$ $(N = 0, 1, \ldots)$. Then for every j $(1 \leqslant j \leqslant s)$*

$$B_{rj}(x, t) = P_r(x-t)\,\omega(x)\left[G_r'(t)\sin\frac{x-x_j}{2} + G_r''(t)\cos\frac{x-x_j}{2}\right], \tag{3.1}$$

and there is a constant $C > 0$ independent of r and t such that

$$|G_r'(t)| \leqslant C\,[\xi_r(t-x_j)]^{-(\alpha_j+1)/2}, \tag{3.2}$$

$$|G_r''(t)| \leqslant C\,[\xi_r(t-x_j)]^{-\alpha_j/2}. \tag{3.3}$$

LEMMA 2. *Let $\Lambda = 2N+1$ $(N = 0, 1, \ldots)$, $1 \leqslant j \leqslant s$ and $x_j = 0$. Then there are positive numbers $0 < r_0 < 1$, $0 < a < 1$ and constants $0 < c' < C'$, $C > 0$, $a_1 < a/(1-r_0)$ with the following properties:*

(i) *If α_j is odd, then*

$$B_{rj}(x, t) = G_{rj}(t)\sin\frac{x-x_r^*(t)}{2}\,\omega(x)\,P_r(x-t), \tag{3.4}$$

where for $1 > r > r_0$ and some positive numbers $0 < \delta_r' < \delta_r'' < C'$

$$|G_{rj}(t)| \geqslant C\,[\xi_r(t)]^{-(\alpha_j+1)/2} \tag{3.5}$$

if $|t| < a(1-r)$,

$$\delta_r'(1-r)^2 + (c'+1)\,t \leqslant x_r^*(t) \leqslant (C'+1)\,t + \delta_r''(1-r)^2 \tag{3.6}$$

if $0 \leqslant t \leqslant a(1-r)$, *and*

$$\delta_r'(1-r)^2 + (C'+1)\,t \leqslant x_r^*(t) \leqslant (c'+1)\,t + \delta_r''(1-r)^2 \tag{3.7}$$

if $0 \geqslant t \geqslant -a(1-r)$.

(ii) *If α_j is even, then*

$$B_{rj}(x, t) = \left|\sin\frac{x_r^*(t)}{2}\right|^{-1} G_{rj}(t)\sin\frac{x-x_r^*(t)}{2}\,\omega(x)\,P_r(x-t), \tag{3.8}$$

where for $1 > r > r_0$

$$|G_{rj}(t)| \geqslant C\,[\xi_r(t)]^{-\alpha_j/2} \quad \text{if } |t| < a(1-r) \tag{3.9}$$

and there is a point b_r, $|b_r| < a_1(1-r)^2$, such that

$$-\pi \leqslant x_r^*(t) < -2a(1-r) \quad \text{if } t \in (b_r,\, a(1-r)), \tag{3.10}$$

$$\pi > x_r^*(t) > 2a(1-r) \quad \text{if } t \in (-a(1-r),\, b_r). \tag{3.11}$$

LEMMA 3. *Let $\Lambda = 2N+1$ $(N = 0, 1, \ldots)$, $1 \leqslant j \leqslant s$, and $x_j = 0$. Then there are positive numbers $0 < r_0 < 1$, $0 < a < 1$, and a constant $C > 0$ with the following properties:*

(i) *If α_j is odd, then the representation* (3.4) *holds, where for $|t| < 1-r_0$*

$$\left(1+\frac{1}{8\alpha_j+1}\right)t \leqslant x_r^*(t) \leqslant 2t \quad \text{if } 0 \leqslant 1-r \leqslant at, \tag{3.12}$$

$$2t \leqslant x_r^*(t) \leqslant \left(1+\frac{1}{8\alpha_j+1}\right)t \quad \text{if } at \leqslant -(1-r) < 0, \tag{3.13}$$

$$|G_{rj}(t)| \geqslant C\,|\sin t/2|^{-(\alpha_j+1)} \quad \text{if } 1-r < a|t|. \tag{3.14}$$

(ii) *If α_j is even, then representation* (3.8) *holds, where estimates* (3.12), (3.13) *for $x_r^*(t)$ are true, and*

$$|G_{rj}(t)| \geqslant C\,|\sin t/2|^{-\alpha_j} \quad \text{if } 1-r < a|t|. \tag{3.15}$$

For the proof of Lemmas 1–3 the following lemma will be needed.

LEMMA 4. *Let $P_r(x)$ be the Poisson kernel and k a natural number. Then*

$$(1-r^2)^{-1}P_r^{(2k)}(x) = \sum_{i=0}^{k} C_i^{(2k)}(r)\sin^{2i}x\cos^{k-i}x\,[\xi_r(x)]^{-(k+i+1)} + \sum_{i=0}^{k-1}\sin^{2i}x\, o\big([\xi_r(x)]^{-(k+i)}\big), \tag{3.16}$$

$$(1-r^2)^{-1}P_r^{(2k-1)}(x) = \sum_{i=0}^{k-1} C_i^{(2k-1)}(r)\sin^{2i+1}x\cos^{k-i-1}x\,[\xi_r(x)]^{-(k+i+1)} + \sum_{i=0}^{k-2}\sin^{2i+1}x\, o\big([\xi_r(x)]^{-(k+i)}\big), \tag{3.17}$$

where $C_i^{(j)}$ satisfy

$$C_i^{(2k+1)}(r) = -(k+i+1)\,2rC_i^{(2k)}(r) + 2(i+1)\,C_{i+1}^{(2k)}(r) \quad (k = 0, 1, \ldots;\ 0 \leqslant i \leqslant k;\ C_{k+1}^{(2k)}(r) = 0), \tag{3.18}$$

$$C_i^{(2k+2)}(r) = (2i+1)\,C_i^{(2k+1)}(r) - (k+i+2)\,2rC_{i-1}^{(2k+1)}(r) \quad (k = 0, 1, \ldots;\ 0 \leqslant i \leqslant k+1;\ C_{-1}^{(2k+1)}(r) = 0,\ C_{k+1}^{(2k+1)}(r) = 0), \tag{3.19}$$

$$\operatorname{sign} C_i^{(j)}(r)\,C_{i+1}^{(j)}(r) = -1 \quad (j = 1, 2, \ldots;\ 0 \leqslant i \leqslant [j/2]-1,\ 0 < r < 1). \tag{3.20}$$

Proof. Equalities (3.16) and (3.17) are proved by induction using differentiation. It is only necessary to observe that in equality (3.16)

$$o\left([\xi_r(x)]^{-(k+i)}\right) = \sum_{j=2}^{k+i-1} [\xi_r(x)]^{-j} \sum_{l=0}^{k} d_{lj}^{(2k)}(r)\cos^l x,$$

and in equality (3.17)

$$o\left([\xi_r(x)]^{-(k+i)}\right) = \sum_{j=2}^{k+i-1} [\xi_r(x)]^{-j} \sum_{l=0}^{k} d_{lj}^{(2k-1)}(r)\cos^l x,$$

where $d_{lj}^{(2k-1)}(r)$, $d_{lj}^{(2k)}(r)$ $(0 \leqslant l \leqslant k)$ are polynomials in r of order at most $2k$. Formulas (3.18) and (3.19) immediately follow from (3.16) and (3.17). Relation (3.20) is easily proved by induction using formulas (3.18), (3.19).

Proof of Lemma 1. Write

$$T_{rj}(x, t) = T(x_j, 0, x) - [P_r(x-t)]^{-1} \sum_{\lambda=0}^{\alpha_j-1} P_r^{(\lambda)}(x_j-t)\, T(x_j, \lambda, x). \tag{3.21}$$

It is obvious that for fixed t and r, $T_{rj}(x, t)$ is a trigonometric polynomial of order $N+1$ and has a zero of order α_i at x_i $(1 \leqslant i \leqslant s,\ i \neq j)$. From the definition it is obvious that $T_{rj}(x_j, t) = 0$. Assume that for a nonnegative integer k $(0 \leqslant k < \alpha_j - 1)$

$$\frac{d^\nu}{dx^\nu} T_{rj}(x, t)\bigg|_{x=x_j} = 0 \quad (0 \leqslant \nu \leqslant k).$$

We will show that

$$\frac{d^{k+1}}{dx^{k+1}} T_{rj}(x, t)\bigg|_{x=x_j} = 0.$$

From our assumption we have

$$\frac{d^{k+1}}{dx^{k+1}} T_{rj}(x, t)\bigg|_{x=x_j} = \sum_{\nu=0}^{k+1} \binom{k+1}{\nu} P_r^{(\nu)}(x_j-t) \frac{d^{k+1-\nu}}{dx^{k+1-\nu}} T_{rj}(x, t)\bigg|_{x=x_j} = P_r(x_j-t)\frac{d^{k+1}}{dx^{k+1}} T_{rj}(x, t)\bigg|_{x=x_j}.$$

Since $(d^{k+1}/dx^{k+1}) B_{rj}(x, t)|_{x=x_j} = 0$ $(0 \leqslant k < \alpha_j - 1)$ and $P_r(x_j - t) \neq 0$ $(0 < r < 1)$, we have $(d^{k+1}/dx^{k+1}) T_{rj}(x, t)|_{x=x_j} = 0$. Thus we have proved that

$$\frac{d^\nu}{dx^\nu} T_{rj}(x, t)\bigg|_{x=x_j} = 0 \quad (0 \leqslant \nu \leqslant \alpha_j - 1). \tag{3.22}$$

Hence it easily follows that the sum

$$T_{rj}(x, t) + [P_r(x-t)]^{-1} P_r^{(\alpha_j-1)}(x_j-t)\, T(x_j, \alpha_j-1, x)$$

for fixed t and r has a zero of order α_i at x_i $(1 \leqslant i \leqslant s,\ i \neq j)$, of order $\alpha_j - 1$ at x_j and is a trigonometric polynomial of order $N+1$. Thus

$$T_{rj}(x, t) + [P_r(x-t)]^{-1} P_r^{(\alpha_j-1)}(x_j-t)\, T(x_j, \alpha_j-1, x) = \omega_j'(x)\sin^{\alpha_j-1}\frac{x-x_j}{2} L_{rj}(x, t), \tag{3.23}$$

where

$$\omega_j'(x) = \prod_{\substack{1 \leqslant k \leqslant s \\ k \neq j}} \sin^{\alpha_k}\frac{x-x_k}{2}, \tag{3.24}$$

$$L_{rj}(x, t) = \beta_{r1}(t) + \beta_{r2}(t)\cos(x-x_j) + \beta_{r3}(t)\sin(x-x_j). \tag{3.25}$$

By formulas (3.16) and (3.17) it is easily seen that

$$|\beta_{rl}(t)| \leqslant C[\xi_r(x_j-t)]^{-\alpha_j/2} \quad (0 < r < 1,\ l = 1, 2, 3). \tag{3.26}$$

From equality (3.23), since

$$T(x_j, \alpha_j-1, x) = b_j\,\omega_j'(x)\sin^{\alpha_j-1}\frac{x-x_j}{2} \tag{3.27}$$

where

$$b_j = \frac{2^{\alpha_j-1}}{(\alpha_j-1)!}[\omega_j'(x_j)]^{-1}, \tag{3.28}$$

we obtain

$$T_{rj}(x, t) = \omega_j'(x)\sin^{\alpha_j-1}\frac{x-x_j}{2}\left[L_{rj}(x, t) - b_j[P_r(x-t)]^{-1} P_r^{(\alpha_j-1)}(x_j-t)\right]. \tag{3.29}$$

Using equality (3.25) we obtain

$$\begin{aligned} L_{rj}(x,t) - b_j[P_r(x-t)]^{-1} P_r^{(\alpha_j-1)}(x_j-t) &= \beta_{r1}(t) - b_j\frac{1+r^2}{1-r^2} P_r^{(\alpha_j-1)}(x_j-t) \\ &\quad + \cos(x-x_j)\left[\beta_{r2}(t) + 2b_j\frac{r}{1-r^2} P_r^{(\alpha_j-1)}(x_j-t)\cos(t-x_j)\right] \\ &\quad + \sin(x-x_j)\left[\beta_{r3}(t) + 2b_j\frac{r}{1-r^2} P_r^{(\alpha_j-1)}(x_j-t)\sin(t-x_j)\right]. \end{aligned} \tag{3.30}$$

In view of (3.22) and (3.29) we have

$$L_{rj}(x_j, t) - b_j[P_r(x_j-t)]^{-1} P_r^{(\alpha_j-1)}(x_j-t) = 0.$$

The identity (3.30) thus implies

$$(3.31)\qquad \beta_{r1}(t)-b_j\frac{1+r^2}{1-r^2}P_r^{(\alpha_j-1)}(x_j-t)$$

$$=-\left[\beta_{r2}(t)+2b_j\frac{r}{1-r^2}P_r^{(\alpha_j-1)}(x_j-t)\cos(t-x_j)\right].$$

By equalities (3.30) and (3.31) we get

$$(3.32)\qquad L_{rj}(x,t)-b_j[P_r(x-t)]^{-1}P_r^{(\alpha_j-1)}(x_j-t)$$

$$=G_r'(t)\sin^2\frac{x-x_j}{2}+G_r''(t)\cdot\tfrac{1}{2}\sin(x-x_j),$$

where

$$(3.33)\qquad G_r'(t)=-2\left[\beta_{r2}(t)+2b_j\frac{r}{1-r^2}P_r^{(\alpha_j-1)}(x_j-t)\cos(t-x_j)\right],$$

$$(3.34)\qquad G_r''(t)=2\left[\beta_{r3}(t)+2b_j\frac{r}{1-r^2}P_r^{(\alpha_j-1)}(x_j-t)\sin(t-x_j)\right].$$

Relations (3.21), (3.29) and (3.32) then force (3.1). Estimates (3.2) and (3.3) can easily be obtained from (3.26), (3.16) and (3.17). Lemma 1 is proved.

