[11] R. Lyons, *A characterization of measures whose Fourier–Stieltjes transforms vanish at infinity*, Thesis, University of Michigan, 1983.
[12] –, *Characterizations of measures whose Fourier–Stieltjes transforms vanish at infinity*, Bull. Amer. Math. Soc. (N.S.) 10 (1984), 93–96.
[13] –, *Fourier–Stieltjes coefficients and asymptotic distribution modulo* 1, Ann. of Math. 122 (1985), 155–170.
[14] O. C. McGehee, *Certain isomorphisms between quotients of a group algebra*, Pacific J. Math. 21 (1967), 133–152.
[15] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, North-Holland, Amsterdam 1972.
[16] –, *Problème de l'unicité et de la synthèse en analyse harmonique*, C. R. Acad. Sci. Paris 266 (1968), A275–A276.
[17] I. I. Pyatetskii-Shapiro, *On the problem of uniqueness of expansion of a function in a trigonometric series* (in Russian), Moscov. Gos. Univ. Uch. Zap. 155, Mat. 5 (1952), 54–72; Supplement to: *On the problem of uniqueness of expansion of a function in a trigonometric series* (in Russian), Moscov. Gos. Univ. Uch. Zap. 165, Mat. 7 (1954), 79–97.
[18] A. Rajchman, *Sur l'unicité du développement trigonométrique*, Fund. Math. (Varsovie) 3 (1922), 287–302; *Rectification et addition à ma note «Sur l'unicité du développement trigonométrique»*, Fund. Math. 4 (1923), 366–367.
[19] –, *Sur la multiplication des séries trigonométriques et sur une classe remarquable d'ensembles fermés*, Math. Ann. 95 (1925), 389–408.
[20] A. Zygmund, *Trigonometric Series*, second ed. reprinted, vols. I, II, Cambridge University Press, Cambridge 1979.

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Added in proof (November 1986). Answers to some of the questions in this paper can be found in the following articles and book to appear: R. Lyons, *Mixing and asymptotic distribution modulo* 1; R. Lyons, *A new type of sets of uniqueness*; G. Debs and J. Saint Raymond, *Ensembles d'unicité et d'unicité au sens large*; A. S. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*.

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A remark on entropy of Abelian groups and the invariant uniform approximation property

by

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Abstract. Let G be a compact Abelian group and let Λ be a finite set of characters on G. We prove that there exists $K \in L_1(G)$ with $\|K\|_1 \leqslant 2$ such that $\hat{K}(\gamma) = 1$ for $\gamma \in \Lambda$ and $\hat{K}(\gamma) \neq 0$ for at most $C^{|\Lambda|}$ characters γ, where C is an absolute constant. In fact, for this type of uniform approximation on G, we obtain more precise estimates in terms of appropriate entropy numbers.

1. Introduction. Let G be a compact Abelian group and Λ a finite subset of the character group $\Gamma = G^*$ (the compactness hypothesis is in fact nonessential and the main result may also be formulated for locally compact groups). Given $\varepsilon > 0$, we consider functions K satisfying the conditions

$$\|K\|_1 < 1+\varepsilon, \tag{1}$$

$$\hat{K}(\gamma) = 1 \quad \text{for each } \gamma \in \Lambda, \tag{2}$$

and where $|\operatorname{supp} \hat{K}|$ (= the size of the support of the Fourier transform of K) is as small as possible. This problem of invariant uniform approximation was considered in [B–P] where an estimate on $|\operatorname{supp} \hat{K}|$ is proved using combinatorial methods.

Associate with Λ the following invariant pseudo-metric on G:

$$d_\Lambda(x, y) = \sup_{\gamma \in \Lambda} |\gamma(x) - \gamma(y)|,$$

and denote by $N_\Lambda(\varrho)$ the corresponding entropy numbers for $\varrho > 0$. The purpose of this note is to show the following fact.

