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Nonfactorization theorems for Hardy and Bergman spaces on bounded symmetric domains

by

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Abstract. We extend the nonfactorization theorems of Rosay, Gowda and others to the case of bounded symmetric domains and some circular product domains. We show that for such domains D, $H^p(D)\cdot H^q(D)$ (resp. $A^{p,\alpha}(D)\cdot A^{q,\alpha}(D)$) is of first category in the Hardy space $H^l(D)$ (resp. the Bergman space $A^{l,\alpha}(D)$), where $1/p+1/q = 1/l$.

1. Introduction. Let U be the unit disc in the complex plane. It is well known that if $0 < p, q, l < \infty$ and $1/p+1/q = 1/l$ then every function f in the Hardy space $H^l(U)$ can be written as a product $f = g\cdot h$ with $g\in H^p(U)$ and $h\in H^q(U)$. The same was shown to be true for weighted Bergman spaces by Horowitz [7]. These results have no generalization to higher dimensions as was shown by Rudin [13], Miles [11] and Rosay [12] for polydiscs and by Gowda [5] for the unit ball. In [17] it was remarked that using just the conclusion of Gowda's theorem (and not its proof) one can easily obtain a generalization of his result to classical symmetric domains of type I. In this paper we show how Gowda's proof can be applied to all bounded symmetric domains and also to some product domains. We obtain exact generalizations of results of [5], namely we show that $H^p(D)\cdot H^q(D)$ (resp. $A^{p,\alpha}(D)\cdot A^{q,\alpha}(D)$) is of first category in $H^l(D)$ (resp. $A^{l,\alpha}(D)$). The proofs are fairly elementary and do not use the Lie algebra machinery, nevertheless they depend on facts that were proved with that technique.

2. Preliminaries. Since we will sometimes refer for details to [5], we will follow as closely as possible the notation of that paper. There will be, however, some additional notation and definitions.

A domain D in C^n is called *symmetric* if each of its points is an isolated fixed point of an involutive biholomorphic automorphism of D. Below we list main well-known facts about bounded symmetric domains.

The group $\mathrm{Aut}(D)$ of holomorphic automorphisms of D acts transitively on D. Each $\Phi\in\mathrm{Aut}(D)$ can be extended to a holomorphic mapping of a neighbourhood of $\bar{D}$. D can be realized as a circular convex domain such

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that the isotropy subgroup $\Gamma_0(D)$ of the component of identity of $\operatorname{Aut}(D)$ is a closed connected subgroup of the unitary group $\mathcal{U}(n)$. The Bergman–Shilov boundary bD of D is a real-analytic submanifold of C^n of real dimension at least n on which $\Gamma_0(D)$ acts transitively, so on bD there exists a unique normalized Γ_0-invariant Borel measure σ_D. In the above-mentioned realization σ_D is the normalized Hausdorff measure for bD. Every bounded symmetric domain is a product of so-called irreducible factors. For a deeper discussion of this subject we refer the reader to [15].

Let $B(z, w)$ be the Bergman kernel for a domain $D \subset C^n$ and let $B(z) = B(z, z)$; then $B(z) > 0$, $z \in D$. For $\alpha \in \boldsymbol{R}$ and $0 < p < \infty$, the Bergman space $A^{p,\alpha}(D)$ consists of all functions holomorphic in D and such that

$$\|f\|_{p,\alpha} = \Big(\int_D |f(z)|^p B(z)^{-\alpha} dv(z)\Big)^{1/p} < \infty,$$

where v is the $2n$-dimensional Lebesgue measure on C^n. We will only consider the case when $B^{-\alpha} dv$ is a finite measure so we define

$$A_D = \Big\{\alpha \in \boldsymbol{R}: \int_D B(z)^{-\alpha} dv < \infty\Big\}.$$

If D is a symmetric domain then $\alpha \in A_D$ if and only if $A^{p,\alpha}(D) \neq \{0\}$ [16, Cor. 2].