Proof of Lemma 2. Write

$$(3.35)\qquad x_r^*(t)=-2\operatorname{arc\,tg}[G_r''(t)/G_r'(t)],$$

where $G_r'(t)$ and $G_r''(t)$ are defined by (3.33) and (3.34). Then we have

(3.36)

$$G_r'(t)\sin\tfrac{1}{2}x+G_r''(t)\cos\tfrac{1}{2}x=\operatorname{sign}G_r'(t)\left[(G_r'(t))^2+(G_r''(t))^2\right]^{1/2}\sin\frac{x-x_r^*(t)}{2}$$

$$=G_{rj}(t)\sin\frac{x-x_r^*(t)}{2}.$$

If $\alpha_j-1=0$, then from (3.23) and (3.25) it is obvious that $\beta_{r2}(t)=\beta_{r3}(t)=0$. Hence in this case from (3.34) and (3.33) we get $G_r''(t)/G_r'(t)=-\operatorname{tg}t$. Thus (3.35) implies

$$(3.37)\qquad x_r^*(t)=2t\quad \text{for } \alpha_j=1.$$

If $\alpha_j-1>0$, then we will first find the representation of $\beta_{r2}(t)$, $\beta_{r3}(t)$. By (3.21), the left side of the identity (3.23) is equal to

$$T(0,0,x)-[P_r(x-t)]^{-1}\sum_{\lambda=0}^{\alpha_j-1}P_r^{(\lambda)}(-t)\,T(0,\lambda,x).$$

On the other hand, $T(0,\lambda,x)$ $(0\leqslant\lambda\leqslant\alpha_j-1)$ and $\omega_j'(x)\sin^{\alpha_j-1}\frac{1}{2}x$, where $\omega_j'(x)$ is defined by (3.24), are trigonometric polynomials of order N. Hence writing $\cos(x-t)=\cos x\cos t+\sin x\sin t$, after the corresponding operations on both sides of (3.23) we get

$$(3.38)\qquad \beta_{rk}(t)=\frac{r}{1-r^2}\cos t\sum_{\nu=0}^{\alpha_j-2}\eta_\nu^{(k)}P_r^{(\nu)}(-t)$$

$$+\frac{r}{1-r^2}\sin t\sum_{\nu=0}^{\alpha_j-2}\theta_\nu^{(k)}P_r^{(\nu)}(-t),$$

where $\eta_\nu^{(k)}$, $\theta_\nu^{(k)}$ $(k=2,3,\ 0\leqslant\nu\leqslant\alpha_j-2)$ are some real numbers. To specify the value of $\eta_{\alpha_j-2}^{(3)}$ we first show that

$$(3.39)\qquad T(0,\alpha_j-2,x)=-\frac{2^{\alpha_j-2}}{(\alpha_j-2)!}\left(\omega_j'(0)\sin\tfrac{1}{2}\varrho\right)^{-1}\omega_j'(x)$$

$$\times\sin^{\alpha_j-2}\frac{x}{2}\sin\frac{x-\varrho}{2},$$

where $\varrho\in[-\pi,\pi)$ and $\varrho\neq x_j=0$. The existence of such a ϱ follows from the fact that $T(0,\alpha_j-2,x)$ is a trigonometric polynomial of order N with real coefficients and $\sum_{\nu=1}^{s}\alpha_\nu-2=2N-1$. It is trivial that $\varrho\neq 0$. For the polynomial defined by (3.39) the condition $T^{(\alpha_j-2)}(0,\alpha_j-2,0)=1$ is true. The value of ϱ is defined from the condition $T^{(\alpha_j-1)}(0,\alpha_j-2,0)=0$. Observe that the following four functions are linearly independent on $[-\pi,\pi)$:

$$(3.40)\qquad \omega_j'(x)\sin^{\alpha_j-2}\tfrac{1}{2}x\sin\tfrac{3}{2}x,\quad \omega_j'(x)\sin^{\alpha_j-2}\tfrac{1}{2}x\cos\tfrac{3}{2}x,$$

$$\omega_j'(x)\sin^{\alpha_j-1}\tfrac{1}{2}x,\quad \omega_j'(x)\sin^{\alpha_j-2}\tfrac{1}{2}x\cos\tfrac{1}{2}x.$$

For every $0<r<1$ the same is true of the functions

$$(3.41)\qquad \left\{\frac{1+r^2}{1-r^2}P_r^{(\nu)}(-t);\ \frac{r}{1-r^2}\cos t\,P_r^{(\nu)}(-t);\ \frac{r}{1-r^2}\sin t\,P_r^{(\nu)}(-t)\right\}_{\nu=0}^{\alpha_j-1}.$$

From the formula

$$\cos(x-t)\sin\frac{x-\varrho}{2}=-\cos\tfrac{3}{2}x\left(\tfrac{1}{2}\sin t\cos\tfrac{1}{2}\varrho+\tfrac{1}{2}\cos t\sin\tfrac{1}{2}\varrho\right)+\ldots$$

and (3.39) we find that on the left side of equality (3.23) the coefficient of the expression

$$P_r^{(\alpha_j-2)}(-t)\cos t\cos\tfrac{3}{2}x\,\omega_j'(x)\sin^{\alpha_j-2}\tfrac{1}{2}x$$

is equal to

$$\frac{r}{1-r^2}\,\frac{2^{\alpha_j-2}}{(\alpha_j-2)!}[\omega_j'(0)]^{-1}.$$

In order to find the coefficient of the same expression on the right of (3.23) we use the linear independence of the functions (3.41) and (3.40). By the equalities

$$\sin\tfrac{1}{2}x\cos x = \tfrac{1}{2}(\sin\tfrac{3}{2}x - \sin\tfrac{1}{2}x),$$
$$\sin\tfrac{1}{2}x\sin x = \tfrac{1}{2}(\cos\tfrac{1}{2}x - \cos\tfrac{3}{2}x)$$

and by (3.38) and (3.34), this coefficient is equal to

$$-\frac{1}{2}\frac{r}{1-r^2}\eta^{(3)}_{\alpha_j-2}.$$

Hence

$$\eta^{(3)}_{\alpha_j-2} = -\frac{2^{\alpha_j-1}}{(\alpha_j-2)!}[\omega_j'(0)]^{-1}. \tag{3.42}$$

Now we estimate $G_r''(t)/G_r'(t)$ in the neighbourhood of the point $x_j = 0$. Let $\alpha_j - 1$ be an even number. Assume that $\alpha_j > 1$, the case $\alpha_j = 1$ being already considered. It is trivial that $P_r^{(\alpha_j-2)}(x)$ is an odd function, and by (3.19), $C_0^{(\alpha_j-1)}(r)$, $C_0^{(\alpha_j-2)}(r)$ have the same sign. Hence (3.33), (3.34), (3.38), (3.42), (3.28) and Lemma 4 imply that there are positive constants $0 < c' < C'$, $a > 0$ and nonnegative numbers δ_r', δ_r'' ($|\delta_r'| < C'$, $|\delta_r''| < C'$) and $1 > r_0 > 0$ such that

$$\delta_r'(1-r)^2 + (c'+1)\operatorname{tg} t \leqslant G_r''(t)/G_r'(t) \leqslant (C'+1)\operatorname{tg} t + \delta_r''(1-r)^2 \tag{3.43}$$

for $0 \leqslant t \leqslant a(1-r)$, $1 > r > r_0$, and

$$\delta_r'(1-r)^2 + (C'+1)\operatorname{tg} t \leqslant G_r''(t)/G_r'(t) \leqslant (c'+1)\operatorname{tg} t + \delta_r''(1-r)^2 \tag{3.44}$$

for $-a(1-r) \leqslant t \leqslant 0$, $1 > r > r_0$. Using estimates (3.43), (3.44), the formula

$$\operatorname{arc\,tg}(\alpha+\beta) = \operatorname{arc\,tg}\alpha + \operatorname{arc\,tg}\frac{\beta}{(\alpha+\beta)\alpha+1}, \tag{3.45}$$

and (3.35) we easily obtain the assertions (3.6) and (3.7) of Lemma 2. Observe that in general the constants in (3.44)–(3.45) and in (3.6)–(3.7) are not the same. Decreasing a if necessary, by (3.33), (3.34), (3.36) and (3.16) we get the assertion (3.5) of Lemma 2.

If $\alpha_j - 1$ is odd, then by (3.33), (3.38), (3.16) and (3.17) it is trivial that the function $G_r'(t)$, for r sufficiently close to 1, is monotone on the interval $(-a(1-r),\, a(1-r))$. Since $\alpha_j - 1$ is odd, from Lemma 4 and the same conditions as above we derive that $G_r'(t)$ is zero at a point b_r which belongs to $(-a(1-r),\, a(1-r))$, and $|b_r| \leqslant a_1(1-r)^2$ for some positive constant a_1. By Lemma 4 and (3.34), (3.38), we have

$$G_r''(t)/[2\eta^{(3)}_{\alpha_j-2}C_0^{(\alpha_j-2)}(r)[\xi_r(t)]^{-\alpha_j/2}] \to 1 \quad \text{as } r \to 1, \quad \frac{t}{1-r} \to 0. \tag{3.46}$$

By (3.18) and (3.20) the coefficients $C_0^{(\alpha_j-2)}(r)$, $C_0^{(\alpha_j-1)}(r)$ have different signs. Hence (3.28), (3.42), (3.33), (3.34), (3.38), (3.46) and (3.35) imply inequalities (3.10) and (3.11). Evidently, one can choose a and r_0 so that all the conditions of Lemma 2 are simultaneously true. By (3.35), $-\operatorname{tg}\frac{1}{2}x_r^*(t) = G_r''(t)/G_r'(t)$. Using (3.36) we now obtain (3.8), where

$$G_{rj}(t) = |G_r''(t)|\operatorname{sign} G_r'(t). \tag{3.47}$$

Now (3.9) follows immediately from (3.46). The proof of Lemma 2 is complete.

Proof of Lemma 3. We will estimate $x_r^*(t)$, defined by (3.35), when t belongs to a neighbourhood of $x_j = 0$ and $r \to 1-$.