THEOREM 1. *If* $0 < \varepsilon \leqslant 1$, *then there exists* K *satisfying* (1), (2) *and*

$$\log |\operatorname{supp} \hat{K}| \leqslant 8 (\log_2 (120/\varepsilon)) \log N_\Lambda(1/20). \tag{3}$$

In particular, we can find K such that (2), $\|K\|_1 \leqslant 2$ and $|\operatorname{supp} \hat{K}| < C^{|\Lambda|}$ where C is a fixed constant. As has been observed by W. B. Johnson (cf. [F–J–S]), this exponential estimate is the best one can hope for. This is clear from the following example (answering also a question at the end of [B–P]).

Let $G = \{1, -1\}^N$ be the Cantor group and $\Lambda = \{\varepsilon_1, \ldots, \varepsilon_n\}$ the first n Rademacher functions. Assume K fulfills (2). Then by Khintchine's inequalities

$$n = \int_G \Big(\sum_{j=1}^{n} \varepsilon_j\Big) K(\varepsilon)\, d\varepsilon \leqslant \Big\|\sum_{j=1}^{n} \varepsilon_j\Big\|_p \|K\|_{p'} \leqslant \sqrt{p}\sqrt{n}\,\|K\|_\infty^{1/p}\|K\|_1^{1/p'},$$

$$\|K\|_\infty \geqslant (n/p\,\|K\|_1^2)^{p/2}$$

and hence for an appropriate choice of p we get

$$\|K\|_\infty \geqslant \exp(n/2e\,\|K\|_1^2).$$

In the case of bounded groups, the estimate (3) is straightforward and the interest of the result is primarily the circle group case $G = T$. The following notion (see [B]) is related to the concept of "arithmetic diameter" introduced by Katznelson and McGehee (see [G–M]):

$$d(\Lambda) = \min\{d = 1, 2, \ldots : C_\Lambda \text{ is 2-isomorphic to a subspace of } l_d^\infty\}.$$

Here C_Λ is the subspace of $\mathscr{C}(G)$ of functions with Fourier transform supported by Λ and l_d^∞ is the d-dimensional complex l^∞-space. The notion "2-isomorphic" has the usual Banach space meaning (see [L–T] for instance for details).

Recall also that a set $\Lambda \subset \Gamma$ is *dissociated* provided Λ does not admit nontrivial $\pm 1, 0$-relations, thus

$$\sum_{\gamma\in\Lambda}{}' \varepsilon_\gamma \gamma = 0, \varepsilon_\gamma = 0, 1, -1 \quad \Rightarrow \quad \varepsilon_\gamma = 0 \text{ if } \gamma \neq 0.$$

The proof of Theorem 1 and the entropy characterizations of Sidon sets obtained in [P] yield

COROLLARY 1. *For given $\delta > 0$ there exists $\delta' > 0$ such that if $\Lambda \subset \Gamma$ is a finite set of characters satisfying $\log d(\Lambda) > \delta|\Lambda|$, then there is a subset A of Λ, A dissociated and $|A| > \delta'|\Lambda|$.*

The main point in proving Theorem 1 is a comparison of the entropy numbers for various invariant pseudo-metrics on G. We make crucial use of the spectral property of a one-point set in T.

2. Inequalities relating entropy numbers. Let $\Lambda \subset \Gamma$ be a finite set. Define also for $x, y \in G$

$$\bar{d}_\Lambda(x, y) = \sup_{f\in C_\Lambda, |f|\leqslant 1} |f(x) - f(y)|.$$

Let, for $0 < \varrho \leqslant 2$, $\bar{N}_\Lambda(\varrho)$ be the corresponding entropy numbers. Obviously,

$$d_\Lambda(x, y) \leqslant \bar{d}_\Lambda(x, y) \quad \text{and} \quad N_\Lambda(\varrho) \leqslant \bar{N}_\Lambda(\varrho).$$

LEMMA 1. *If $0 < \varepsilon \leqslant 2$, then*

$$\log N_\Lambda(\varepsilon) \leqslant (\log_2(4/\varepsilon)) \log N_\Lambda(1/20). \tag{4}$$

LEMMA 2. *If $\varepsilon > 0$, then*

$$\bar{N}_\Lambda(\varepsilon) \leqslant N_\Lambda(\varepsilon/5). \tag{5}$$

Lemma 1 appears in [B]. Lemma 2 seems to be new and in fact clarifies some points in [B].