In the sequel, unless otherwise stated, we will always assume that D is balanced, i.e. $z \in D \Rightarrow \lambda z \in D$ for $\lambda \in C$, $|\lambda| \leqslant 1$.

Suppose D is such that $rD \Subset D$ for $r < 1$ and μ is a probabilistic circularly invariant measure on the boundary ∂D of D. For $0 < p < \infty$ the Hardy space $H^p(D, \mu)$ consists of all functions holomorphic in D and such that

$$\|f\|_p = \sup_{0<r<1} \Big(\int_{\partial D} |f(rz)|^p d\mu(z)\Big)^{1/p} < \infty.$$

It was shown in [1] that if $f \in H^p(D, \mu)$ then f has radial limits μ-almost everywhere on ∂D. The function f^* thus obtained is in $L^p(\mu)$ and $\|f^*\|_{L^p(\mu)} = \|f\|_p$.

Definition. We will say that a pair (D, μ) satisfies *condition* (A) ($(D, \mu) \in (A)$ for short) if:

(1) D is a balanced domain such that $rD \Subset D$ for $r < 1$;

(2) μ is a probabilistic circularly invariant regular Borel measure on ∂D such that if $K \Subset D$ then for every nonnegative function u continuous on $\bar{D}$ and plurisubharmonic in D we have

$$u(z) \leqslant C_K \int_{\partial D} u(z)\, d\mu(z) \quad \text{for } z \in K,$$

with some constant C_K.

Example. (i) Let D be a balanced domain with C^2-boundary and such that $rD \Subset D$ for $r < 1$. If μ is the normalized Hausdorff measure for ∂D then $(D, \mu) \in (A)$ (cf. [14]).

(ii) If D is a bounded symmetric domain then $(D, \sigma_D) \in (A)$. This can easily be seen from the well-known properties of the Poisson kernel for D.

The following can be proved by standard methods:

Fact 1. *If $(D, \mu) \in (A)$ then:*

(i) *Every bounded sequence in $H^p(D, \mu)$ has a subsequence uniformly convergent on compact subsets of D;*

(ii) *$H^p(D, \mu)$ is a Banach space for $p \geqslant 1$ and an F-space for $p < 1$;*

(iii) *The analytic polynomials are dense in $H^p(D, \mu)$.*

An analogous fact holds for the Bergman spaces $A^{p,\alpha}(D)$, $\alpha \in A_D$.

Remark. If D is strictly pseudoconvex and balanced then our $H^p(D)$ coincides with the one defined by harmonic majorants.

In the sequel we will need some well-known properties of the Bergman kernel $B(z, w)$.

Fact 2. (i) *$B(z, w)$ is holomorphic in $z \in D$ and $B(z, w) = \overline{B(w, z)}$.*

(ii) *If $v(D) < \infty$ then for all $z \in D$ we have $B(z) > 0$.*

(iii) *If $\Phi\colon D_1 \to D_2$ is biholomorphic then*

$$B_{D_1}(z, w) = B_{D_2}(\Phi(z), \Phi(w)) \det \Phi'(z) \overline{\det \Phi'(w)}, \qquad z, w \in D_1.$$

(iv) *If $v(D) < \infty$ and D is balanced then for every $w \in D$, $B(0, w) = (v(D))^{-1}$.*

(v) *If D is balanced and $rD \Subset D$ for $r < 1$ then for every $w \in D$ the function $z \mapsto B(z, w)$ can be extended onto a neighbourhood of $\bar{D}$, in fact $B(r^{-1}z, rw) = B(z, w)$.*

In the above Φ' denotes the complex Jacobi matrix of Φ. The real Jacobian will be denoted by $J\Phi$. Recall that $J\Phi = |\det \Phi'|^2$. To complete the notation let $\boldsymbol{T}$ stand for the unit circle in the complex plane $\boldsymbol{C}$ and m for the normalized Lebesgue measure on $\boldsymbol{T}$. $C(X)$ will be the space of all continuous functions on X, $H(D)$ the space of all holomorphic functions on D and $A(D) = C(\bar{D}) \cap H(D)$. I will stand for the identity mapping of C^n.