Let α_j be an odd number. By (3.19) we have

$$\begin{aligned}\sum_{i=0}^{(\alpha_j-1)/2} C_i^{(\alpha_j-1)}(1) &= \sum_{i=0}^{(\alpha_j-1)/2}(2i+1)C_i^{(\alpha_j-2)}(1) \\ &\quad - \sum_{i=0}^{(\alpha_j-1)/2} 2\left(\frac{\alpha_j-1}{2}+i+1\right)C_{i-1}^{(\alpha_j-2)}(1) \\ &= -\alpha_j\sum_{i=0}^{(\alpha_j-1)/2-1} C_i^{(\alpha_j-2)}(1).\end{aligned} \tag{3.48}$$

Relations (3.42) and (3.28) imply

$$\eta^{(3)}_{\alpha_j-2} = -(\alpha_j-1)b_j. \tag{3.49}$$

Writing $\xi_r(t) = (1-r)^2 + 4r\sin^2\frac{1}{2}t$ and setting $r = 1$ in (3.33) and (3.34), we deduce by (3.38), (3.16) and (3.17) that there is a $\delta > 0$ such that

$$\begin{aligned}\sigma_j(1-1/(2\alpha_j))4b_j &\sum_{i=0}^{(\alpha_j-1)/2} C_i^{(\alpha_j-1)}(1)\cos t/|4\sin^2\tfrac{1}{2}t|^{(\alpha_j-1)/2+1} \\ &\leqslant -\sigma_j G_1'(t) \leqslant \sigma_j(1+1/(2\alpha_j))4b_j \\ &\quad\times\sum_{i=0}^{(\alpha_j-1)/2} C_i^{(\alpha_j-2)}(1)\cos t/|4\sin^2\tfrac{1}{2}t|^{(\alpha_j-1)/2+1},\end{aligned} \tag{3.50}$$

$$\begin{aligned}\frac{2\alpha_j+1}{2\alpha_j+2}&\Big[4b_j\sum_{i=0}^{(\alpha_j-1)/2} C_i^{(\alpha_j-1)}(1) \\ &\quad - 2\eta^{(3)}_{\alpha_j-2}\sum_{i=0}^{(\alpha_j-1)/2-1} C_i^{(\alpha_j-2)}(1)\Big]\sigma_j/|4\sin^2\tfrac{1}{2}t|^{(\alpha_j-1)/2+1} \\ &\leqslant \frac{G_1''(t)}{\sin t}\sigma_j \leqslant \frac{2\alpha_j+3}{2\alpha_j+2}\Big[4b_j\sum_{i=0}^{(\alpha_j-1)/2} C_i^{(\alpha_j-1)}(1) \\ &\quad - 2\eta^{(3)}_{\alpha_j-2}\sum_{i=0}^{(\alpha_j-1)/2-1} C_i^{(\alpha_j-2)}(1)\Big]\sigma_j/|4\sin^2\tfrac{1}{2}t|^{(\alpha_j-1)/2+1}\end{aligned} \tag{3.51}$$

for $|t| < \delta$, where $\sigma_j = \operatorname{sign}\left[b_j \sum_{i=0}^{(\alpha_j-1)/2} C_i^{(\alpha_j-1)}(1)\right]$. By (3.48)–(3.50), there is a $\delta_1 > 0$ such that

$$(3.52)\qquad -\frac{4\alpha_j+6}{8\alpha_j-1}\operatorname{tg} t \leqslant \frac{G_1''(t)}{G_1'(t)} \leqslant -\frac{4\alpha_j+2}{8\alpha_j+1}\operatorname{tg} t \quad \text{for } 0 \leqslant t \leqslant \delta_1,$$

$$(3.53)\qquad -\frac{4\alpha_j+2}{8\alpha_j+1}\operatorname{tg} t \leqslant \frac{G_1''(t)}{G_1'(t)} \leqslant -\frac{4\alpha_j+6}{8\alpha_j-1}\operatorname{tg} t \quad \text{for } -\delta_1 < t < 0.$$

Obviously, if $0 < \delta' < a < 1$, $1-r < a|t|$ and $|t| < \delta'$ then

$$\frac{1-a}{1+a^2} < \frac{\xi_1(t)}{\xi_r(t)} < \frac{1}{1-a^2}.$$

Hence, for $a > 0$ small, by (3.52), (3.53), the definitions of $G_r'(t)$, $G_r''(t)$ and equalities (3.35), (3.45), we easily infer (3.12) and (3.13), where $1-r_0 \leqslant \delta$. Writing

$$G_{rj}(t) = \operatorname{sign} G_r'(t)\,\{[G_r'(t)]^2 + [G_r''(t)]^2\}^{1/2},$$

from equalities (3.1), (3.34) and relations (3.16), (3.17) and (3.33)–(3.35) we derive (3.14).

Let α_j be an even number. Then by (3.18)

$$(3.54)\qquad \sum_{i=0}^{(\alpha_j-2)/2} C_i^{(\alpha_j-1)}(1) = -\sum_{i=0}^{(\alpha_j-2)/2} 2\left(\frac{\alpha_j-2}{2}+i+1\right) C_i^{(\alpha_j-2)}(1) + \sum_{i=0}^{(\alpha_j-2)/2} 2i C_i^{(\alpha_j-2)}(1) = -\alpha_j \sum_{i=0}^{(\alpha_j-2)/2} C_i^{(\alpha_j-2)}(1).$$

Hence, as above, we obtain (3.12) and (3.13). Relations (3.1) and (3.36) imply equality (3.8), where $G_{rj}(t)$ is defined by (3.47). In order to estimate $G_r''(t)$ from below, we use (3.34), (3.38), Lemma 4 and equalities (3.54) and (3.51). The proof of Lemma 3 is complete.

4. Proof of the implications (a) $\Rightarrow$ (b) **and** (a′) $\Rightarrow$ (b′). Both proofs will be done simultaneously. If we stipulate that $p > 1$ resp. $p = 1$, it means that we are in the course of proving (a) $\Rightarrow$ (b) resp. (a′) $\Rightarrow$ (b′).

If (a) resp. (a′) is true, then (2.4), (2.2) and (1.4) easily imply that there are positive numbers $1 > R_1 > 0$ and $\delta > 0$ such that for every $f \in L^p_{[-\pi,\pi]}(\psi)$ ($p > 1$ resp. $p = 1$)

$$(4.1)\qquad \left\|\int_{-\pi}^{\pi} f(x) B_{ri}(x,\cdot)\,dx\right\|_{L^p_{\Delta_i}(\psi)} \leqslant \tfrac{3}{2} C_p \|f\|_{L^p_{[-\pi,\pi]}(\psi)} \quad (R_1 < r < 1,\ 1 \leqslant i \leqslant s),$$

where $\Delta_i = (x_i - \delta_i, x_i + \delta_i)$.

Let $1 \leqslant j \leqslant s$. We will show that there is a positive constant $\varepsilon_j > 0$ such that (b) resp. (b′) is true for every interval contained in $\Delta_j' = (x_j - \varepsilon_j, x_j + \varepsilon_j)$. Without loss of generality assume that $x_j = 0$.

First we consider the case of α_j odd. By (3.4) and (4.1), for every $f \in L^p_{[-\pi,\pi]}(\psi)$

$$(4.2)\qquad \left\|G_{rj}(\cdot)\int_{-\pi}^{\pi} P_r(x-\cdot) f(x)\omega(x) \sin\frac{x - x_r^*(\cdot)}{2}\,dx\right\|_{L^p_{\Delta_j}(\psi)} \leqslant \tfrac{3}{2} C_p \|f\|_{L^p_{[-\pi,\pi]}(\psi)}$$

for $R_1 < r < 1$ and $\Lambda = 2N+1$; $G_{rj}(t)$ satisfies (3.5), and (3.6)–(3.7) are true. Write

$$(4.4)\qquad r_0' = \max\left[r_0,\ 1 - \frac{a\min(c',1)}{2(C'+1)},\ R_1\right],$$

$$(4.5)\qquad \varepsilon_j = \min\left[\frac{a}{4(C'+1)}(1-r_0'),\ \frac{\pi}{2(C'+1)},\ \tfrac{1}{2}\min_{\substack{i \neq j\\ 1 \leqslant i \leqslant s}} |x_i - x_j|,\ \delta,\ \delta'\right],$$

where the constants satisfy (3.5)–(3.7); the value of δ' will be specified later. Fix an interval Q and a number r_Q such that

$$(4.6)\qquad Q = (0, d) \subset (0, \eta), \quad \eta = 4(C'+1)\varepsilon_j, \quad a(1-r_Q) = d.$$

By (4.4)–(4.6), (3.6) and (3.7) we infer that

$$(4.7)\qquad -\pi \leqslant x - x_{r_Q}^*(t) \leqslant \pi \quad \text{if } t \in Q,\ x \in (0, \varepsilon_j).$$

For clarity we first consider the case $\alpha_j = 1$. By (3.7) we then have $x_r^*(t) = 2t$. Let Q_1, Q_2 be the left and right halves of Q, and $Q_1^{(1)}$, $Q_1^{(2)}$ the left and right halves of Q_1. If a function f is zero outside Q_2 and $f(x)\omega(x) \geqslant 0$, then by (3.4), (3.5) and (4.4)–(4.7),

$$(4.8)\qquad \left|\int_{-\pi}^{\pi} B_{rj}(x,t) f(x)\,dx\right| \geqslant C\left(\frac{1}{|Q|^{\alpha_j+2}}\int_{Q_2} f(x)\omega(x)\sin\frac{x-t_Q}{2}\,dx\right)\chi_{Q_1^{(1)}}(t),$$

where t_Q is the centre of Q and $\chi_{Q_1^{(1)}}(t)$ is the characteristic function of $Q_1^{(1)}$. Hence, by (4.2) and (4.3),

$$(4.9)\qquad \int_{Q_1^{(1)}} \psi(t)\,dt \left(\frac{1}{|Q|^{\alpha_j+2}}\int_{Q_2} f(x)\omega(x)\sin\frac{x-t_Q}{2}\,dx\right)^p \leqslant B_p \int_{Q_2} |f|^p \psi\,dt.$$

Setting $f(x) = \operatorname{sign}\omega(x)$ in (4.9) we obtain

$$(4.10)\qquad \int_{Q_1^{(1)}} \psi\,dt \leqslant C_p \int_{Q_2} \psi\,dt.$$

As above, for every function f such that $f(x)\omega(x) \geqslant 0$ and $f(x)\omega(x) = 0$ outside $Q_1^{(1)}$,

$$(4.11)\qquad \int_{Q\setminus Q_1^{(1)}} \psi\,dt \left(\frac{1}{|Q|^{\alpha_j+1}}\int_{Q_1^{(1)}} f\omega\,dt\right) \leqslant C_p \int_{Q_1^{(1)}} |f|^p \psi\,dt.$$

If $p>1$, then by (4.10) and (4.11), setting $f=(|\omega|/\psi)^{1/(p-1)}\operatorname{sign}\omega$, we get

$$(4.12)\qquad \int_{Q_1^{(1)}}\psi\,dt\left[\int_{Q_1^{(1)}}\left(\frac{|\omega|^p}{\psi}\right)^{\frac{1}{p-1}}dt\right]^{p-1}\leqslant C_p\Big(\int_{Q_1^{(1)}}|\omega|\,dt\Big)^p.$$

Thus, for $\alpha_j=1$, we have proved that ψ satisfies condition $(A_p^{|\omega|})$ for every interval $Q_1^{(1)}=(0,\,d/2)\subset(0,\,\frac{1}{2}\eta)$.