Proof of Lemma 1. Denote $N_\Lambda(\varrho)$ by $N(\varrho)$ for simplicity. Since

$$N(\varepsilon) = \prod_{j=0}^{m} \frac{N(2^j\varepsilon)}{N(2^{j+1}\varepsilon)}$$

where $m = [\log_2(2/\varepsilon)]$, there exists some $\varepsilon \leqslant \varrho \leqslant 1$ such that

$$\log\frac{N(\varrho)}{N(2\varrho)} \geqslant \frac{\log N(\varepsilon)}{\log_2(2/\varepsilon)} \equiv M.$$

From this fact, it is easy to derive the existence of a subset P of G with

$$\log|P| \geqslant M$$

such that

$$\varrho \leqslant d_\Lambda(x, y) \leqslant 4\varrho \quad \text{if } x \neq y \text{ in } P. \tag{6}$$

We shall find a positive integer s such that

$$d_\Lambda(sx, sy) > 1/10 \quad \text{if } x \neq y \text{ in } P.$$

This will complete the proof of the lemma, because

$$N_\Lambda(1/20) \geqslant |\{sx\colon x \in P\}| = |P| \geqslant \exp M.$$

By (6), we may assume that $0 < \varrho \leqslant 1/10$. Let $s = [1/4\varrho]$. Then

$$1/4 \geqslant s\varrho > 1/4 - \varrho > 3/20.$$

For any $\gamma \in \Gamma$ and $x \in G$ we can write

$$1 - \gamma(sx) = (1 - \gamma(x))\big(1 + \gamma(x) + \gamma(2x) + \ldots, + \gamma((s-1)x)\big).$$

Hence

$$|1 - \gamma(sx)| \geqslant s|1 - \gamma(x)| - (1/2)s(s-1)|1 - \gamma(x)|^2. \tag{7}$$

For $x \neq y$ in P, let $\gamma \in \Lambda$ satisfy

$$|\gamma(x) - \gamma(y)| = d_\Lambda(x, y).$$

It follows from (6) and (7) that

$$d_\Lambda(sx, sy) \geqslant s d_\Lambda(x, y)\big(1 - (1/2)s d_\Lambda(x, y)\big) \geqslant s\varrho(1 - 2s\varrho).$$

By our choice of s, the right-hand side is $\geqslant 21/200$. This completes the proof.

Proof of Lemma 2. We show that

(8) $$\bar{d}_A(0,x) \leqslant (3\pi/4)\, d_A(0,x).$$

From the invariance property, this clearly implies (5). We will use the spectral synthesis property of the point 0 in the unit circle $\boldsymbol{T}$. Thus

$$1-e^{i\theta} = \sum_{k\in \boldsymbol{Z}} c_k e^{ik\theta} \quad \text{if } |\theta| \leqslant \tau,$$

where

(9) $$\sum_{k\in\boldsymbol{Z}} |c_k| = \tau + O(\tau^2).$$

See [G–M] (p. 417) for this result and a proof which the authors attribute to N. Wiener. In fact, specific computations show that for all $\tau > 0$ (9) is true with $\tau+O(\tau^2)$ replaced by $3\tau/2$.

Assume now $x\in G$ satisfies $d_A(0,x) = 2\sin(\tau/2)$, $0<\tau\leqslant\pi$. Then for each $\gamma\in\Lambda$ we may write $\gamma(x) = e^{i\theta}$ where $|\theta|\leqslant\tau$. From (9) it follows that