3. Main results. If X and Y are two spaces of functions on D then $X \cdot Y$ will denote the set $\{f \cdot g\colon f \in X,\ g \in Y\}$.

Theorem 1. *Let $D \subset C^n$ $(n > 1)$ be a bounded symmetric domain, $\alpha \in A_D$, $0 < p, q, l < \infty$ and $1/l = 1/p + 1/q$. Then $A^{p,\alpha}(D) \cdot A^{q,\alpha}(D)$ (resp. $H^p(D) \cdot H^q(D)$) is of first category in $A^{l,\alpha}(D)$ (resp. $H^l(D)$).*

Theorem 2. *Let $D \subset C^n$ be a balanced domain and $D_1 = U \times D$. Then Theorem 1 holds for the spaces $A^{p,\alpha}(D_1)$.*

THEOREM 3. *If* $(D, \mu) \in (A)$ *then Theorem* 1 *holds for the spaces* $H^p(U \times D, m \times \mu)$.

Theorems 2 and 3 remain valid for any bounded symmetric domain in place of U but for dimensions higher than one they are trivial corollaries of Theorem 1.

Theorem 1 has an analogue for symmetric Siegel domains of type II. It can be obtained from Theorem 1 by an application of suitable isometries of Bergman and Hardy spaces.

In [3] it was shown that every $f \in H^1(B_n)$ is an infinite sum of the form $\sum_{i=1}^{\infty} g_i h_i$ with $g_i, h_i \in H^2(B_n)$. In [2] the same was proved for Bergman spaces on bounded symmetric domains. It is not known whether these results hold for Hardy spaces on general bounded symmetric domains or even on the polydisc.

4. Proofs and other results. The proofs for Bergman spaces and for Hardy spaces are very similar but the H^p case is a little more complicated so we will usually restrict ourselves to this case.

If $K(z)$ is a holomorphic function in a balanced domain D then $K(z) = \sum_{i=0}^{\infty} K_i(z)$, where K_i is a homogeneous polynomial of degree i and the series is uniformly convergent on compact subsets of D. For such expansions we have:

LEMMA 1. *Let* $0 < t < \infty$. *There exists a constant* M_t *such that if* D *is balanced,* $\alpha \in A_D$ *(resp.* $(D, \mu) \in (A)$*) and* $K(z) = \sum_{i=N-1}^{\infty} K_i(z)$ *then*

$$\|K_N\|_{t,\alpha} \leqslant M_t \|K\|_{t,\alpha} \quad (\textit{resp. } \|K_N\|_t \leqslant M_t \|K\|_t).$$

The proof uses only the circular invariance of measures and is a standard application of integration by slices (see the proof of Lemma 2 in [5]).

LEMMA 2. *Suppose* μ *is a finite regular Borel measure on some compact set* X *and* $\operatorname{supp} \mu = X$. *Let* $f \in C(X)$, $|f(x)| \leqslant 1$ *on* X *and* $\mu\{x: |f(x)| = 1\} = 0$. *Then for* $0 < l < t < \infty$

$$\lim_{N \to \infty} \|f^N\|_{L^t(\mu)} / \|f^N\|_{L^l(\mu)} = \infty.$$

We omit the elementary proof.

PROPOSITION 1. *Let* $D \subset C^n$ $(n > 1)$ *be a bounded symmetric domain,* $\alpha \in A_D$, $0 < p, q, l < \infty$, $1/l = 1/p + 1/q$. *Then the product map* $(h, k) \mapsto h \cdot k$ *from* $A^{p,\alpha}(D) \times A^{q,\alpha}(D)$ *to* $A^{l,\alpha}(D)$ *(resp. from* $H^p(D) \times H^q(D)$ *to* $H^l(D)$*) is not open at the origin.*

Proof. We will follow an idea originating from Rudin [13] and applied in all proofs of nonfactorization theorems. The proof of [5, Lemma 3] should be referred to for some details.