The proof of (4.12) for α_j odd is analogous. Let, as above, Q_1, Q_2 be the halves of Q, and $Q_1^{(1)}=(0,\,d')\subset(0,\,\varepsilon_j)$ an interval such that

$$(4.13)\qquad |Q_1^{(1)}|/|Q_1|=1/[2(C'+1)].$$

Defining r_Q from the equation $a(1-r_Q)=d'$, by (4.4) and (4.5) we will have

$$(4.14)\qquad (1-r_Q)^2(C'+1)<\tfrac{1}{2}a\min(c',\,1)(1-r_Q).$$

Thus if we fix a function f such that $f(x)\omega(x)\geqslant 0$ and $f(x)\omega(x)=0$ outside Q_2, then (4.7), (4.2), (4.3) and (3.5)–(3.7) yield (4.8). Hence, as above, inequality (4.10) follows. Observe that by (4.14), (3.6) and (4.13), $x-x_r^*(t)\leqslant -|Q|/4$ for $x\in Q_1^{(1)}$, $t\in Q_2$. Thus for every f such that $f(x)\omega(x)=0$ outside $Q_1^{(1)}$ and $f(x)\omega(x)\geqslant 0$, by (4.2), (4.3), (3.5), (4.6) and (4.7) we obtain

$$(4.15)\qquad \int_{Q_2}\psi\,dt\left(\frac{1}{|Q_1^{(1)}|^{\alpha_j+1}}\int_{Q_1^{(1)}}f\omega\,dt\right)^p\leqslant C_p\int_{Q_1^{(1)}}|f|^p\psi\,dt.$$

Hence, as above, we get (4.12) for $p>1$. For $p=1$, by (4.10), taking into account that inequality (4.15) is true for every $f\in L_{Q_1^{(1)}}(\psi)$, we get

$$(4.16)\qquad \Big(\int_{Q_1^{(1)}}\psi\,dt\Big)\|\omega/\psi\|_{L^\infty_{Q_1^{(1)}}}\leqslant C\int_{Q_1^{(1)}}|\omega|\,dt.$$

Obviously, a similar proof can be done for every interval $(-d',\,0)\subset(-\varepsilon_j,\,0)$. If inequality (4.16) is true for every interval $(0,\,d')\subset(0,\,\varepsilon_j)$ and $(-d',\,0)\subset(-\varepsilon_j,\,0)$, then we easily prove it for every interval $(-d',\,d')\subset(-\varepsilon_j,\,\varepsilon_j)$. Thus when α_j is an odd number, for every interval $(-d',\,d')\subset(-\varepsilon_j,\,\varepsilon_j)$ conditions $(A_p^{|\omega|})$ and $(A_1^{|\omega|})$ are satisfied.

Now consider the case of α_j even. Fix any interval $Q=(-d,\,d)\subset(-\varepsilon_j,\,\varepsilon_j)$, and define the number r_Q by $a(1-r_Q)=d$. Observe that the expression $\sin\frac{1}{2}(x-x_r^*(t))/\sin\frac{1}{2}x_r^*(t)$ does not change sign for $x\in(0,\,\varepsilon_j)$, $t\in(0,\,\varepsilon_j)$, and its absolute value is greater than some positive constant. Hence for every f such that $f(x)\omega(x)=0$ outside Q and $f(x)\omega(x)\geqslant 0$, by (3.8)–(3.11) we get

$$\Big|\int_{-\pi}^{\pi}B_{rj}(x,\,t)f(x)\,dx\Big|\geqslant C\left(\frac{1}{|Q|^{\alpha_j}}\int_Q f(x)P_r(x-t)\,\omega(x)\,dx\right)\chi_Q(t).$$

Hence (4.1) implies

$$(4.17)\qquad \int_Q\psi\,dt\left(\frac{1}{|Q|^{\alpha_j+1}}\int_Q f\omega\,dt\right)^p\leqslant B_p\int_Q|f|^p\psi\,dt.$$

For $p>1$ setting $f=(\omega/\psi)^{1/(p-1)}\operatorname{sign}\omega$ in (4.17), and for $p=1$ taking into account that

$$\sup_{f\in L_Q(\psi)}\Big(\int_Q f\omega\,dt\Big/\int_Q|f|\,\psi\,dt\Big)=\|\omega/\psi\|_{L^\infty_Q},$$

we conclude that for every interval $Q=(-d,\,d)\subset(-\varepsilon_j,\,\varepsilon_j)$ conditions (b) and (b′) respectively are true.

Obviously, if conditions $(A_p^{|\omega|})$ resp. $(A_1^{|\omega|})$ are true for every interval $(-d,\,d)\subset(-\varepsilon_j,\,\varepsilon_j)$, then, by changing the constants, they are true for all intervals $(d_1,\,d_2)\subset(-\varepsilon_j,\,\varepsilon_j)$ with $|(d_1,\,d_2)|/\max(|d_1|,\,|d_2|)\geqslant\tau>0$, where

$$(4.18)\qquad \tau=\min\{1/[2(8\alpha_j+2)],\quad a/(1+a)\}.$$

Consider any interval Q such that

$$(4.19)\qquad Q\equiv(d_1,\,d_2)\subset(-\varepsilon_j,\,\varepsilon_j),\qquad \frac{|(d_1,\,d_2)|}{\max(|d_1|,\,|d_2|)}<\tau.$$

By (4.18), $\tau\leqslant 1/[2(8\alpha_j+2)]$, hence every interval which satisfies (4.19) lies either entirely to the right of $x_j=0$ or entirely to the left. Without loss of generality, we can assume that Q lies to the right of $x_j=0$, i.e. $0<d_1<d_2$. Define the number r_Q by

$$(4.20)\qquad 1-r_Q=d_2-d_1.$$

Then by (4.18)–(4.20) we get

$$1-r_Q=\frac{d_2-d_1}{d_2}d_2\leqslant\tau\frac{d_2}{d_1}d_1<\tau\frac{1}{1-r}d_1\leqslant ad_1.$$

Thus we can apply Lemma 3. By (4.18) and (4.19) we have $\frac{9}{8}d_1>d_2+d_2/16$. So there is a positive constant $\delta'>0$ such that for $d_1<d_2<\delta'$ we have, by (3.12),

$$(4.21)\qquad x_r^*(t)-x>\frac{d_2}{6\alpha_j}\quad\text{for } t,x\in(d_1,\,d_2).$$

In virtue of Lemma 3 and using (3.4), (3.8), (3.5), (3.9), (4.19) and (4.5) we have

$$-\frac{B_{r_Qj}(x,t)\operatorname{sign}G_{rj}(t)}{P_{r_Q}(x-t)\,\omega(x)}\geqslant C|\sin\tfrac{1}{2}t|^{-\alpha_j}\quad\text{for } x,\,t\in(d_1,\,d_2).$$

Hence for any function f such that $f(x)\omega(x)\geqslant 0$ and $f(x)\omega(x)=0$ outside

(d_1, d_2), we easily obtain

$$(4.22)\qquad \Big|\int_{-\pi}^{\pi} B_{r_Q j}(x, t) f(x)\,dx\Big| \geqslant C d_2^{-\alpha_j} \frac{1}{|Q|} \int_Q f\omega\,dx.$$

Relations (4.1), (4.4), (4.5), (4.19) and (4.22) then force

$$\Big(\int_Q \psi\,dx\Big)\Big[\Big(\int_Q |\omega|\,dx\Big)^{-1} \int_Q f\omega\,dx\Big]^p \leqslant C_p \int_Q |f|^p \psi\,dx.$$

As above, it follows that $(A_p^{|\omega|})$ and $(A_1^{|\omega|})$ respectively are true for every interval which satisfies (4.19). Thus we have proved that $(A_p^{|\omega|})$ and $(A_1^{|\omega|})$ are true for every interval which lies in $(x_j-\varepsilon_j, x_j+\varepsilon_j)$.

Since x_j was chosen arbitrarily among x_i $(1 \leqslant i \leqslant s)$, we conclude that $(A_p^{|\omega|})$ and $(A_1^{|\omega|})$ respectively are satisfied for every interval which lies in one of the sets $(x_i-\varepsilon_i, x_i+\varepsilon_i)$ $(1 \leqslant i \leqslant s)$.

We will now show that the same is true for every interval Q, $|Q| < \sigma$, where σ is a positive constant which will be defined later. First, we require that

$$(4.23)\qquad \sigma < \tfrac{1}{2} \min_{1 \leqslant i \leqslant s} \varepsilon_i = \varepsilon.$$

Obviously, it is sufficient to consider only intervals Q for which

$$(4.24)\qquad |Q| < \sigma, \qquad \min_{1 \leqslant i \leqslant s} \inf_{x \in Q} |x-x_i| > \varepsilon.$$

Fix any interval Q for which (4.24) is true. Then we easily infer that there is a positive number R_2, $0 < R_2 < 1$, independent of Q such that for every function f which is zero outside Q

$$(4.25)\qquad \Big\|\int_Q f(x) \sum_{j=1}^{s} \sum_{\lambda=0}^{\alpha_j-1} P_r^{(\lambda)}(x_j-\cdot)\, T(x_j, \lambda, x)\,dx\Big\|_{L_Q^p(\psi)} \leqslant \tfrac{1}{2} B_p \|f\|_{L_Q^p(\psi)},$$

for every $1 > r > R_2$. Now we can state the precise value of σ:

$$(4.26)\qquad \sigma = \min\{\varepsilon, \tfrac{1}{2}(1-R_2)\}.$$

Relations (1.3), (1.4) and (4.25) immediately imply

$$(4.27)\qquad \Big\|\int_Q P_r(x-\cdot) f(x)\,dx\Big\|_{L_Q^p(\psi)} \leqslant \tfrac{3}{2} B_p \|f\|_{L_Q^p(\psi)} \quad \text{for } 1 > r > R_2 > 0.$$

Trivially, by (1.2), (4.26) and (4.24) there is a positive constant C independent of Q such that $|Q| \geqslant C \int_Q |\omega|\,dx$. Hence if we take $r_Q = 1-|Q|$ then as above we infer that condition $(A_p^{|\omega|})$ is satisfied for Q. Thus conditions (β_1) and (β_1') respectively are satisfied for every interval Q, $|Q| < \sigma$. By increasing the corresponding constants, it follows that these conditions are true for every interval.

To complete the proof of the implications (a) $\Rightarrow$ (b) and (a′) $\Rightarrow$ (b′) we now have to prove that conditions (β_2) and (β_2') respectively are true. Consider the case of α_j odd. Let ε_j be defined by (4.5). Write $Q_h = (-h, h)$ and $Q_{h'} = (-h', h')$, where

$$(4.28)\qquad 0 < h < \varepsilon_j, \qquad h' = \frac{h}{4(C'+1)}.$$

Relations (4.5) and (4.1) imply

$$(4.29)\qquad \Big\|\int_{-\pi}^{\pi} f(x) B_{rj}(x, \cdot)\,dx\Big\|_{L_{Q_{h'}}^p(\psi)} \leqslant \tfrac{3}{2} B_p \|f\|_{L_{[-\pi,\pi]}^p(\psi)} \qquad (R_1 < r < 1).$$

Taking $r_h = 1-h$, for every function f which is zero on Q_h and $f(x)\omega(x)(\sin\frac{1}{2}x)^{-1} \geqslant 0$ by (3.4)–(3.7), (4.28) and (4.29) we get

$$h^{-\alpha_j p} \int_{Q_{h'}} \psi\,dt \Big(\int_{Q_h^c} f(x)\omega(x)(\sin\tfrac{1}{2}x)^{-1}\,dx\Big)^p \leqslant C_p \int_{Q_h^c} |f|^p \psi\,dt,$$

where $Q_h^c = [-\pi, \pi] - Q_h$. Hence, setting

$$f(x) = \left|\frac{\omega(x)}{\sin\frac{1}{2}x\,\psi(x)}\right|^{\frac{1}{p-1}} \operatorname{sign}[\omega(x)(\sin\tfrac{1}{2}x)^{-1}],$$

we immediately obtain

$$(4.30)\qquad h^{-\alpha_j p} \int_{Q_{h'}} \psi\,dt \left[\int_{Q_h^c} \left[\frac{|\omega(x)(\sin\frac{1}{2}x)^{-1}|^p}{\psi(x)}\right]^{\frac{1}{p-1}} dx\right]^{p-1} \leqslant C_p.$$

Since $Q_{h'} \subset Q_h$, by (β_1) and (β_1') respectively we have

$$(4.31)\qquad \int_{Q_{h'}} |\omega|\,dx \leqslant \Big(\int_{Q_{h'}} \psi\,dx\Big)^{1/p} \left(\int_{Q_h} \left(\frac{|\omega|^p}{\psi}\right)^{\frac{1}{p-1}} dx\right)^{\frac{p-1}{p}}$$
$$\leqslant C\Big(\int_{Q_{h'}} \psi\,dx \Big/ \int_{Q_h} \psi\,dx\Big)^{1/p} \int_{Q_h} |\omega|\,dx.$$

Relations (1.2), (4.28) and (4.30) now imply that (β_2) and (β_2') are true when α_j is an odd number.