(10) $$1-\gamma(x) = \sum_{k\in\boldsymbol{Z}} c_k \gamma(kx).$$

Let $f\in C_A$. Then by (10)

$$f(0)-f(x) = \sum_{\gamma\in\Lambda} \hat{f}(\gamma)(1-\gamma(x)) = \sum_{k\in\boldsymbol{Z}} c_k \Big(\sum_{\gamma\in\Lambda} \hat{f}(\gamma)\gamma(kx)\Big) = \sum_{k\in\boldsymbol{Z}} c_k f(kx),$$

$$|f(0)-f(x)| \leqslant \Big(\sum_{k\in\boldsymbol{Z}} |c_k|\Big)\|f\|_\infty \leqslant (3\tau/2)\|f\|_\infty.$$

Therefore $\bar{d}_A(0,x)\leqslant 3\tau/2$. Since $\tau\leqslant\pi\sin(\tau/2)$, this proves the lemma.

3. Proof of Theorem 1. Fix $\Lambda\subset\Gamma$, Λ finite and $\varepsilon>0$. Put $\eta=\varepsilon/6$. It follows from (4) and (5) that

(11) $$\log \bar{N}_A(\eta) \leqslant (\log_2(20/\eta))\log N_A(1/20).$$

From the definition of $\bar{d}_A$, there is an expectation operator E satisfying

(12) $$\operatorname{rank} E \leqslant \bar{N}_A(\eta) \equiv S,$$

(13) $$|Ef-f| \leqslant \eta\|f\|_\infty \quad \text{if } f\in C_A.$$

Average E to obtain a convolution operator. Thus let K_1 be the convolution kernel of the operator

$$\int_G (R_{x^{-1}} E R_x)\,dx \quad (R_x = \text{translation by } x\in G).$$

From (12) and (13)

(14) $$\|K_1\|_1 \leqslant 1,$$

(15) $$\|K_1\|_\infty \leqslant \|E\|_{N(\mathscr{C}(G),\mathscr{C}(G))} \leqslant S \quad (\|\cdot\|_{N(\cdot,\cdot)} = \text{nuclear norm}),$$

(16) $$\|f-(f*K_1)\|_\infty \leqslant \eta\|f\|_\infty \quad \text{if } f\in C_A.$$

(16) also means that

$$\|1-\hat{K}_1|_A\|_{B(A)} \leqslant \eta.$$

Let $\mu\in M(G)$ fulfill $\|\mu\|\leqslant\eta$ and $\hat{\mu}|_A = 1-\hat{K}_1|_A$. Define

$$K_2 = K_1 * \Big(\sum_{j=0}^{\infty} \mu^{(j)}\Big), \quad \mu^{(j)} \equiv \mu*\mu*\ldots*\mu \quad (j\text{-fold convolution}).$$

Then

(17) $$\|K_2\|_1 \leqslant \|K_1\|_1(1-\|\mu\|)^{-1} \leqslant (1-\eta)^{-1},$$

(18) $$\hat{K}_2(\gamma) = 1 \quad \text{if } \gamma\in\Lambda,$$

(19) $$\|K_2\|_\infty \leqslant (1-\eta)^{-1}\|K_1\|_\infty \leqslant (1-\eta)^{-1}S.$$

Finally, let

$$\hat{K}(\gamma) = \begin{cases} \hat{K}_2(\gamma)^3 & \text{if } |\hat{K}_2(\gamma)| > S^{-3}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by (19)

$$|K-(K_2*K_2*K_2)| \leqslant \sum_{|\hat{K}_2(\gamma)|\leqslant S^{-3}} |\hat{K}_2(\gamma)|^3 \leqslant (1-\eta)^{-2} S S^{-3} < \eta;$$

by (17)

$$\|K\|_1 \leqslant \|K_2\|_1^3 + \eta \leqslant (1-\eta)^{-3}+\eta < 1+6\eta = 1+\varepsilon.$$

Clearly, K still satisfies (2) and

$$|\operatorname{supp}\hat{K}| < S^6 \sum |\hat{K}_2(\gamma)|^2 \leqslant (1-\eta)^{-2} S S^6 < S^8,$$

which together with (11) gives the desired estimate.