Recall that $\Gamma_0(D) \subset \mathcal{U}(n)$. It is known that $\Gamma_0(D)$ contains a toral subgroup of dimension at least two but we will only need the fact that it contains a circular subgroup other than $\{\lambda I: \lambda \in T\}$, which is true for every connected closed subgroup of $\mathcal{U}(n)$ not equal to $\{I\}$ or $\{\lambda I\}$. Let $T \ni \lambda \mapsto k'_\lambda \in \Gamma_0(D)$ be a homomorphism onto such a subgroup. Then there exists a unitary matrix V such that for every λ we have

$$k'_\lambda = V^* \begin{bmatrix} \lambda^{s_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{s_n} \end{bmatrix} V$$

where $s_i \in Z$, $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n$, $s_1 < s_n$. Define $k_\lambda = \lambda^{-s_1} k'_\lambda$. Then $k_\lambda \in \Gamma_0(D)$ and passing to the new variables defined by V we get

$$k_\lambda(z_1, \ldots, z_n) = (z_1, \lambda^{m_2} z_2, \ldots, \lambda^{m_n} z_n)$$

where $0 \leqslant m_2 \leqslant \ldots \leqslant m_n$ and $0 < m_n$.

Let $A_i = \{z \in bD: |z_i| = \max_{z \in \bar{D}} |z_i|\}$. A_i is a level set of the real-analytic function $|z_i|^2$ on the real-analytic manifold bD so it is either of measure zero or equal to the whole bD.

Suppose at the beginning that $A_i = bD$ for every i. We may assume that $|z_i| = 1$ on bD for every i, so $bD \subset T^n$. But bD is a compact manifold and $\dim bD \geqslant n$ so $bD = T^n$. This implies that $D = U^n$, for instance because $\bar{D} = \operatorname{conv}(bD)$. For the polydisc our proposition was proved in [5] so we omit this case and will assume that $\sigma(A_j) = 0$ for some j. Since $bD = \operatorname{supp}(\mu)$ we deduce, by Lemma 2, that for $t > l$

$$(1) \qquad \|z_j^N\|_t / \|z_j^N\|_l \to \infty \quad \text{as } N \to \infty.$$

Let

$$s = \begin{cases} n & \text{if } m_j = 0, \\ 1 & \text{if } m_j > 0, \end{cases}$$

and $F(z) = a z_s^{N-1} + z_j^N$ where a is such that

$$(2) \qquad \|F\|_l \leqslant 2 \|z_j^N\|_l$$

(in Gowda's proof $s = 1$, $j = 2$, $a = 1$). Assume $F(z) = H(z) \cdot K(z)$ and let $H = \sum_{i=0}^{\infty} H_i$, $K = \sum_{i=0}^{\infty} K_i$ be the homogeneous expansions. Then, as in [5], writing $H_0 = A$ we get $K_i = 0$ for $0 \leqslant i < N-1$ and

$$A K_N(z) = z_j^N - \left(a H_1(z) z_s^{N-1}\right) / A.$$

Hence

$$AK_N(k_\lambda(z)) = \lambda^{Nm_j} z_j^N - \left(aH_1(k_\lambda(z))\lambda^{(N-1)m_s} z_s^{N-1}\right)/A.$$

Assume for a moment that $m_j = 0$. Then

$$AK_N(k_\lambda(z)) = z_j^N - \left(aH_1(k_\lambda(z))\lambda^{(N-1)m_n} z_n^{N-1}\right)/A.$$

Therefore z_j^N is the constant term of the polynomial $\lambda \mapsto AK_N(k_\lambda(z))$ and, by subharmonicity,