Let α_j be even and $A = 2N+1$, $0 < h < \varepsilon_j$, $r_h = 1-h$. Write $Q = (b_{r_h}, ah)$. For every function f which is zero outside $(h, \pi/2)$ and $f(x)\omega(x)(\sin\frac{1}{2}x)^{-1} \geqslant 0$, by (4.29) and (3.8)–(3.11) we get

$$h^{-\alpha_j p} \int_Q \psi\,dt \Big(\int_h^{\pi/2} f(x)\omega(x)(\sin\tfrac{1}{2}x)^{-1}\,dx\Big)^p \leqslant C_p \int_h^{\pi/2} |f|^p \psi\,dx.$$

Hence, setting

$$f(x) = \left|\frac{\omega(x)(\sin\frac{1}{2}x)^{-1}}{\psi(x)}\right|^{\frac{1}{p-1}} \operatorname{sign}[\omega(x)(\sin\tfrac{1}{2}x)^{-1}]$$

we will have

$$(4.32)\qquad h^{-\alpha_j p}\int_Q \psi\,dt\left[\int_h^{\pi/2}\left[\frac{|\omega(x)(\sin\frac{1}{2}x)^{-1}|^p}{\psi(x)}\right]^{\frac{1}{p-1}}dx\right]^{p-1}\leqslant C_p.$$

Taking into account that $b_{r_h}\leqslant a_1 h^2$, from (4.31) we derive

$$(4.33)\qquad h^{-\alpha_j p}\int_{-h}^{h}\psi\,dt\left[\int_h^{\pi}\left[\frac{|\omega(x)(\sin\frac{1}{2}x)^{-1}|^p}{\psi(x)}\right]^{\frac{1}{p-1}}dx\right]^{p-1}\leqslant C_p.$$

Similarly we can prove the inequality

$$(4.34)\qquad h^{-\alpha_j p}\int_{-h}^{h}\psi\,dt\left[\int_{-\pi}^{-h}\left[\frac{|\omega(x)(\sin\frac{1}{2}x)^{-1}|^p}{\psi(x)}\right]^{\frac{1}{p-1}}dx\right]^{p-1}\leqslant C_p.$$

Now (4.33) and (4.34) imply conditions (β_2), (β_2') for $p>1$ and $p=1$ respectively.

5. Proof of the implication (b) $\Rightarrow$ (a). Fix any function $f\in L^p_{[-\pi,\pi]}(\psi)$ and periodically continue it on the whole real line. Write

$$(5.1)\qquad U_r(f,t)=\int_{-\pi}^{\pi} f(x)P_{\mathfrak{X},\mathfrak{N},r}(x,t)\,dx,$$

where $P_{\mathfrak{X},\mathfrak{N},r}(x,t)$ is defined by (1.3). Relations (2.4) and (2.2) imply

$$(5.2)\qquad U_r(f,t)=\sum_{j=1}^{s}\int_{-\pi}^{\pi} f(x)B_{rj}(x,t)\,dx=\sum_{j=1}^{s}U_{rj}(f,t).$$

We wish to estimate the norms of $U_{rj}(f,t)$ in $L^p_{[-\pi,\pi]}(\psi)$. Fix any natural number $j\,(1\leqslant j\leqslant s)$. Without loss of generality, we can assume that $x_j=0$ and

$$(5.3)\qquad f(x)\,\omega(x)\geqslant 0.$$

With this assumption, we prove several lemmas.

LEMMA 5. *Let $M^{|\omega|}(f,t)$ be the maximal function defined by* (2.1). *Then for every $0<r<1$*

$$\left|\int_{-2(1-r)}^{2(1-r)} f(x+t)B_{rj}(x+t,t)\,dx\right|\leqslant CM^{|\omega|}(f,t),\qquad t\in[-\pi,\pi],$$

where C is an absolute constant depending only on j, and $\omega(x)$ is defined by (1.2).

Proof is analogous to the classical case (see [25], p. 249): we write $F_t(x)=\int_0^x f(\tau+t)\,\omega(\tau+t)\,d\tau$ and integrate by parts. The desired estimates are obtained by use of Lemma 1. We omit the technical details.

LEMMA 6. *For every r $(0<r<1)$ and $t\in[-\pi,\pi]$ such that $2(1-r)<|t|$*

$$(5.4)\qquad \left|\int_{2(1-r)<|x|<|t|} f(x+t)B_{rj}(x+t,t)\,dx\right|\leqslant CM^{|\omega|}(f,t).$$

Proof is analogous to that of Lemma 5.

LEMMA 7. *Condition (β_2) resp. (β_2') implies that for every $f\in L^p_{[-\pi,\pi]}(\psi)$ $(p>1$ resp. $p=1)$*

$$(5.5)\qquad \int_{|t|<1-r}\psi(t)\left|\int_{2(1-r)<|x|<\pi} f(x+t)B_{rj}(x+t,t)\,dx\right|^p dt\leqslant C_p\int_{-\pi}^{\pi}|f|^p\psi\,dt.$$

Proof. Consider the case $t\geqslant 0$, $-2(1-r)>x>-\pi$. In virtue of Lemma 1, by changing the variable, the expression

$$\left|\int_{-2(1-r)>x\geqslant-\pi} f(x+t)B_{rj}(x+t,t)\,dx\right|=\left|\int_{-\pi\leqslant y-t<-2(1-r)} f(y)B_{rj}(y,t)\,dy\right|$$

is less than or equal to

$$\int_{-\pi\leqslant y\leqslant t-2(1-r)} f(y)\,\omega(y)\frac{1-r^2}{(1-r)^2+4r\sin^2\frac{1}{2}(y-t)}\times|\sin\tfrac{1}{2}y\,G_r'(t)+\cos\tfrac{1}{2}y\,G_r''(t)|\,dy.$$

Since $0<t<1-r$, by (3.2) and (3.3) we thus get

$$(5.6)\qquad \left|\int_{-\pi\leqslant x\leqslant-2(1-r)} f(x+t)B_{rj}(x+t,t)\,dx\right|\leqslant\frac{C}{(1-r)^{\alpha_j}}\int_{-\pi}^{t-2(1-r)} f(y)\,\omega(y)|\sin\tfrac{1}{2}y|^{-1}\,dy.$$

Now we will show that

$$(5.7)\qquad \left[\int_0^{1-r}\frac{\psi(t)}{(1-r)^{p\alpha_j}}\Big(\int_{-\pi}^{t-2(1-r)} f(y)\,\omega(y)|\sin\tfrac{1}{2}y|^{-1}\,dy\Big)^p dt\right]^{1/p}\leqslant C_p\Big[\int_{-\pi}^{-(1-r)}|f|^p\psi\,dt\Big]^{1/p}.$$

Putting $\sigma=-y$ and $\tau=2(1-r)-t$, and changing the variables, we can rewrite this inequality in the following form:

$$(5.8)\qquad \left[\int_{1-r}^{2(1-r)}\frac{\psi[2(1-r)-\tau]}{(1-r)^{p\alpha_j}}\Big(\int_{\tau}^{\pi} f(-\sigma)\,\omega(-\sigma)|\sin\tfrac{1}{2}\sigma|^{-1}\,d\sigma\Big)^p d\tau\right]^{1/p}\leqslant C\Big[\int_{1-r}^{\pi}|f(-\tau)|^p\psi(-\tau)\,d\tau\Big]^{1/p}.$$

Let

$$U(\tau)=\begin{cases}(\psi[2(1-r)-\tau])^{1/p}(1-r)^{-\alpha_j} & \text{for } 1-r\leqslant\tau\leqslant 2(1-r),\\ 0 & \text{for } \tau\in \boldsymbol{R}^1\setminus[1-r,\,2(1-r)],\end{cases}$$

$$V(\tau)=\begin{cases}\sin\tfrac12\tau\,[\psi(-\tau)]^{1/p}[\omega(-\tau)]^{-1} & \text{for } \tau\in[1-r,\,\pi],\\ \infty & \text{for } \tau\in \boldsymbol{R}^1\setminus[1-r,\,\pi].\end{cases}$$

Hence by Theorem E, inequality (5.8) is true if condition (H'_p) is satisfied. Substituting $U(\tau)$ and $V(\tau)$ in (H'_p), we infer that (5.8) is true if

$$D'=\sup_{1-r<a\leqslant 2(1-r)}\Big[\int_{1-r}^{a}\frac{\psi[2(1-r)-\tau]}{(1-r)^{p\alpha_j}}\,d\tau\Big]^{1/p}\times\Big[\int_a^{\pi}\frac{[\psi(-\tau)]^{-q/p}}{\sin^q\frac12\tau}|\omega(-\tau)|^q\,d\tau\Big]^{1/q}<\infty.$$

This condition is equivalent to the following:

$$(5.9)\qquad D''=\sup_{0<a\leqslant 1-r}\Big[\int_a^{1-r}\frac{\psi(x)}{(1-r)^{p\alpha_j}}\,dx\Big]^{1/p}\times\Big[\int_{-\pi}^{a-2(1-r)}\frac{|\omega(x)|^q}{|\sin\frac12 x|^q[\psi(x)]^{q/p}}\,dx\Big]^{1/q}<\infty.$$

But (5.9) immediately follows from (β_2). Hence inequality (5.7) is proved. By (5.6) we now get

$$\Big[\int_0^{1-r}\psi(t)\Big|\int_{-\pi}^{-2(1-r)}f(x+t)B_{rj}(x+t,\,t)\,dx\Big|^p dt\Big]^{1/p}\leqslant C_p\Big[\int_{-\pi}^{-(1-r)}|f|^p\psi\,dt\Big]^{1/p}.$$

The proof of analogous inequalities for the other cases is similar, and we omit it.

LEMMA 8. *If condition* (b) *resp.* (b′) *is true, then there is a constant* C *independent of* $f\in L^p_{[-\pi,\pi]}(\psi)$ $(p>1$ *resp.* $p=1)$ *such that*

$$(5.10)\qquad \Big[\int_{1-r\leqslant|t|\leqslant\pi}\psi(t)\Big|\int_{E_r(t)}f(x+t)B_{rj}(x+t,\,t)\,dx\Big|^p dt\Big]^{1/p}\leqslant C\Big[\int_{-\pi}^{\pi}|f|^p\psi\,dt\Big]^{1/p},$$

where $E_r(t)=\{x\colon |t|\leqslant x\leqslant\pi\}\cap\{x\colon |x|>2(1-r)\}$.