Remark. Let E be a finite-dimensional normed space. Define

$$s_1(E) = \max\{m\colon E \text{ contains a 2-isomorphic copy of } l_m^1\},$$

$$s_\infty(E) = \max\{m\colon E \text{ contains a 2-isomorphic copy of } l_m^\infty\}.$$

It is proved in [B] that in general if E is a subspace of l_d^∞, $\dim E > d^\varepsilon$, then $s_\infty(E) > \varepsilon^C(\dim E)^{1/2}$ where C is a constant. Combining this fact and Theorem 1, the next property for invariant spaces is deduced:

$$s_1(C_A)s_\infty(C_A) > |\Lambda|^\tau$$

for a fixed constant $\tau>0$. In [B] an example is given of an n-dimensional space E for which $d(E, l_n^2)\sim\sqrt{n}$ and

$$s_1(E)s_\infty(E) \leqslant (\log n)^2.$$

References

[B] J. Bourgain, *Subspaces of l_d^∞, arithmetic diameter and Sidon sets*, in: Proc. Tufts Conf. Probability Theory, Lecture Notes in Math., Springer, 1986.

[B–P] M. Bożejko and A. Pełczyński, *An analogue in commutative harmonic analysis of the uniform bounded approximation property of a Banach space*, Sém. Maurey–Schwartz 1978–79, École Polytechnique.

[F–J–S] T. Figiel, W. B. Johnson and G. Schechtman, *Natural embeddings of l_p^n into L_r, I*, to appear.

[G–M] C. Graham and O. E. McGehee, *Essays in Commutative Harmonic Analysis*, Grundlehren Math. Wiss. 238, Springer, 1979.

[L–T] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, 1977.

[P] G. Pisier, *Conditions d'entropie et caractérisations des ensembles de Sidon*, preprint, Univ. Paris VI.

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Nonfactorization theorems for Hardy and Bergman spaces on bounded symmetric domains

by

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Abstract. We extend the nonfactorization theorems of Rosay, Gowda and others to the case of bounded symmetric domains and some circular product domains. We show that for such domains D, $H^p(D)\cdot H^q(D)$ (resp. $A^{p,\alpha}(D)\cdot A^{q,\alpha}(D)$) is of first category in the Hardy space $H^l(D)$ (resp. the Bergman space $A^{l,\alpha}(D)$), where $1/p+1/q = 1/l$.

1. Introduction. Let U be the unit disc in the complex plane. It is well known that if $0 < p, q, l < \infty$ and $1/p+1/q = 1/l$ then every function f in the Hardy space $H^l(U)$ can be written as a product $f = g\cdot h$ with $g \in H^p(U)$ and $h \in H^q(U)$. The same was shown to be true for weighted Bergman spaces by Horowitz [7]. These results have no generalization to higher dimensions as was shown by Rudin [13], Miles [11] and Rosay [12] for polydiscs and by Gowda [5] for the unit ball. In [17] it was remarked that using just the conclusion of Gowda's theorem (and not its proof) one can easily obtain a generalization of his result to classical symmetric domains of type I. In this paper we show how Gowda's proof can be applied to all bounded symmetric domains and also to some product domains. We obtain exact generalizations of results of [5], namely we show that $H^p(D)\cdot H^q(D)$ (resp. $A^{p,\alpha}(D)\cdot A^{q,\alpha}(D)$) is of first category in $H^l(D)$ (resp. $A^{l,\alpha}(D)$). The proofs are fairly elementary and do not use the Lie algebra machinery, nevertheless they depend on facts that were proved with that technique.

2. Preliminaries. Since we will sometimes refer for details to [5], we will follow as closely as possible the notation of that paper. There will be, however, some additional notation and definitions.

A domain D in $\mathbf{C}^n$ is called *symmetric* if each of its points is an isolated fixed point of an involutive biholomorphic automorphism of D. Below we list main well-known facts about bounded symmetric domains.

The group $\operatorname{Aut}(D)$ of holomorphic automorphisms of D acts transitively on D. Each $\Phi \in \operatorname{Aut}(D)$ can be extended to a holomorphic mapping of a neighbourhood of $\bar{D}$. D can be realized as a circular convex domain such

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