(3) $$|z_j^N|^t \leqslant \int_T |AK_N(k_\lambda(z))|^t\, dm(\lambda).$$

Now, if $m_j > 0$ then

$$AK_N(k_\lambda(z)) = \lambda^{Nm_j} z_j^N - \left(aH_1(k_\lambda(z)) z_1^{N-1}\right)/A.$$

The expression $aH_1(k_\lambda(z))z_1^{N-1}$ is a polynomial in λ of degree not larger than m_n. Hence for every $t > 0$ there exists a C_t such that for $N \geqslant m_n$ we have

(4) $$|z_j^N|^t = \int_T |\lambda^{Nm_j} z_j^N|^t\, dm(\lambda) \leqslant C_t^t \int_T |AK_N(k_\lambda(z))|^t\, dm(\lambda).$$

By the Γ_0-invariance of μ, both (3) and (4) imply

$$\|z_j^N\|_t \leqslant C_t |A|\, \|K_N\|_t.$$

Since $A = H(0)$, by subharmonicity we get $|A| \leqslant \|H\|_t$ and, by Lemma 1, $\|K_N\|_t \leqslant M_t \|K\|_t$, so that finally

$$\|z_j^N\|_t \leqslant M_t C_t \|H\|_t \|K\|_t.$$

Taking $t = \min(p, q)$ we get

$$\|z_j^N\|_t \leqslant M_t C_t \|H\|_p \|K\|_q$$

and so, by (2),

$$\|H\|_p \|K\|_q / \|F\|_l \geqslant (2M_t C_t)^{-1} \|z_j^N\|_t / \|z_j^N\|_l.$$

By (1), the ratio on the left-hand side can be arbitrarily large, which proves our claim.

Remark. By using a fairly strong result describing rational inner functions on bounded symmetric domains ([10]) the proof could be slightly simplified. Namely, if $D \subset C^n$ $(n > 1)$ is irreducible then there are no linear nonconstant inner functions on D so for every i, $|z_i|$ attains its maximum on a set of measure zero.

Proposition 2. *Suppose D_i, $i = 1, 2$, are balanced and $\alpha \in A_{D_1} \cap A_{A_2}$ (resp. $(D_i, \mu_i) \in$ (A)). Then the product map from $A^{p,\alpha}(D_1 \times D_2) \times A^{q,\alpha}(D_1 \times D_2)$ to $A^{l,\alpha}(D_1 \times D_2)$ (resp. from $H^p(D_1 \times D_2, \mu_1 \times \mu_2) \times H^q(D_1 \times D_2, \mu_1 \times \mu_2)$ to $H^l(D_1 \times D_2, \mu_1 \times \mu_2)$ is not open at the origin.*

Proof. The argument is very much like the one just presented so we will merely emphasize the important points.

Let $X_i = \operatorname{supp} \mu_i$. We will show that $\bar{D}_i \subset \operatorname{conv} X_i$. Since $\operatorname{conv} X_i$ is closed it is enough to show that $D_i \subset \operatorname{conv} X_i$. Assume to the contrary that $z_0 \in D_1 \setminus \operatorname{conv} X_1$. Then there exists a linear functional φ such that $\varphi(z_0) = 1$ and $|\varphi(z)| \leqslant c < 1$ for $z \in X_1$. Since $(D_1, \mu_1) \in$ (A) we get

$$1 = |\varphi^N(z_0)| \leqslant C_{\{z_0\}} \int_{X_1} |\varphi^N(z)|\, d\mu_1(z) \leqslant C_{\{z_0\}} c^N$$

which gives a contradiction. Hence for each i we can find a linear functional φ_i such that $|\varphi_i(z)| \leqslant 1$ for $z \in \bar{D}_i$ and $|\varphi_i(z)| = 1$ only for $z = \lambda w_i$ where $\lambda \in T$ and w_i is some fixed point of X_i. Write $E_i = \{\lambda w_i \colon \lambda \in T\}$. It may happen that $\mu_i(E_i) > 0$ for both i. In this case assume additionally that $\varphi_i(w_i) = 1$ and put, for $f \in H^t(D_1 \times D_2, \mu_1 \times \mu_2)$ and $\xi, \eta \in U$,