Proof. Consider the case $1-r\leqslant t\leqslant\pi$. Obviously

$$(5.11)\qquad \int_{E_r(t)}f(x+t)B_{rj}(x+t,\,t)\,dx=\int_{\tilde E_r(t)}f(y)B_{rj}(y,\,t)\,dy,$$

where $\tilde E_r(t)=\{y\colon |t|\leqslant|y-t|\leqslant\pi\}\cap\{y\colon |y-t|>2(1-r)\}$. If $y\leqslant 0$, then for $2(1-r)<t\leqslant\pi$, y varies from $t-\pi$ to 0, and for $1-r\leqslant t\leqslant 2(1-r)$ it varies from $t-\pi$ to $t-2(1-r)$. If $y>0$, then for $2(1-r)<t\leqslant\pi$, y varies from $2t$ to $t+\pi$, and for $1-r\leqslant t\leqslant 2(1-r)$ it varies from $t+2(1-r)$ to $t+\pi$. For fixed t the kernel $B_{rj}(y,\,t)$ is a 2π-periodic function of y. Thus the integral on the right-hand side of (5.11) does not change if we change the limits of integration in the following manner: for $\pi/2\leqslant t\leqslant\pi$, $2(1-r)<t<\pi/2$ and $1-r\leqslant t\leqslant 2(1-r)$ respectively we integrate from $2t-2\pi$ to 0; from $-\pi$ to 0 and from $2t$ to π; from $-\pi$ to $t-2(1-r)$ and from $t+2(1-r)$ to π respectively. Hence

$$(5.12)\qquad \Big|\int_{\tilde E_r(t)}f(y)B_{rj}(y,\,t)\,dy\Big|\leqslant\Big|\int_{-\pi}^{0}f(y)B_{rj}(y,\,t)\,dy\Big|+\Big|\int_{2t}^{\pi}f(y)B_{rj}(y,t)\,dy\Big|$$
$$\leqslant\Big|\int_{-\pi}^{-t}f(y)B_{rj}(y,\,t)\,dy\Big|+\Big|\int_{-t}^{0}f(y)B_{rj}(y,\,t)\,dy\Big|+\Big|\int_{2t}^{\pi}f(y)B_{rj}(y,\,t)\,dy\Big|.$$

From Lemma 1 we derive that

$$(5.13)\qquad \Big|\int_{-\pi}^{-t}f(y)B_{rj}(y,\,t)\,dy\Big|\leqslant C(1-r)|G'_r(t)|\int_{-\pi}^{-t}|f(y)\omega(y)|\,|\sin\tfrac12 y|^{-1}\,dy$$
$$+C(1-r)|G''_r(t)|\int_{-\pi}^{-t}|f(y)\omega(y)|(\sin\tfrac12 y)^{-2}\,dy$$
$$\leqslant C(1-r)|\sin\tfrac12 t|^{-(\alpha_j+1)}\int_{-\pi}^{-t}|f(y)\omega(y)|\,|\sin\tfrac12 y|^{-1}\,dy.$$

By a change of variable we have

$$(5.14)\qquad \int_{-\pi}^{-t}|f(y)\omega(y)|\,|\sin\tfrac12 y|^{-1}\,dy=\int_t^{\pi}|f(-\sigma)\omega(-\sigma)|\,|\sin\tfrac12\sigma|^{-1}\,d\sigma.$$

As in Lemma 7, one can show that

$$(5.15)\qquad \Big[\int_{1-r}^{\pi}\Big[(\psi(t))^{1/p}(1-r)|\sin\tfrac12 t|^{-(\alpha_j+1)}\int_t^{\pi}|f(-\sigma)\omega(-\sigma)|\times(\sin\tfrac12\sigma)^{-1}|\,d\sigma\Big]^p dt\Big]^{1/p}\leqslant C\Big[\int_0^{\pi}|f(-\sigma)|^p\psi(-\sigma)\,d\sigma\Big]^{1/p}$$

provided that

$$(5.16)\qquad D=\sup_{1-r<a<\pi}\Big[\int_{1-r}^{a}\frac{(1-r)^p}{|\sin\frac12 x|^{p(\alpha_j+1)}}\psi(x)\,dx\Big]^{1/p}\times\Big[\int_{-\pi}^{-a}\frac{|\omega(x)|^q}{|\sin\frac12 x|^q[\psi(x)]^{q/p}}\,dx\Big]^{1/q}<\infty.$$

Thus we have to prove that (5.16) follows from (β_2). For $0\leqslant b<a\leqslant\pi$ it is

trivial that

$$\int_b^a \frac{\psi(t)}{|\sin\frac12 t|^{p(\alpha_j+1)}}\,dt \leqslant \sum_{k=0}^{L(a,b)} \int_{2^k b}^{2^{k+1}b} \frac{\psi(t)}{|\sin\frac12 t|^{p(\alpha_j+1)}}\,dt \leqslant \sum_{k=0}^{L(a,b)} |\sin 2^{k-1}b|^{-p(\alpha_j+1)} \int_{2^k b}^{2^{k+1}b}\psi(t)\,dt,$$

where $L(a, b)$ denotes the integer part of $\log_2 a/b$. From (β_2) we now derive

(5.17)
$$\int_a^b \frac{\psi(t)}{|\sin\frac12 t|^{p(\alpha_j+1)}}\,dt \leqslant C_p\sum_{k=0}^{L(a,b)} \frac{(2^{k+1}b)^{p\alpha_j}}{|\sin 2^{k-1}b|^{p(\alpha_j+1)}}\Big[\int_{-\pi}^{-2^{k+1}b} \frac{|\omega(x)|^q}{|\sin\frac12 x|^q[\psi(x)]^{q/p}}\,dx\Big]^{-p/q} \leqslant C_p(2\pi)^{p(\alpha_j+1)}\sum_{k=0}^{L(a,b)}\frac{1}{(2^k b)^p}\Big[\int_{-\pi}^{-a}\frac{|\omega(x)|^q}{|\sin\frac12 x|^q[\psi(x)]^{q/p}}\,dx\Big]^{-p/q}.$$

By (5.17) we have

$$\Big[\int_a^b \frac{b^p\psi(t)}{|\sin\frac12 t|^{p(\alpha_j+1)}}\,dt\Big]^{1/p}\Big[\int_{-\pi}^{-a}\frac{|\omega(x)|^q}{|\sin\frac12 x|^q[\psi(x)]^{q/p}}\,dx\Big]^{1/q} \leqslant (2\pi)^{p(\alpha_j+1)}C_p\Big[\sum_{k=0}^{L(a,b)}\frac{1}{2^{kp}}\Big]^{1/p} = D.$$

The inequality

(5.18)
$$\Big[\int_{1-r}^{\pi}\psi(t)\Big|\int_{-\pi}^{-t} f(y)B_{rj}(y,t)\,dy\Big|^p dt\Big]^{1/p} \leqslant C_p\Big[\int_{-\pi}^0 |f|^p\psi\,dt\Big]^{1/p}$$

now follows from (5.13)–(5.15).

Similarly we estimate the second expression on the right of (5.12). Lemma 1 implies

(5.19)
$$\Big|\int_{-t}^0 f(y)B_{rj}(y,t)\,dy\Big| \leqslant C_p(1-r)|\sin\tfrac12 t|^{-(\alpha_j+2)}\int_{-t}^0|f(y)\,\omega(y)|\,dy.$$

Hence in virtue of Theorem D

(5.20)
$$\Big[\int_{1-r}^\pi \psi(t)\Big|\int_{-t}^0 f(y)B_{rj}(y,t)\,dy\Big|^p dt\Big]^{1/p}\leqslant C_p\Big[\int_{-\pi}^0|f|^p\psi\,dy\Big]^{1/p}$$

provided that

(5.21)
$$D=\sup_{1-r\leqslant a\leqslant\pi}\Big[\int_a^\pi \frac{(1-r)^p}{|\sin\frac12 x|^{p(\alpha_j+2)}}\psi(x)\,dx\Big]^{1/p}\Big[\int_{-a}^0\frac{|\omega(x)|^q}{[\psi(x)]^{q/p}}\,dx\Big]^{1/q}<+\infty.$$

We will derive (5.21) from (β_1). First observe that for arbitrary intervals I, I', $I'\subset I$, there is a constant C_p independent of I and I' such that

(5.22)
$$\int_I\psi\,dx\Big/\int_{I'}\psi\,dx \leqslant C_p\Big(\int_I|\omega|\,dx\Big/\int_{I'}|\omega|\,dx\Big)^p.$$

Indeed, by condition $(A_p^{|\omega|})$ we have

$$\Big(\int_{I'}|\omega|\,dx\Big)^p\leqslant \int_{I'}\psi\,dx\Big(\int_I(|\omega|^q/\psi^{q/p})\,dx\Big)^{p/q} \leqslant C_p\int_{I'}\psi\,dx\Big(\int_I\psi\,dx\Big)^{-1}\Big(\int_I|\omega|\,dx\Big)^p$$

and (5.22) follows. By (5.22) we obtain

$$\int_a^\pi\frac{\psi(t)}{|\sin\frac12 t|^{p(\alpha_j+2)}}\,dt\leqslant\sum_{k=0}^{L(\pi,a)}|\sin 2^{k-1}a|^{-p(\alpha_j+2)}\int_{a\cdot2^k}^{a\cdot2^{k+1}}\psi\,dt \leqslant C\sum_{k=0}^{L(\pi,a)}(2^k a)^{-p(\alpha_j+2)}\int_{-a}^{a\cdot 2^{k+1}}\psi\,dt \leqslant C\sum_{k=0}^{L(\pi,a)}(2^k a)^{-p(\alpha_j+2)}\int_{-a}^0\psi\,dt\Big(\int_{-a}^{a\cdot2^{k+1}}|\omega|\,dx\Big/\int_{-a}^0|\omega|\,dx\Big)^p.$$

Hence there is a positive number $\delta>0$ such that for $0<a<\delta$

(5.23)
$$\int_a^\pi \frac{\psi(t)}{|\sin\frac12 t|^{p(\alpha_j+2)}}\,dt\leqslant C\Big(\sum_{k=0}^{L(\pi,a)}\frac{1}{2^{pk}}\Big)a^{-p(\alpha_j+2)}\int_{-a}^0\psi\,dt.$$

By (β_1) and (5.23) we trivially get (5.21). Thus inequality (5.20) is true.

Finally we will show that

(5.24)
$$\Big[\int_{1-r}^{\pi/2}\psi(t)\Big|\int_{2t}^\pi f(y)B_{rj}(y,t)\,dy\Big|^p dt\Big]^{1/p}\leqslant C_p\Big[\int_0^\pi|f|^p\psi\,dt\Big]^{1/p}.$$

Taking into account that

$$\sin\frac y2=\sin\frac{y-t}{2}\cos\frac t2+\cos\frac{y-t}{2}\sin\frac t2$$

we get

$$\Big|\sin\frac y2\Big(\sin\frac{y-t}{2}\Big)^{-1}\Big|\leqslant\Big|\cos\frac t2\Big|+\Big|\cos\frac{y-t}{2}\Big|\leqslant 2.$$

Hence in virtue of Lemma 1

$$\Big|\int_{2t}^\pi f(y)B_{rj}(y,t)\,dy\Big|\leqslant C(1-r)|\sin\tfrac12 t|^{-(\alpha_j+1)}\int_{2t}^\pi\Big|\frac{f(y)\,\omega(y)}{\sin\frac12 y}\Big|\,dy.$$

Theorem E implies that (5.24) is true provided that

$$(5.25)\qquad D=\sup_{1-r<a<\pi/2}\Big[\int_{1-r}^{a}\frac{(1-r)^p\psi(x)}{|\sin\frac12 x|^{p(\alpha_j+1)}}dx\Big]^{1/p}\times\Big[\int_{2a}^{\pi}\frac{|\omega(x)|^q}{|\sin\frac12 x|^q[\psi(x)]^{q/p}}dx\Big]^{1/q}<\infty.$$

One can easily observe that (5.25) follows from (β_2) in the same way as inequality (5.21). Hence inequality (5.24) is true.

By (5.12), (5.13), (5.18), (5.20) and (5.24) we immediately obtain

$$\Big[\int_{1-r}^{\pi}\psi(t)\Big|\int_{E_r(t)}f(x+t)B_{rj}(x+t,t)\,dx\Big|^p dt\Big]^{1/p}\leqslant C_p\Big[\int_{-\pi}^{\pi}|f|^p\psi\,dt\Big]^{1/p}.$$

In the case $-\pi\leqslant t\leqslant-(1-r)$, the proof of the corresponding inequality is analogous. The proof of Lemma 8 is complete.

Lemmas 5–8 and Theorem C complete the proof of the implication (b)$\Rightarrow$(a). Indeed, using these results we conclude that

$$\begin{aligned}\int_{-\pi}^{\pi}\psi(t)\Big|\int_{-\pi}^{\pi}f(x+t)B_{rj}(x+t,t)\,dx\Big|^p dt&\leqslant 2^pC_p\int_{-\pi}^{\pi}\psi(t)[M^{|\omega|}(f,t)]^p dt\\&+2^p\int_{|t|\leqslant1-r}\psi(t)\Big|\int_{2(1-r)<|x|\leqslant\pi}f(x+t)B_{rj}(x+t,t)\,dx\Big|^p dt\\&+2^p\int_{1-r\leqslant|t|\leqslant\pi}\psi(t)\Big|\int_{2(1-r)<x\leqslant\pi}f(x+t)B_{rj}(x+t,t)\,dx\Big|^p dt\\&\leqslant C_p\int_{-\pi}^{\pi}|f|^p\psi\,dt\\&+4^p\int_{1-r\leqslant|t|\leqslant\pi}\psi(t)\Big|\int_{E_r(t)}f(x+t)B_{rj}(x+t,t)\,dx\Big|^p dt\\&+4^pC_p\int_{2(1-r)\leqslant|t|\leqslant\pi}\psi(t)[M^{|\omega|}(f,t)]^p dt\\&\leqslant C_p\int_{-\pi}^{\pi}|f|^p\psi\,dt.\end{aligned}$$

6. Proof of the remaining implications. First we will prove the implication (b′)$\Rightarrow$(a′). In the case $p=1$ there are no strong estimates for the maximal function, therefore Lemmas 5 and 6 are useless.