$$(Pf)(\xi, \eta) = f(\xi w_1, \eta w_2).$$

Then P is a bounded operator from $H^t(D_1 \times D_2)$ to $H^t(U^2)$. It is enough to check the boundedness for $f \in A(D_1 \times D_2)$, and indeed,

$$\begin{aligned}\int_{\partial D_1 \times \partial D_2} |f(z, s)|^t\, d\mu_1\, d\mu_2 &\geqslant \int_{E_1 \times E_2} |f(z, s)|^t\, d\mu_1\, d\mu_2 \\ &= \int_{E_1 \times E_2} \int_{T^2} |f(\xi z, \eta s)|^t\, dm(\xi)\, dm(\eta)\, d\mu_1(z)\, d\mu_2(s).\end{aligned}$$

Since $z \in E_1$ and $s \in E_2$, the inner integral does not depend on z or s and we just get $\mu_1(E_1)\mu_2(E_2)\|Pf\|^t_{H^t(U^2)}$.

Now for $g \in H^t(U^2)$ let

$$(Sg)(z, s) = g(\varphi_1(z), \varphi_2(s)).$$

Then S is bounded to $H^t(D_1 \times D_2)$. Indeed, if again $g \in A(U^2)$ then

$$\begin{aligned}\int_{\partial D_1 \times \partial D_2} |g(\varphi_1(z), \varphi_2(s))|^t\, d\mu_1(z)\, d\mu_2(s) \\ = \int_{\partial D_1 \times \partial D_2} \int_{T^2} |g(\xi\varphi_1(z), \eta\varphi_2(s))|^t\, dm(\xi)\, dm(\eta)\, d\mu_1(z)\, d\mu_2(s).\end{aligned}$$

The inner integral is bounded by $\|g\|^t_{H^t(U^2)}$ so $\|S\| \leqslant 1$. Since $P \circ S = \mathrm{Id}_{H^t(U^2)}$, it follows that P is onto the space $H^t(U^2)$, hence open. That reduces the proposition to the case of the polydisc.

Therefore we may assume that $\mu_1(E_1) = 0$ and use φ_1 as we used z_j in the proof of Prop. 1.

Lemma 3. *Suppose D is a domain in C^n, $\alpha \in A_D$ (resp. $(D, \mu) \in$ (A)) and*

$E \subset \partial D$ *is a peak set for* $A(D)$. *Let* $A_E(D) = \{f \in A(D): f|_E = 0\}$. *Then*:

(i) $\mu(E) = 0$;

(ii) $A_E(D)$ *is dense in* $A^{p,\alpha}(D)$ (*resp.* $H^p(D, \mu)$), $0 < p < \infty$;

(iii) *If for some* $z_0 \in D$ *and* $w_0 \in E$ *there exists a sequence* $\Phi_m \in \mathrm{Aut}(D)$ *such that* $\Phi_m(z_0) \to w_0$ *then for every* $z \in D$ *all cluster points of* $\{\Phi_m(z)\}$ *lie in* E.

Proof. Choose an $f \in A(D)$ which peaks on E. Then, by the plurisubharmonicity of $\log|1-f|$ and the circular invariance of μ we get

$$-\infty < \log|1-f(0)| \leqslant \int_{\partial D} \log|1-f(z)|\, d\mu(z)$$

so $\mu(E) = 0$, which proves (i). We also get $f^N \to 0$ μ-a.e. on ∂D. Let $h \in H^p(D,\mu)$ and take $g \in A(D)$ such that $\|h-g\|_p < \varepsilon$. Then $(1-f^N)g \to g$ in H^p so $\|(1-f^N)g-h\|_p < \varepsilon$ for large N and obviously $(1-f^N)g \in A_E(D)$.

Now suppose $\Phi_{m_k}(z) \to \xi \notin E$. Then $|f(\xi)| < 1$ and $f(w_0) = 1$. Let

$$d(z,w) = \sup\{\varrho(g(z),g(w)):\ g: D \to U,\ g \in H(D)\}$$

be the Carathéodory distance in D (cf. [8]; ϱ is the hyperbolic distance in U). Then d is $\mathrm{Aut}(D)$-invariant. If $\alpha < 1$ then $\alpha f: D \to U$ and

$$d(z,z_0) = d(\Phi_{m_k}(z), \Phi_{m_k}(z_0)) \geqslant \varrho\big(\alpha f(\Phi_{m_k}(z)), \alpha f(\Phi_{m_k}(z_0))\big) \to \varrho(\alpha f(\xi), \alpha),$$

and $\varrho(\alpha f(\xi), \alpha)$ can be arbitrarily large if α is close to one, which gives a contradiction.

LEMMA 4. *Let* $D \subset \mathbf{C}^n$ *be a domain, not necessarily balanced but starlike with respect to some* $z_0 \in D$, *and let* $h(z) = P(z)/Q(z)$ *be a rational function defined and nonvanishing in* D. *Suppose* g *is a continuous branch of the argument of* h. *Then*

$$|g(z)-g(z_0)| \leqslant \pi(\deg P + \deg Q), \qquad z \in D.$$

Proof. We may assume that $Q \equiv 1$. Let $\deg P = N$. Assume first that $D \subset \mathbf{C}$. Since then $P(\lambda) = \alpha \prod_{i=1}^{N} (\lambda - \lambda_i)$, $\lambda_i \notin D$, we easily find a branch of the argument satisfying the claim. Any other branch must have the same property.

Now return to the general case and for $z \in D$ define

$$D_z = \{\lambda \in \mathbf{C}: \lambda(z-z_0)+z_0 \in D\}.$$

Then D_z is a domain in $\mathbf{C}$, starlike with respect to the origin, $P_z(\lambda) = P\big(\lambda(z-z_0)+z_0\big)$ is a polynomial nonvanishing in D_z and $\deg P_z \leqslant N$. Moreover, $g_z(\lambda) = g\big(\lambda(z-z_0)+z_0\big)$ is a continuous branch of the argument of P_z, in particular

$$|g(z)-g(z_0)| = |g_z(1)-g_z(0)| \leqslant \pi \deg P_z \leqslant \pi N.$$

Now we return once again to bounded symmetric domains. For simplicity we will assume that $v(D) = 1$.

As we have mentioned, $\mathrm{Aut}(D)$ is transitive on D; but proving this as in [6, p. 170] one concludes in fact that for any two points $z, w \in D$ there is an involution $\Phi \in \mathrm{Aut}(D)$ which interchanges them. For $a \in D$ we will denote by Φ_a any involution interchanging a with the origin.

Define also

$$K_\alpha(a,z) = \left(\frac{|B(a,z)|^2}{B(a)}\right)^{\alpha+1}, \qquad a, z \in D,\ \alpha \in A_D.$$

LEMMA 5. (i) $K_\alpha(a, \Phi_a(z)) K_\alpha(a, z) = 1$.

(ii) $\int_D f(z) B(z)^{-\alpha} dv(z) = \int_D f(\Phi_a(z)) K_\alpha(a,z) B(z)^{-\alpha} dv(z)$, $\quad f \in C(\bar{D})$.