LEMMA 9. *Let $p=1$ and let condition (β_1') be satisfied. Then there is a constant $C>0$ independent of $f\in L_{[-\pi,\pi]}(\psi)$ such that*

$$(6.1)\qquad \int_{-\pi}^{\pi}\psi(t)\Big|\int_{-2(1-r)}^{2(1-r)}f(x+t)B_{rj}(x+t,t)\,dx\Big|dt\leqslant C\int_{-\pi}^{\pi}|f|\psi\,dt.$$

Proof. In virtue of Lemma 1 we get

$$\begin{aligned}(6.2)\qquad &\Big|\int_{-2(1-r)}^{2(1-r)}f(x+t)B_{rj}(x+t,t)\,dx\Big|\\&\leqslant C(1-r)^{-1}(1-2r\cos t+r^2)^{-\alpha_j/2}\int_{-2(1-r)}^{2(1-r)}f(x+t)\,\omega(x+t)\,dt\\&=C(1-r)^{-1}(1-2r\cos t+r^2)^{-\alpha_j/2}\int_{t-2(1-r)}^{t+2(1-r)}f(y)\,\omega(y)\,dy.\end{aligned}$$

Taking into account the periodicity of the functions and changing the variable we obtain

$$\begin{aligned}(6.3)\qquad &\int_{-\pi}^{\pi}\psi(t)(1-r)^{-1}(1-2r\cos t+r^2)^{-\alpha_j/2}\int_{t-2(1-r)}^{t+2(1-r)}f(y)\,\omega(y)\,dy\,dt\\&=\int_{-\pi}^{\pi}f(y)\,\omega(y)\int_{y-2(1-r)}^{y+2(1-r)}\psi(t)\,[(1-r)(1-2r\cos t+r^2)^{\alpha_j/2}]^{-1}\,dt\,dy\\&\leqslant\int_{-\pi}^{\pi}f(y)\,\omega(y)(1-r)^{-1}\max_{\varepsilon=\pm1}\Big[(1-r)^2+4r\sin^2\frac{y+\varepsilon2(1-r)}{2}\Big]^{-\alpha_j/2}\int_{y-2(1-r)}^{y+2(1-r)}\psi\,dt\,dy\end{aligned}$$

The weight function ψ satisfies condition $(A_1^{|\omega|})$, hence

$$\int_{y-2(1-r)}^{y+2(1-r)}\psi\,dt\leqslant C\int_{y-2(1-r)}^{y+2(1-r)}|\omega|\,dt\cdot\psi(y)\,|\omega(y)|^{-1}.$$

From (6.2) and (6.3) we now easily derive (6.1).

LEMMA 10. *Let $p=1$ and let condition (β_1') be satisfied. Then there is a constant $C>0$ independent of $f\in L_{[-\pi,\pi]}(\psi)$ such that*

$$(6.4)\qquad \int_{2(1-r)<|t|\leqslant\pi}\psi(t)\Big|\int_{2(1-r)<|x|\leqslant|t|}f(x+t)B_{rj}(x+t,t)\,dx\Big|dt\leqslant C\int_{-\pi}^{\pi}|f|\psi\,dt.$$

Proof. Changing the variable we deduce that for $2(1-r)<|t|<\pi$

$$(6.5)\qquad \int_{2(1-r)<|x|\leqslant|t|}f(x+t)B_{rj}(x+t,t)\,dx=\int_{2(1-r)<|y-t|\leqslant|t|}f(y)B_{rj}(y,t)\,dy.$$

In virtue of Lemma 1 we have for $|y|\leqslant2|t|$

$$\begin{aligned}(6.6)\qquad &\int_{2(1-r)<|y-t|\leqslant|t|}|f(y)B_{rj}(y,t)|\,dy\\&\leqslant C[1-2r\cos t+r^2]^{-\alpha_j/2}\int_{2(1-r)<|y-t|\leqslant|t|}|f(y)\,\omega(y)|\,P_r(y-t)\,dy.\end{aligned}$$

Consider the case $2(1-r) < t < \pi$. By Fubini's theorem,

$$(6.7)\quad \int_{2(1-r)}^{\pi} \psi(t)\left[1-2r\cos t+r^2\right]^{-\alpha_j/2} \int_{2(1-r)<|y-t|\leqslant t} |f(y)\,\omega(y)|\, P_r(y-t)\,dy\,dt$$
$$= \int_0^{2\pi} |f(y)\,\omega(y)| \int_{F_r(y)} P_r(y-t)\,\psi(t)\left[1-2r\cos t+r^2\right]^{-\alpha_j/2}\,dt\,dy,$$

where $F_r(y) = \{t\colon\ y/2 \leqslant t \leqslant \pi\} \cap \{t\colon\ |y-t| > 2(1-r)\}$. By a change of variable we obtain for $0 \leqslant y \leqslant 2\pi$

$$(6.8)\quad \int_{F_r(y)} P_r(t-y)\left[1-2r\cos t+r^2\right]^{-\alpha_j/2} \psi(t)\,dt$$
$$= \int_{\tilde{F}_r(y)} P_r(x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2} \psi(y+x)\,dx$$
$$= \int_{2(1-r)}^{\pi-y} P_r(x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2} \psi(y+x)\,dx$$
$$+ \int_{-y/2}^{-2(1-r)} P_r(x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2} \psi(y+x)\,dx,$$

where $\tilde{F}_r(y) = \{x\colon\ -y/2 \leqslant x \leqslant \pi-y\} \cap \{x\colon\ |x| > 2(1-r)\}$. Integrating by parts we easily get

$$(6.9)\quad \int_{2(1-r)}^{\pi-y} P_r(x)\,\psi(y+x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2} dx$$
$$= \int_0^{y+x} \psi(t)\,dt \cdot P_r(x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2}\Big|_{2(1-r)}^{\pi-y}$$
$$- \int_{2(1-r)}^{\pi-y} \frac{d}{dx} P_r(x)\cdot\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2} \int_y^{y+x} \psi(t)\,dt\,dx$$
$$+ r\alpha_j \int_{2(1-r)}^{\pi-y} P_r(x)\sin(y+x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2-1} \int_y^{y+x} \psi(t)\,dt\,dx.$$

The first term on the right-hand side of (6.9) is equal to

$$(6.10)\quad \int_y^{\pi} \psi(t)\,dt\cdot P_r(\pi-y)\left(1-2r\cos\pi+r^2\right)^{-\alpha_j/2}$$
$$- \int_y^{y+2(1-r)} \psi(t)\,dt\cdot P_r\big(2(1-r)\big)\left[1-2r\cos\left[y+2(1-r)\right]+r^2\right]^{-\alpha_j/2}.$$

From (β_1') it easily follows that the absolute values of both integrals in (6.10) do not exceed $C\psi(y)\,|\omega(y)|^{-1}$, where C is an absolute constant.

It is obvious that the absolute value of the sum of the second and third

terms on the right-hand side of (6.9) does not exceed

$$\sup_{2(1-r)\leqslant x\leqslant \pi-y} \Big[\int_y^{y+x} \psi(t)\,dt\Big(\int_y^{y+x} |\omega(t)|\,dt\Big)^{-1}\Big]$$
$$\times \int_{2(1-r)}^{\pi-y} \Big\{\Big|\frac{d}{dx} P_r(x)\cdot\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2}\Big|$$
$$+ r\alpha_j\,|\sin(y+x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2}| \int_y^{y+x} |\omega|\,dt\Big\}\,dx,$$

which is less than or equal to

$$C \sup_{2(1-r)\leqslant x\leqslant \pi-y} \int_y^{y+x} \psi(t)\,dt\Big(\int_y^{y+x} |\omega(t)|\,dt\Big)^{-1},$$

where C is an absolute constant, in virtue of the inequality

$$\int_y^{y+x} |\omega(t)|\,dt \leqslant x\sin^{\alpha_j}\tfrac{1}{2}(y+x).$$

The expression obtained is in turn less than $C\psi(y)\,|\omega(y)|^{-1}$.

Thus we have proved that

$$(6.11)\quad \Big|\int_{2(1-r)}^{\pi-y} P_r(x)\,\psi(y+x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2} dx\Big| \leqslant C\psi(y)\,|\omega(y)|^{-1}$$

Similarly,

$$(6.12)\quad \Big|\int_{-y/2}^{-2(1-r)} P_r(x)\,\psi(y+x)\left[1-2r\cos(y+x)+r^2\right]^{-\alpha_j/2} dx\Big| \leqslant C\psi(y)\,|\omega(y)|^{-1}.$$

Now (6.7), (6.8), (6.11) and (6.12) imply

$$(6.13)\quad \int_{2(1-r)}^{\pi} \psi(t)\left[1-2r\cos t+r^2\right]^{-\alpha_j/2} \int_{2(1-r)<|y-t|\leqslant|t|} |f(y)\,\omega(y)|\, P_r(t-y)\,dy\,dt$$
$$\leqslant C\int_0^{2\pi} |f(y)|\,\psi(y)\,dy.$$

Analogously we can prove that

$$(6.14)\quad \int_{-\pi}^{-2(1-r)} \psi(t)\left[1-2r\cos t+r^2\right]^{-\alpha_j/2}$$
$$\times \int_{2(1-r)<|y-t|\leqslant|t|} |f(y)\,\omega(y)|\, P_r(t-y)\,dy\,dt \leqslant C\int_{-2\pi}^{0} |f(y)|\,\psi(y)\,dy.$$

Relations (6.5), (6.6), (6.13) and (6.14) immediately imply (6.4). The proof of Lemma 10 is complete.

By using Lemmas 7–10, the implication (b′) ⇒ (a′) is proved just as the implication (b) ⇒ (a).

Proof of the implication (b) ⇒ (c).

Lemma 11. *Let condition* (β_2) *be satisfied for a weight function* $\psi(x) \geqslant 0$. *Let* $\omega(x)$ *be defined by* (1.2) *and* $x_j = 0$. *Then there is a number* $D_p > 1$ *such that for every* a, $0 < a < \min\{1, \frac{1}{2}\min_{i \neq j}|x_i - x_j|\}$,

$$\int_{a/2}^{\pi}\left[\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right]^{\frac{1}{p-1}} dt \Big/ \int_{a}^{\pi}\left[\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right]^{\frac{1}{p-1}} dt \geqslant D_p > 1. \tag{6.15}$$

Proof. By (β_2) we obviously have

$$\int_{a/2}^{\pi}\left[\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right]^{\frac{1}{p-1}} dt \Big/ \int_{a}^{\pi}\left|\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right|^{\frac{1}{p-1}} dt$$

$$= 1 + \int_{a/2}^{a}\left[\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right]^{\frac{1}{p-1}} dt \Big/ \int_{a}^{\pi}\left[\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right]^{\frac{1}{p-1}} dt$$

$$\geqslant 1 + B_p^{-1/(p-1)} \int_{a/2}^{a}\left[\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right]^{\frac{1}{p-1}} dt$$

$$\times a^{-p\alpha_j/(p-1)}\Big[\int_0^a \psi(t)\,dt\Big]^{1/(p-1)}$$

$$\geqslant 1 + B_p^{-1/(p-1)} a^{-p\alpha_j/(p-1)} \left(\frac{2}{a}\right)^{\frac{p}{p-1}} \int_{a/2}^{a}\left[\frac{|\omega(t)|^p}{\psi(t)}\right]^{\frac{1}{p-1}} dt$$

$$\times\Big[\int_{a/2}^{a}\psi(t)\,dt\Big]^{1/(p-1)}.$$

Taking into account that

$$\Big(\int_I|\omega(t)|\,dt\Big)^p \leqslant \left[\int_I\left[\frac{|\omega(t)|^p}{\psi(t)}\right]^{\frac{1}{p-1}} dt\right]^{p-1} \int_I \psi(t)\,dt, \tag{6.16}$$

we conclude that the last expression is greater than

$$1 + C_p a^{-p(\alpha_j+1)/(p-1)}\Big(\int_{a/2}^{a}|\omega(t)|\,dt\Big)^{p/(p-1)} \geqslant D_p > 1.$$

The proof of Lemma 11 is complete.

Fix any natural number j $(1 \leqslant j \leqslant s)$. Obviously without loss of generality we can assume that $x_j = 0$. One can easily observe that in order to prove (c) it is sufficient to show that the function $w(x) = \psi(x)/|\omega(x)|^p$ satisfies condition (A_p) for intervals which lie in a neighbourhood of $x_j = 0$. We will

prove that $w(x)$ satisfies condition (A_p) for intervals which lie in $(-\delta_j, \delta_j)$, where $\delta_j = \min\{1, \frac{1}{2}\min_{i \neq j}|x_i - x_j|\}$.

For intervals $(b_1, b_2) \subset (0, \delta_j)$, $b_1 > b_2/2$, this follows immediately from (β_1). Hence we have to prove it for intervals $(-a, a) \subset (-\delta_j, \delta_j)$. Observe that in virtue of Lemma 11 there is $D_p > 1$ such that

$$\int_{-\pi}^{-a/2}\left[\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right]^{\frac{1}{p-1}} dt \left[\int_{-\pi}^{-a}\left[\frac{|\omega(t)|^p}{|\sin\frac{1}{2}t|^p\psi(t)}\right]^{\frac{1}{p-1}} dt\right]^{-1} \geqslant D_p > 1. \tag{6.17}$$

By inequalities (6.15), (6.17) and (β_2) we get

$$\int_{-a}^{a}\frac{\psi(x)}{|\sin\frac{1}{2}x|^{p\alpha_j}}\,dx = \sum_{i=0}^{\infty}\int_{a/2^{i+1}}^{a/2^i}\frac{\psi(x)}{|\sin\frac{1}{2}x|^{p\alpha_j}}\,dx + \sum_{i=0}^{\infty}\int_{-a/2^i}^{-a/2^{i+1}}\frac{\psi(x)}{|\sin\frac{1}{2}x|^{p\alpha_j}}\,dx$$

$$\leqslant \sum_{i=0}^{\infty}\frac{8^{p\alpha_j}}{a^{p\alpha_j}}2^{p\alpha_j(i+1)}\int_{a/2^{i+1}}^{a/2^i}\psi(x)\,dx$$

$$+ \sum_{i=0}^{\infty}\frac{8^{p\alpha_j}}{a^{p\alpha_j}}2^{p\alpha_j(i+1)}\int_{-a/2^i}^{-a/2^{i+1}}\psi(x)\,dx$$

$$\leqslant C_p\sum_{i=0}^{\infty}\left\{\left[\int_{a/2^i}^{\pi}\left[\frac{|\omega(x)|^p}{|\sin\frac{1}{2}x|^p\psi(x)}\right]^{\frac{1}{p-1}} dx\right]^{1-p}\right.$$

$$\left. + \left[\int_{-\pi}^{-a/2^i}\left[\frac{|\omega(x)|^p}{|\sin\frac{1}{2}x|^p\psi(x)}\right]^{\frac{1}{p-1}} dx\right]^{1-p}\right\}$$

$$\leqslant C_p'\left\{\left[\int_{a}^{\pi}\left[\frac{|\omega(x)|^p}{|\sin\frac{1}{2}x|^p\psi(x)}\right]^{\frac{1}{p-1}} dx\right]^{1-p}\right.$$

$$\left. + \left[\int_{-\pi}^{-a}\left[\frac{|\omega(x)|^p}{|\sin\frac{1}{2}x|^p\psi(x)}\right]^{\frac{1}{p-1}} dx\right]^{1-p}\right\}.$$

By (β_1), (6.16) and (5.22) we now easily obtain

$$\int_{-a}^{a}\frac{\psi(x)}{|\sin\frac{1}{2}x|^{p\alpha_j}}\,dx \leqslant C_p\left\{a^p\left[\int_a^{2a}\left[\frac{|\omega|^p}{\psi}\right]^{\frac{1}{p-1}} dx\right]^{1-p} + a^p\left[\int_{-2a}^{-a}\left[\frac{|\omega|^p}{\psi}\right]^{p-1} dx\right]^{1-p}\right\}$$

$$\leqslant C_p'\Big\{a^p\Big(\int_a^{2a}|\omega|\,dx\Big)^{-p}\int_a^{2a}\psi\,dx + a^p\Big(\int_{-2a}^{-a}|\omega|\,dx\Big)^{-p}\int_{-2a}^{-a}\psi\,dx\Big\}$$

$$\leqslant C_p a^p\Big(\int_{-a}^{a}|\omega|\,dx\Big)^{-p}\int_{-a}^{a}\psi\,dx.$$

The proof of the implication (b) ⇒ (c) is complete.

Proof of the implication (b′)⇒(c′). For an arbitrary natural number j $(1 \leqslant j \leqslant s)$, define a 2π-periodic function $\theta_j(x)$ by

$$(6.18)\qquad 1/\theta_j(x) = \left\| \frac{\omega(t)}{\sin\frac{1}{2}t\,\psi(t)} \right\|_{L^\infty_{(x, x_j+\pi)}} \quad \text{for } x_j \leqslant x \leqslant x_j + \pi/2,$$

$$(6.19)\qquad 1/\theta_j(x) = \left\| \frac{\omega(t)}{\sin\frac{1}{2}t\,\psi(t)} \right\|_{L^\infty_{(x, x_j+\pi)}} \quad \text{for } x_j - \pi/2 \leqslant x < x_j,$$

and continue it linearly. We will now prove that the function

$$(6.20)\qquad w(x) = \psi(x)\Big[\prod_{i=1}^{s} |\sin^{\alpha_i - 1}\tfrac{1}{2}(x - x_i)|\,\theta_i(x)\Big]^{-1}$$

satisfies condition (A_1).

Fix any j $(1 \leqslant j \leqslant s)$ and assume that $x_j = 0$. Obviously it is sufficient to consider only intervals $(0, a) \subset (0, \delta_j)$, where $\delta_j = \min\{1, \frac{1}{2}\min_{i \neq j}|x_i - x_j|\}$. By (6.18)–(6.20) and by condition (β_2') we get

(6.21)

$$\int_0^a w(x)\,dx = \sum_{i=0}^{\infty} \int_{a/2^{i+1}}^{a/2^i} w(x)\,dx \leqslant C \sum_{i=0}^{\infty} \left(\frac{a}{2^{i+2}}\right)^{1-\alpha_j} \left[\theta_j\left(\frac{a}{2^{i+1}}\right)\right]^{-1} \int_{a/2^{i+1}}^{a/2^i} \psi(x)\,dx$$

$$\leqslant CB_1 \sum_{i=0}^{\infty} \left(\frac{a}{2^{i+2}}\right)^{1-\alpha_j} \left[\theta_j\left(\frac{a}{2^{i+1}}\right)\right]^{-1} \theta_j\left(\frac{a}{2^i}\right)\left(\frac{a}{2^i}\right)^{\alpha_j}$$

$$= CB_1 a \sum_{i=0}^{\infty} \frac{4^{\alpha_j}}{2^{i+2}} \left[\theta_j\left(\frac{a}{2^{i+1}}\right)\right]^{-1} \theta_j\left(\frac{a}{2^i}\right).$$

By the definition of $\theta_j(x)$ we have

$$(6.22)\qquad \left[\theta_j\left(\frac{a}{2^{i+1}}\right)\right]^{-1} \theta_j\left(\frac{a}{2^i}\right) \leqslant \max\left\{1, \left\| \frac{\omega(t)}{\sin\frac{1}{2}t\,\psi(t)} \right\|_{L^\infty_{(a/2^{i+1}, a/2^i)}} \left\| \frac{\omega(t)}{\sin\frac{1}{2}t\,\psi(t)} \right\|^{-1}_{L^\infty_{(a/2^i, \pi)}} \right\}.$$

By (5.22) and (β_1') we infer that

$$\left\| \frac{\omega(t)}{\sin\frac{1}{2}t\,\psi(t)} \right\|_{L^\infty_{(a/2^{i+1}, a/2^i)}} \left\| \frac{\omega(t)}{\sin\frac{1}{2}t\,\psi(t)} \right\|^{-1}_{L^\infty_{(a/2^i, \pi)}} \leqslant C\left(\frac{a}{2^{i+1}}\right)^{-1} \Big(\int_{a/2^{i+1}}^{a/2^i} \psi\,dt\Big)^{-1} \left(\frac{a}{2^i}\right)^{\alpha_j + 1} \frac{a}{2^{i-1}}$$

$$\times \int_0^{a/2^{i-1}} \psi(t)\,dt \left(\frac{a}{2^{i-1}}\right)^{-(\alpha_j+1)} \leqslant C'.$$

Hence (6.22) and (6.21) imply

$$(6.23)\qquad \int_0^a w(x)\,dx \leqslant Ca.$$

By (6.18)–(6.20) it is obvious that $[w(x)]^{-1}$ is bounded. From (6.23) we now conclude that condition (A_1) is satisfied for all intervals $(0, a)$. The required conditions on θ_j $(1 \leqslant j \leqslant s)$ are also easily verified.

The proofs of the implications (c)⇒(b) and (c′)⇒(b′) are trivial. The proofs of Theorems 1 and 2 are complete.

The author expresses his gratitude to J. Trzeciak for his editorial care.

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Received June 21, 1983 (1904)
New version September 12, 1984
Revised version March 3, 1986

STUDIA MATHEMATICA, T. LXXXVI. (1987)

Some results on the convergence of weighted sums of random elements in separable Banach spaces

by

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Abstract. Let X_n, $n \geqslant 1$, be a sequence of random elements taking values in a separable Banach space, A_n, $n \geqslant 1$, a sequence of real random variables and a_{nk}, $n \geqslant 1$, $k \geqslant 1$, a double array of real numbers. Under some conditions, we show that $\sum_{k\geqslant 1} a_{nk} A_k X_k$, $n \geqslant 1$, converges to 0 in the mean if and only if $\sum_{k\geqslant 1} a_{nk} f(A_k X_k)$, $n \geqslant 1$, converges to 0 in probability for every continuous linear functional f from the Banach space to the real line (Section 3). The main result in Section 3 unifies many results in the literature on the convergence of weighted sums of sequences of random elements. In Section 4, results on strong convergence are established. Marcinkiewicz–Zygmund–Kolmogorov's and Brunk–Chung's Strong Laws of Large Numbers are extended to separable Banach spaces. Using a certain stability theorem, a general result on strong convergence for weighted sums is proved from which many results in the literature follow as special cases under much less restrictive conditions.

1. Introduction. This paper is devoted to a study of limit theorems for weighted sums of sequences of random elements in separable Banach spaces. Section 2 presents some preliminaries needed in the subsequent sections. Section 3 concentrates on the convergence in probability and convergence in the mean of weighted sums of random elements. Let X_n, $n \geqslant 1$, be a sequence of random elements defined on some probability space $(\Omega, \mathscr{B}, P)$ taking values in a separable Banach space B, A_n, $n \geqslant 1$, a sequence of real random variables defined on Ω and a_{nk}, $n \geqslant 1$, $k \geqslant 1$, a double array of real numbers. Under some conditions, we show that $\sum_{k\geqslant 1} a_{nk} A_k X_k$, $n \geqslant 1$, converges to 0 in the mean if and only if $\sum_{k\geqslant 1} a_{nk} f(A_k X_k)$, $n \geqslant 1$, converges to 0 in probability for every continuous linear functional f from B to the real line $\boldsymbol{R}$ (Theorem 3.3). This result unifies many results in the literature on the underlying theme of Theorem 3.3. Moreover, the conditions imposed in

AMS Subject Classification Scheme 1979: Primary 60B12, Secondary 60F15, 60F05.
Key words and phrases: random elements, Banach space, weighted sums, convergence in probability, in mean and almost surely, uniform integrability and uniform tightness.
* Part of the work of this author was supported by the Air Force Office of Scientific Research under contract F49620-82-K-0001. Reproduction in whole or in part is permitted for any purpose of the United States Government.