(iii) *If* $w \in bD$ *and* $a \to w$ *then for* $f \in C(\bar{D})$

$$\int_D f(\Phi_a(z)) B(z)^{-\alpha} dv(z) \to f(w).$$

(iv) *There exists a* C *depending only on* D *and* α *such that for every* $a \in D$ *there exists a function* $\psi_a \in A(D)$ *such that*

$$\max\{1, K_\alpha(a,z)\} \leqslant |\psi_a(z)| \leqslant C\{1+K_\alpha(a,z)\}, \qquad z \in \bar{D}.$$

Proof. Using Fact 2 (ii), (iv) we get

$$|B(a,z)|^2/B(a) = J\Phi_a(z),$$

so

$$|B(a, \Phi_a(z))|^2/B(a) = J\Phi_a(\Phi_a(z)) = J(\Phi_a^{-1})(\Phi_a(z)) = (J\Phi_a(z))^{-1}$$

which proves (i). Next

$$K_\alpha(a,z) B(z)^{-\alpha} = J\Phi_a(z)\big(|B(a,z)|^2/B(a)B(z)\big)^\alpha = J\Phi_a(z) B(\Phi_a(z))^{-\alpha}$$

and (ii) follows.

Since bD is $\Gamma_0(D)$-homogeneous, every point of bD is a peak point for $A(D)$, and (iii) follows by Lemma 3 (iii).

To prove (iv) we will first show that $B(z, w)$ has in $D \times D$ a bounded continuous branch of the argument. Let D_1 be a Siegel domain of type II biholomorphic to D, $\Phi: D \to D_1$ the Cayley transform (i.e. a biholomorphic map of D onto D_1) and B_1 the Bergman kernel for D_1. B_1 was explicitly computed by Gindikin [4] (cf. also [9]) and is a rational function in z, $\bar{w}$, nonvanishing in $D_1 \times D_1$. Since D_1 is convex, by Lemma 4 every branch of the argument of B_1 is bounded in $D_1 \times D_1$. Using again Fact 2 (ii), (iv) we get

$$B(z,w) = J\Phi(0) \frac{B_1(\Phi(z), \Phi(w))}{B_1(\Phi(z), \Phi(0)) B_1(\Phi(0), \Phi(w))}$$

so any branch of the argument of B must also be bounded. This shows that there is a β such that if $a \in D$ then one can find a $\lambda \in \boldsymbol{T}$ such that

$$\operatorname{Re}\left(\lambda B(z, a)^{\beta}\right) \geqslant 0 \quad \text{for } z \in \bar{D}$$

(recall that $B(z, a)$ extends onto a neighbourhood of $\bar{D}$). Then

$$\psi_a(z) = \left(1 + \lambda \frac{B(z, a)^{\beta}}{B(a)^{\beta/2}}\right)^{2(\alpha+1)/\beta}$$

satisfies the claim with $C = 2^{2(\alpha+1)/\beta}$ (cf. [5, Lemma 5]).

Remark. If we replace $B(z, w)$ by the Cauchy–Szegő kernel $C(z, w)$ and define

$$K(a, z) = |C(a, z)|^2 / C(a, a)$$

then $K(a, z)$ is the Poisson kernel for D and Lemma 5 extends to this function with $d\sigma_D$ in place of $B(z)^{-\alpha} dv$. (i) and (iv) follow easily from their analogues for the Bergman kernel because for irreducible D one has $C(z, w) = B(z, w)^{\gamma}$ for some γ [9]. The remaining assertions are consequences of the well-known properties of the Poisson kernel (cf. [15] or [9]).

The proof of Theorem 1 can now be finished exactly as in [5]. Instead of the point e_1 we use any point $w \in bD$ and apply Lemma 3 to get the density of $A_{\{w\}}(D)$ in $H^p(D)$.

In the proofs of Theorems 2 and 3 we use Proposition 2 for $D_1 = U$ and an analogue of Lemma 5 for $a = (a_1, 0, \ldots, 0)$, $a_1 \in U$. For such a there exist involutions Φ_a. Instead of Lemma 1 (iii) we prove that if $f \in C(\overline{U \times D})$ and $f|_{\{1\} \times \bar{D}} = 0$ then for $a \to (1, 0, \ldots, 0)$ we have $\int f \circ \Phi_a \to 0$. Also, $A_{\{1\} \times \bar{D}}(U \times D)$ is dense in H^p and in $A^{p,\alpha}$. The rest of Gowda's proof can be applied with minor modifications.

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